Integrability and Chaos in Multicomponent Systems, 2-7 October 2017

Far Eastern Federal University, Vladivostok

Finite size effects in QFTs

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MTA Wigner Research Centre for Physics, Budapest

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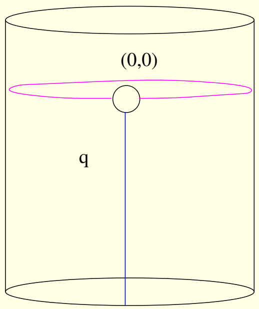
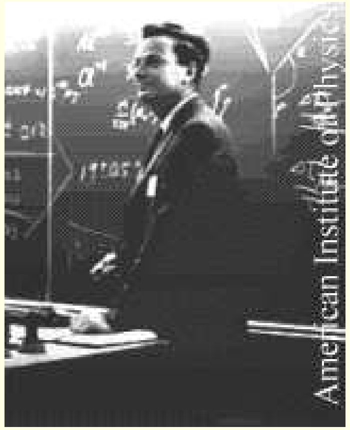
Leading Lüscher correction for a non-diagonal form factor



$$\langle 0|\mathcal{O}|q\rangle_L = \frac{\sqrt{2\pi}}{\sqrt{\rho_1^{(1)}}}\Big\{F_1 + \int_{-\infty}^{\infty} d\theta\, F_3^{\mathrm{reg}}(\theta+i\pi,\theta,\theta_1 - i\frac{\pi}{2})e^{-mL\cosh\theta} + \dots\Big\}$$

In collaboration with János Balog, Márton Lájer and Chao Wu

Introduction



R. P. Feynman

(1918–1988)

If you want to learn about nature, to appreciate nature, it is necessary to understand the language she speaks in: Quantum Field Theory

Fundamental interactions

interaction	particles	gauge theory
electromagnetic	photon+electron	$U(1)$
electroweak	$W^{\pm}, Z$ μ,ν+Higgs	$SU(2) \times U(1)$
strong	gluon+quarks	$SU(3)$

Effective theories: solid state systems, statistical physics...

Strongly coupled gauge theories?

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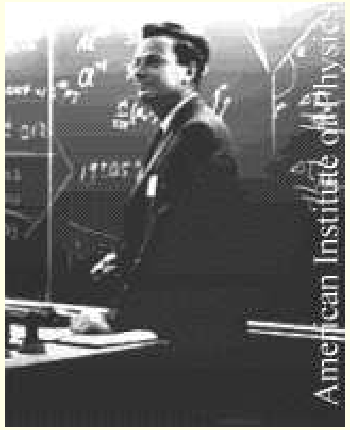

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maximally supersymmetric gauge theory (harmonic oscillator) or QCD

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$\leftrightarrow$ Lattice simulations

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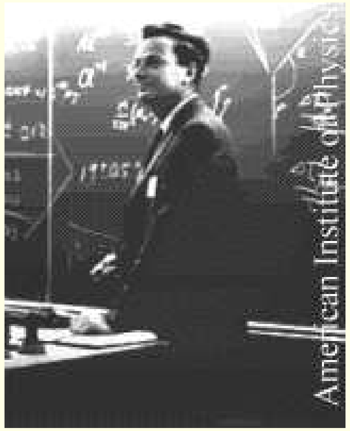

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$\leftrightarrow$ Lattice simulations

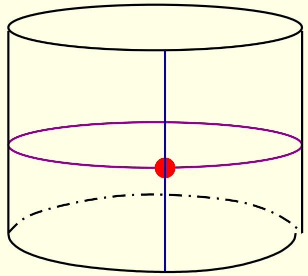
Finite size effects are unavoidable

II_B superstring on $AdS_5 \times S^5$: finite J charge $\leftrightarrow$ Finite size lattices

Finite size effects

Finite size energy corrections:

finite volume mass from lattice simulations: Lüscher



$$m(L) = -\int \frac{d\theta}{2\pi} \cosh\theta \left(S(\theta + \frac{i\pi}{2}) - 1 \right) e^{-mL\cosh\theta}$$

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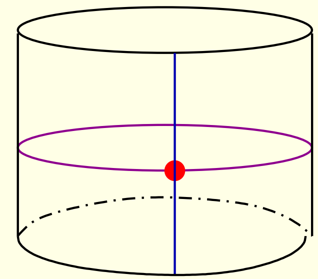

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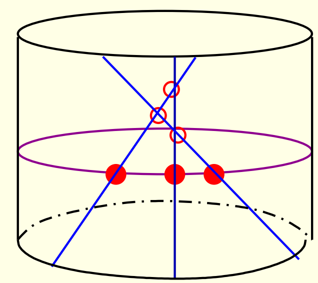
Perturbative dimension of $\mathrm{Tr}(\Phi^2)$ at 4loop from strings:

$$(324+864\zeta_3-1440\zeta_5)g^8$$

conjectured multiparticle Lüscher correction [ZB,Janik]



$$E(\theta_1,\ldots,\theta_n) = \sum_i m\cosh(\theta_i + \delta\theta_i)$$
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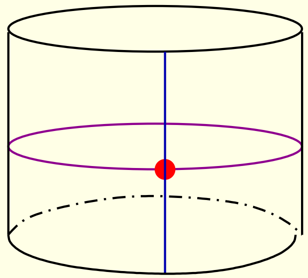

