Emlékeztető

hatás: $S[q(t)] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)\,dt$

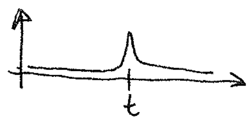
mozgásegyenletek származtatása: $S[q+\delta q] - S[q]$ lineáris része

$$S[q+\delta q] - S[q] \approx \delta S[q] =$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial q} \delta q \right) dt = \frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2}$$

$$+ \int_{t_1}^{t_2} \left(-\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \right) \delta q \, dt$$

Hamilton-elv ("legkisebb hatás elve") $\delta S = 0$ a valódi pálya mentén minden olyan δq variációra, melyre $\delta q(t_1) = \delta q(t_2) = 0$

Ennek a feltétele: ami az integrálban δq-t szorozza, az 0 legyen (t pillanatban $\delta q =$  → ott $(\ldots) = 0$ kell)

$$-\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

(variációs probléma Euler-Lagrange-egyenlete v. Lagrange-féle másodfajú ME)

Ha $L(q, q, t) = L(q, q, \cancel{t})$ explicite nem függ az időtől
$\Rightarrow$ Beltrami fv.

$p = \frac{\partial L}{\partial q}$ $\qquad H = p\dot{q} - L$ $\qquad$ → energia megmarad

Hamilton-féle (kanonikus) mozgásegyenletek

$$H = p\dot{q} - L \qquad p = \frac{\partial L}{\partial \dot{q}} \qquad -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} = 0$$

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \quad \Rightarrow \quad \boxed{\dot{p} = -\frac{\partial L}{\partial q}}$$

$$\frac{\partial H}{\partial p} = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - \underbrace{\frac{\partial L}{\partial \dot{q}}}_{p}\frac{\partial \dot{q}}{\partial p} = \dot{q} \quad \Rightarrow \quad \boxed{\dot{q} = \frac{\partial L}{\partial p}}$$

$$\frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial q}}_{-\dot{p}}\dot{q} + \underbrace{\frac{\partial H}{\partial p}}_{\dot{q}}\dot{p} + \frac{\partial H}{\partial t} \quad \Rightarrow \quad \frac{dH}{dt} = \frac{\partial H}{\partial t}$$

$$\frac{\partial H}{\partial t} = p\frac{\partial \dot{q}(q,p,t)}{\partial t} - \frac{\partial L}{\partial t} - \underbrace{\frac{\partial L}{\partial \dot{q}}}_{p}\frac{\partial \dot{q}(q,p,t)}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

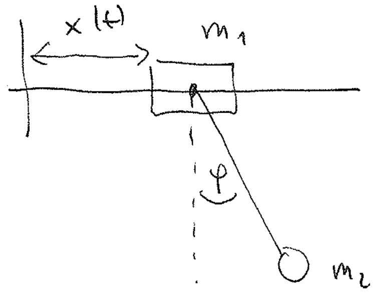
Problémamegoldás receptje

1. Jó q általános koordináták választása
2. L-fv felírása (pl. $L = K - V$)

↙ ↘

3. E–L egy. felírása	3. $p = \frac{\partial L}{\partial \dot{q}} \to H$ (pl. $H = K + V$)
4. E–L-mego.	4. Hamilton-egy. felírása
	5. Hamilton-egy. mego.

1. Példa a Lagrange- (Hamilton-...) függvény alkalmazására



rúd mentén szabadon elcsúszó test

a testre felfüggesztett inga

① általános koordináták választása

$x_1 = x(t)$ $\qquad\qquad$ $\underline{x, \varphi}$

$z_1 = 0$

$x_2 = x + l\sin\varphi$

$\cancel{z}_2 = -l\cos\varphi$

② L-fgv felírása:

