Házi feladatok megoldása - 2. feladatsor

1. Divergencia, Gauss-tétel

a.)

$$\underline{F}(\underline{r}) = F_0\,\underline{r} = \begin{pmatrix} F_0 x \\ F_0 y \\ F_0 z \end{pmatrix} \qquad F_0 = \text{áll.}$$

$$\operatorname{div} \underline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 3F_0$$

b.) Gauss-tétel teljesülésének ellenőrzése

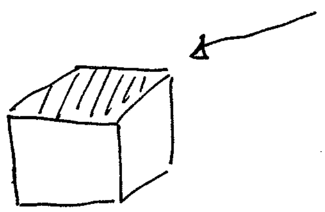
$$\int_K \operatorname{div} F\, dV = \int_{-a}^{a} dx \int_{-a}^{a} dy \int_{-a}^{a} dz\, \operatorname{div} F =$$

$$= \int_{-a}^{a} dx \int_{-a}^{a} dy \int_{-a}^{a} dz\; 3F_0 = 3F_0 \underbrace{\int\ldots\int\ldots\int\ldots}_{\text{kocka térf.}}$$

$$= 3F_0 \cdot 8a^3 = 24 F_0 a^3$$

$$\int_{\partial K} \underline{F}\, d\underline{f} \qquad \text{a lapokra vett felület integrálok összege}$$

szimmetria $\Rightarrow$ a 6 lapra vett integrál egyenlő, számoljunk ki csak egyet



ezen a lapon: $z = a$

normálvektor: $\underline{N} = \underline{k}$ $\qquad d\underline{f} = \underline{N}\, df = \underline{k}\, df$

$$df = dx\, dy$$

$$\int_{\text{felső lapra}} \underline{F}\, d\underline{f} = \int_{-a}^{a}\int_{a}^{a} \underline{F}\cdot\underline{k}\; dx\, dy \qquad \underline{F}\cdot\underline{k} = \begin{pmatrix} F_0 x \\ F_0 y \\ F_0 z \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = F_0 z \underset{\text{a lapon}}{=} F_0 a$$

$$\int_{-a}^{a} dx \int_{-a}^{a} dy \; F_0 a = F_0 a \int_{-a}^{a} dx \int_{-a}^{a} dy = F_0 a \cdot 4a^2 = 4F_0 a^3$$

6 lap van $\Rightarrow$

$$\int_{\partial K} \underline{F}\, d\underline{f} = 6 \cdot (4F_0 a^3) = 24 F_0 a^3$$

a Gauss-tételnek megfelelően azt kaptuk, hogy

$$\int_K \operatorname{div} \underline{F}\, dV = 24 F_0 a^3 = \int_{\partial K} \underline{F}\, d\underline{f}$$

2. Nem potenciálos vektormező

$$\underline{F}(\underline{r}) = \underline{r} \times \underline{A} \qquad \underline{A} = (0, 0, A)$$

$$\underline{F} = \underline{r} \times \underline{A} = \begin{pmatrix} yA_z - zA_y \\ zA_x - xA_z \\ xA_y - yA_x \end{pmatrix} = \begin{pmatrix} yA \\ -xA \\ 0 \end{pmatrix}$$

a görbe mentén

$$\underline{r}(t) = \begin{pmatrix} R\cos t \\ R\sin t \\ 0 \end{pmatrix}$$

$$\dot{\underline{r}}(t) = \begin{pmatrix} -R\sin t \\ R\cos t \\ 0 \end{pmatrix} \qquad \underline{F}(\underline{r}(t)) = \begin{pmatrix} AR\sin t \\ -AR\cos t \\ 0 \end{pmatrix}$$

$$\underline{F}(\underline{r}(t)) \cdot \dot{\underline{r}}(t) = -AR^2 \; , \text{ így}$$

a munka:

$$\oint \underline{F}\, d\underline{r} = \int_0^{2\pi} \underline{F}(\underline{r}(t))\, \dot{\underline{r}}(t)\, dt = -AR^2 \int_0^{2\pi} dt = \underline{-2\pi A R^2}$$

3. Gradiensszámolás

a.) $\phi(\underline{r}) = r = |\underline{r}| = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial \phi}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$\frac{\partial \phi}{\partial y} = \frac{y}{r}$ $\frac{\partial \phi}{\partial z} = \frac{z}{r}$ teljesen hasonlóan, így

$$\nabla \phi(\underline{r}) = \frac{1}{r}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{\underline{r}}{r}$$

b.) $\phi(\underline{r}) = r^2 = x^2+y^2+z^2$

$\frac{\partial \phi}{\partial x} = 2x$, $\frac{\partial \phi}{\partial y} = 2y$ $\frac{\partial \phi}{\partial z} = 2z$ $\nabla \phi(\underline{r}) = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\underline{r}$

c.) $\phi(\underline{r}) = r^3 = (x^2+y^2+z^2)^{3/2}$

$\frac{\partial \phi}{\partial x} = \frac{3}{2}(x^2+y^2+z^2)^{1/2}\, 2x$ $\nabla \phi(r) = 3r\underline{r}$

d.) $\phi(\underline{r}) = r^n$

$$\frac{\partial \phi}{\partial x} = n \cdot r^{n-1} \frac{\partial r}{\partial x} = n \cdot r^{n-1} \cdot \frac{x}{r} = n \cdot r^{n-2} x$$

$\frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$ hasonlóan $\nabla \phi(\underline{r}) = n \cdot r^{n-2}\, \underline{r}$

e.) $\phi(\underline{r}) = f(r)$

$$\frac{\partial \phi}{\partial x} = f'(r)\frac{\partial r}{\partial x} = f'(r)\frac{x}{r} \rightarrow \nabla \phi(\underline{r}) = \frac{f'(r)}{r}\underline{r}$$

y, z teljesen hasonlóan