$\dot{x}_1 = \dot{x}$

$\dot{z}_1 = 0$

$\dot{x}_2 = \dot{x} + l\cos\varphi\,\dot{\varphi}$

$\dot{z}_2 = l\sin\varphi\,\dot{\varphi}$

$$K = \frac{1}{2} m_1(\dot{x}_1^2 + \dot{z}_1^2) + \frac{1}{2} m_2(\dot{x}_2^2 + \dot{z}_2^2) =$$

$$= \frac{1}{2} m_1(\dot{x}^2 + 0) + \frac{1}{2} m_2\left((\dot{x} + l\cos\varphi\,\dot{\varphi})^2 + (l\sin\varphi\,\dot{\varphi}^2)\right)$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\varphi}^2 + \cancel{\frac{1}{2}}\cdot\cancel{2}\, m_2 l\cos\varphi\,\dot{\varphi}\dot{x}$$

$$V = m_1 g z_1 + m_2 g z_2 = -m_2 g l\cos\varphi$$

$$L = K - V = \frac{m_1+m_2}{2}\dot{x}^2 + \frac{1}{2} m_2 l^2\dot{\varphi}^2 + m_2 l\cos\varphi\,\dot{\varphi}\dot{x} + m_2 l g\cos\varphi$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 l\cos\varphi\,\dot{\varphi}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m_2 l^2\dot{\varphi} + m_2 l\cos\varphi\,\dot{x}$$

$$\frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad x \text{ ciklikus koordináta}, \; P_x = \text{áll.}$$

P_x = összimpulzus! $\quad \underline{P_x = (m_1+m_2)\dot{x} + m_2 l \dot{\varphi}\cos\varphi}$

$$\frac{d}{dt} P_x = \frac{\partial L}{\partial x} = 0 \qquad P_x = \text{áll.} \qquad \dot{P}_x = (m_1+m_2)\ddot{x} + m_2 l \ddot{\varphi}\cos\varphi - m_2 l \dot{\varphi}^2 \sin\varphi$$

$$\frac{\partial L}{\partial \varphi} = -m_2 l \sin\varphi \, \dot{\varphi}\dot{x} - m_2 l g \sin\varphi$$

$$\dot{P}_x = 0$$

$$\dot{P}_\varphi = m_2 l^2 \ddot{\varphi} + m_2 l \cos\varphi \, \ddot{x} - \cancel{m_2 l \sin\varphi \, \dot{\varphi}\dot{x}}$$

$$\overset{!}{=} -\cancel{m_2 l \sin\varphi \, \dot{\varphi}\dot{x}} - m_2 l g \sin\varphi$$

rendezve:

$$\underbrace{\begin{pmatrix} m_1+m_2 & m_2 l\cos\varphi \\ m_2 l \cos\varphi & m_2 l^2 \end{pmatrix}}_{B} \begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} = \begin{pmatrix} m_2 l \dot{\varphi}^2 \sin\varphi \\ -m_2 l g \sin\varphi \end{pmatrix}$$

ki tudjuk-e fejezni $\ddot{x}$-ot, $\ddot{\varphi}$-ot ?

$$\det B = \underbrace{(m_1+m_2) m_2 l^2}_{m_1 m_2 l^2 + \underbrace{m_2^2 l^2}_{m_2^2 l^2 (\sin^2\varphi + \cos^2\varphi)}} - m_2^2 l^2 \cos^2\varphi > 0 \Rightarrow \text{igen, kifejezhető}$$

$$\det B = m_1 m_2 l^2 + m_2^2 l^2 \sin^2\varphi$$

$$B^{-1} = \frac{1}{\det B} \begin{pmatrix} m_2 l^2 & -m_2 l\cos\varphi \\ -m_2 l \cos\varphi & m_1+m_2 \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} = \frac{1}{m_1 m_2 \ell^2 + m_2^2 \ell^2 \sin^2\varphi} \cdot \begin{pmatrix} m_2 \ell^2 & -m_2 \ell \cos\varphi \\ -m_2 \ell \cos\varphi & m_1 + m_2 \end{pmatrix} \begin{pmatrix} \dot{\varphi}^2 \\ -g \end{pmatrix} m_2 \ell \sin\varphi$$

$$= \frac{\sin\varphi}{m_1 \cancel{\ell} + m_2 \cancel{\ell} \sin\varphi} \begin{pmatrix} m_2 \ell^{\cancel{2}} \dot{\varphi}^2 + m_2 \cancel{\ell} \cos\varphi \, g \\ -m_2 \cancel{\ell} \cos\varphi \, \dot{\varphi}^2 \quad -\frac{1}{\ell} g (m_1 + m_2) \end{pmatrix}$$

$$\ddot{x} = \frac{m_2 \sin\varphi}{m_1 + m_2 \sin\varphi} \left(\dot{\varphi}^2 + g \cos\varphi \right)$$

$$\ddot{\varphi} = \frac{-\sin\varphi}{m_1 + m_2 \sin\varphi} \left(m_2 \cos\varphi \, \dot{\varphi}^2 + g \frac{m_1 + m_2}{\ell} \right)$$

<u>Hamilton-fv:</u>

$$H = p_x \dot{x} + p_\varphi \dot{\varphi} - L$$

$$\begin{pmatrix} p_x \\ p_\varphi \end{pmatrix} = \dots$$

itt nem érdemes kiszámolni, hosszú

<u>Egyensúly körüli kis rezgések</u> $\varphi = 0$ a potenciál minimuma

akörül sorfejtés $\cos\varphi \approx 1 - \frac{\varphi^2}{2}$

$$L \approx \frac{m_1 + m_2}{2} \dot{x}^2 + \frac{1}{2} m_2 \ell^2 \dot{\varphi}^2 + m_2 \ell \dot{\varphi} \dot{x} + m_2 \ell g \left(1 - \frac{\varphi^2}{2}\right)$$

EOM: $p_x \approx (m_1 + m_2) \dot{x} + m_2 \ell \dot{\varphi}$ $\qquad \dot{p}_x \approx (m_1 + m_2) \ddot{x} + m_2 \ell \ddot{\varphi}$

$p_\varphi \approx m_2 \ell^2 \dot{\varphi} + m_2 \ell \dot{x}$

$$\underbrace{\begin{pmatrix} m_1+m_2 & m_2 \ell \\ m_2 \ell & m_2 \ell^2 \end{pmatrix}}_{\underline{\underline{M}}} \begin{pmatrix}\ddot{x}\\ \ddot{\varphi}\end{pmatrix} = \begin{pmatrix}0\\ -m_2 \ell g \varphi\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}0 & 0\\ 0 & -m_2 \ell g\end{pmatrix}}_{-\underline{\underline{K}}}\begin{pmatrix}x\\ \varphi\end{pmatrix}$$

~~stétvá~~

$$\begin{pmatrix}\ddot{x}\\ \ddot{\varphi}\end{pmatrix} = -\underline{\underline{M}}^{-1}\underline{\underline{K}}\begin{pmatrix}x\\ \varphi\end{pmatrix} \qquad \underline{\underline{M}}^{-1} = \frac{1}{\det M}\,\mathrm{adj}M$$

$$\ddot{x} = 0 \qquad \det M = (m_1+m_2)m_2\ell^2 - m_2^2\ell^2$$

$$\ddot{\varphi} = -\frac{1}{m_1\cancel{m_2}\ell^{\cancel{2}}}(m_1+m_2)\cancel{m_2}\ell g\varphi \qquad = m_1 m_2 \ell^2$$

$$\mathrm{adj}\,M = \frac{1}{m_1 m_2 \ell^2}\begin{pmatrix} m_2\ell^2 & -m_2\ell \\ -m_2\ell & m_1+m_2\end{pmatrix}$$

$$\omega_1^2 = 0$$

$$\omega_2^2 = \frac{m_1+m_2}{m_1}\,\frac{g}{\ell}$$ a másik vergési frekv.

2. Példa Hamilton-egyenletekre

térbeli oscillátor $\quad L = K - V$

$$K = \frac{1}{2}m\dot{\underline{r}}^2 \qquad V = \frac{1}{2}k\,\underline{r}^2$$

$$L = K - V = \frac{1}{2}m\,\dot{\underline{r}}^2 - \frac{1}{2}k\,\underline{r}^2$$

Euler-Lagrange: $$\frac{\partial L}{\partial r_i} = -\frac{1}{2}k\cdot 2r_i = -k r_i$$

$$p_i = \frac{\partial L}{\partial \dot{r}_i} = m\dot{r}_i$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} - \frac{\partial L}{\partial r_i} = 0 \quad \rightarrow \quad m\ddot{r}_i + k r_i = 0 \qquad \omega^2 = k/m\text{-el}$$

$$\ddot{r}_i + \omega^2 r_i = 0$$

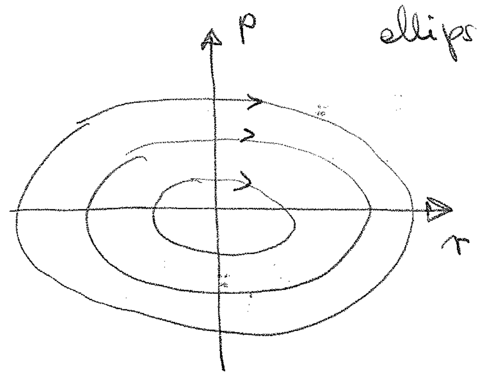
$$H = \underline{p}\,\dot{\underline{r}} - L \qquad \dot{\underline{r}} = \frac{\underline{p}}{m}$$

$$H = \frac{\underline{p}^2}{m} - \left(\frac{1}{2} m \left(\frac{\underline{p}}{m}\right)^2 - \frac{1}{2} k\, \underline{r}^2\right) = \frac{p^2}{2m} + \frac{1}{2} k r^2$$

$$= \underline{\frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2}$$

fázistér - trajektóriák:
$H = \mathrm{const}$

ellipszisek



$$\dot{\underline{r}} = \frac{\partial H}{\partial p} = \frac{\underline{p}}{m}$$

$$\dot{\underline{p}} = -\frac{\partial H}{\partial \underline{r}} = -k\,\underline{r}$$

3. Centrális mozgás Lagrange - függvényes tárgyalása

$$L = K - V$$

$$K = \frac{1}{2} m \dot{\underline{r}}^2$$

- síkmozgás: $\underline{r} = r \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$

$$\dot{\underline{r}} = r\dot{\varphi} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} + \dot{r} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$\dot{\underline{r}}^2 = r^2 \dot{\varphi}^2 + \dot{r}^2 + \dot{r} r \dot{\varphi} \underbrace{(-\sin\varphi\cos\varphi + \cos\varphi\sin\varphi)}_{0}$$

- azzal $K = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2)$

- a teljes Lagrange - függvény

$$L = K - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) - V(r)$$

Euler - Lagrange - egyenletek

$$p_r = \frac{\partial L}{\partial \dot r} = m \dot r \qquad \frac{d}{dt} p_r = m \ddot r$$

$$\frac{\partial L}{\partial r} = m r \dot\varphi^2 - V'(r)$$

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = m r^2 \dot\varphi \qquad \frac{d}{dt} p_\varphi = 2m \dot r \dot\varphi + m r^2 \ddot\varphi$$

$$\frac{\partial L}{\partial \varphi} = 0 \quad \to \varphi$$ ciklikus koordináta

$$\boxed{p_\varphi = N_z = \text{áll.}}$$

$$\dot\varphi = \frac{p_\varphi}{m r^2} \qquad \dot\varphi^2 = \frac{p_\varphi^2}{m^2 r^4}$$

$$\to \quad \frac{\partial L}{\partial r} = m r \frac{p_\varphi^2}{m^2 r^4} - V'(r) = \frac{p_\varphi^2}{m r^3} - V'(r) = -V_{eff}'(r)$$

$$V_{eff} = V(r) + \frac{p_\varphi^2}{2 m r^2}$$

visszakaptuk, amit már egyszer levezettünk, csak sokkal egyszerűbben

Félig áttérünk Hamiltoni formalizmusba: Routh - fv.

$$R = p_\varphi \dot\varphi - L = \frac{p_\varphi^2}{m r^2} - \left(\frac{p_\varphi^2}{2 m r^2} + \frac{1}{2} m \dot r^2 - V(r) \right)$$

$$p_\varphi = m r^2 \dot\varphi$$

$$= \frac{p_\varphi^2}{2 m r^2} - \frac{1}{2} m \dot r^2 + V(r)$$

$$\dot\varphi = \frac{p_\varphi}{m r^2}$$

φ mozgásegyenlete Hamilton-féle

$$\dot{p}_\varphi = -\frac{\partial R}{\partial \varphi} = 0$$

$$\dot{\varphi} = \frac{\partial R}{\partial p_\varphi} = \frac{p_\varphi}{m r^2}$$ valóban visszakaptuk

r mozgásegyenlete Lagrange-féle

$$-p_r = \frac{\partial R}{\partial \dot{r}} = -\frac{\partial L}{\partial \dot{r}} = -m\dot{r}$$

$$p_r = m\dot{r} \qquad \dot{p}_r = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = -\frac{\partial R}{\partial r} = \frac{p_\varphi^2}{m r^3} - V'(r)$$

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{r}} = \frac{\partial R}{\partial r} \qquad m\ddot{r} = \frac{p_\varphi^2}{m r^3} - V'(r)$$

<u>Hamilton-függvény</u>

$$H = \dot{r} p_r + \dot{\varphi} p_\varphi - L = \frac{p_r}{m} p_r + \frac{p_\varphi}{m r^2} p_\varphi$$
$$- \left(\frac{1}{2} m \left(\frac{p_r}{m}\right)^2 + \frac{1}{2} m r^2 \left(\frac{p_\varphi}{m r^2}\right)^2 - V(r) \right) =$$
$$= \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2 m r^2} + V(r)$$ = a teljes energia

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2} \qquad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \underbrace{\frac{p_\varphi^2}{mr^3} - V'(r)}_{V_{eff}'(r)}$$

tehát a Lagrange- és a Hamilton-formalizmusból is visszakapjuk a jól ismert eredményeket, de sokkal rövidebben.

4. Poisson-zárójelek

Hamiltoni rendszer: $q_i, p_i \quad (i=1,\dots,N) \qquad H = H(q,p,t)$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

tekintsünk egy általános $f = f(q,p,t)$ függvényt!

$$\dot{f} = \frac{df}{dt} = \frac{\partial f(q,p,t)}{\partial t} + \sum_{i=1}^{N}\left[\frac{\partial f(q,p,t)}{\partial q_i}\dot{q}_i + \frac{\partial f(q,p,t)}{\partial p_i}\dot{p}_i\right]$$

$$= \frac{\partial f}{\partial t} + \sum_{i=1}^{N}\left[\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right]$$

$$= \frac{\partial f}{\partial t} + \{f, H\}$$

ahol

$$\underbrace{\{f,g\} = \sum_{i=1}^{N}\left[\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right]}_{\text{Poisson-zárójel}}$$

(Goldstein, Landauban fordítva)

A PZ-ek tulajdonságai:

1. linearitás $\{f+g, h\} = \{f, h\} + \{g, h\}$

$c \in \mathbb{R}$ $\{cf, h\} = c\{f, h\}$

2. Leibniz-szabály

$$\{f \cdot g, h\} = f\{g, h\} + \{f, h\}g$$

ez a differenciálás Leibniz-szabályából triviálisan adódik.

3. $\{f, g\} = -\{g, f\}$

4. Jacobi-azonosság

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

elvben: $\{f_1, f_2\} = \sum_{i=1}^{N} \left(\frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q_i} \right) =$

$$= \underbrace{\sum_{i=1}^{N} \left(\frac{\partial f_2}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f_2}{\partial q_i} \frac{\partial}{\partial p_i} \right)}_{D_2} f_1$$

$$\{\{f, g\}, h\} + \underbrace{\{\{h, f\}, g\}}_{-\{\{f, h\}, g\}} = D_h D_g f - D_g D_h f$$

$$D_h = \sum_k \xi_k \frac{\partial}{\partial x_k} \qquad \xi_k = \left(\frac{\partial h}{\partial p_i}, -\frac{\partial h}{\partial q_i} \right)$$

$$D_g = \sum_k \eta_k \frac{\partial}{\partial x_k} \qquad \frac{\partial}{\partial x_k} = \left(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i} \right)$$

$D_h D_g f - D_g D_h f$: másodrendű deriváltak kiesnek

$$D_h D_g f - D_g D_h f = \sum_{k,l=1}^{2N} \left(\xi_k \frac{\partial \eta_l}{\partial x_k} - \eta_k \frac{\partial \xi_l}{\partial x_k} \right) \frac{\partial f}{\partial x_l}$$

így $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$

baloldalt: a másodrendű tagok kölcsönösen kiejtik egymást. De akkor ugyanígy g, h második deriváltjai is $\rightarrow$ az egész eltűnik.

MJ: Lie-algebra

Példa: egy koordináta függvénye:

$$f = f(q_j) \qquad \{f,g\} = ?$$

$$\{f,g\} = \sum_{i=1}^{N} \Big(\underbrace{\frac{\partial f}{\partial q_i}}_{f'(q)\,\underbrace{\delta_{ij}}_{\frac{\partial q_j}{\partial q_i}}} \frac{\partial g}{\partial p_i} - \underbrace{\frac{\partial f}{\partial p_i}}_{0} \frac{\partial g}{\partial q_i} \Big) = f'(q_j)\{q_j, g\}$$

$$\frac{\partial q_j}{\partial p_i} = 0$$

Még egy érdekesség:

$$\boxed{\begin{aligned} \{q_i, q_j\} &= 0. \\ \{p_i, p_j\} &= 0. \end{aligned}}$$

ui. $\dfrac{\partial q_i}{\partial q_j} = \delta_{ij} \qquad \dfrac{\partial q_i}{\partial p_j} = 0$

$\dfrac{\partial p_i}{\partial p_j} = \delta_{ij} \qquad \dfrac{\partial p_i}{\partial q_j} = 0$

$$\boxed{\{q_i, p_j\}} = \sum_{l=1}^{N} \Big(\underbrace{\frac{\partial q_i}{\partial q_l} \frac{\partial p_j}{\partial p_l}}_{\delta_{il}\delta_{jl}} - \underbrace{\frac{\partial q_i}{\partial p_l} \frac{\partial p_j}{\partial q_l}}_{0} \Big) = \boxed{\delta_{ij}}$$

$$\dot q_i = \{q_i, H\} = \frac{\partial H}{\partial p_i} \qquad \dot p_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}$$

5. Poisson-tétele

ismétlés: $f = f(q,p,t)$

$$\dot{f} = \frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}$$

időtől expliciten nem függő $\left(\frac{\partial f}{\partial t} = 0\right)$ mozgásállandó $\left(\frac{df}{dt} = 0\right)$

PZ-je H-val 0

$$\frac{df}{dt} = 0 \;\&\; \frac{\partial f}{\partial t} = 0 \quad \Rightarrow \quad \{f, H\} = 0$$

Tétel: ha f, g 2 időtől expliciten nem függő mozgásállandó, akkor $\{f, g\}$ is az

Biz: Jacobi-azonosság

$$\{\{f,g\},H\} = -\{\underbrace{\{g,H\}}_{0},f\} - \{\underbrace{\{H,f\}}_{0},g\}$$

általános eset:

Tétel: $\dot{f} = \dot{g} = 0 \quad \Rightarrow \quad \frac{d}{dt}\{f,g\} = 0$

Biz: deriváltak szorzásszabálya

$$\frac{d}{dt}\{f,g\} = \frac{\partial}{\partial t}\{f,g\} + \underbrace{\{\{f,g\},H\}}_{-\{\{g,H\},f\} - \{\{H,f\},g\}}$$

$$-\{\underbrace{\{g,H\}}_{-\frac{\partial g}{\partial t}},f\} - \{\underbrace{\{H,f\}}_{+\frac{\partial f}{\partial t}},g\}$$

$$= \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\} + \left\{\frac{\partial g}{\partial t}, f\right\} - \left\{\frac{\partial f}{\partial t}, g\right\} = 0$$

kiesik

6. Megmaradó mennyiségek

$$\frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial H}{\partial x} = 0$$

$$p_x = \frac{\partial L}{\partial \dot{x}}$$

$$\{p_x, H\} = -\frac{\partial H}{\partial x} = 0 \qquad p_x = \text{áll.}$$