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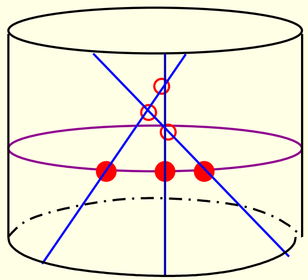
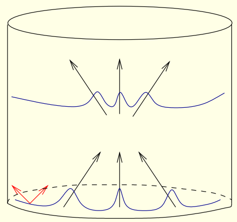
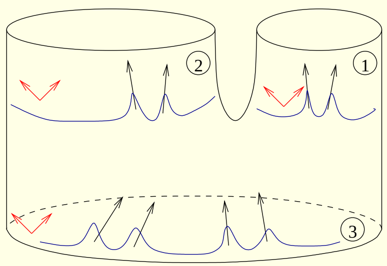


Decay rates from the lattice or 3pt functions from strings

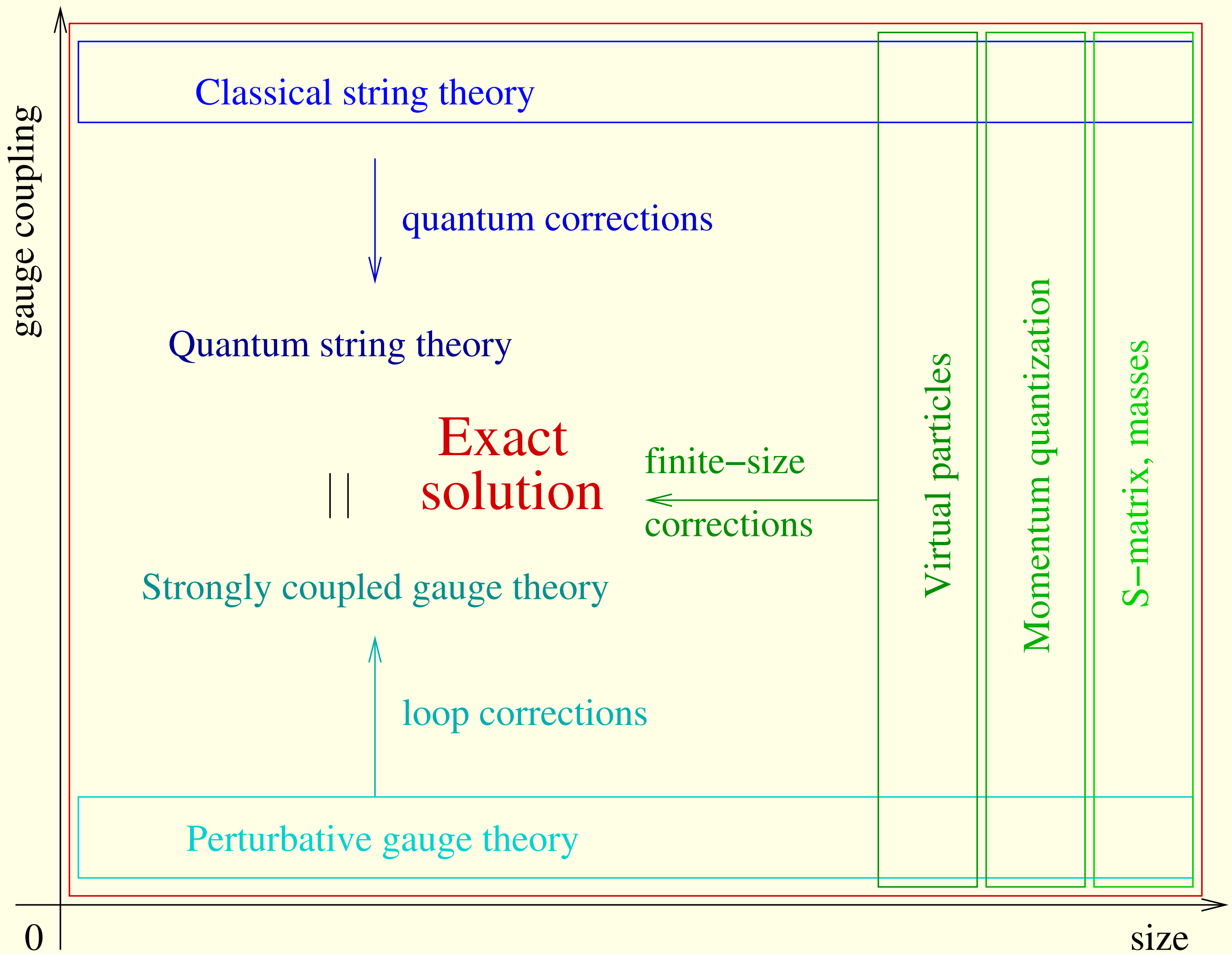
Finite size form factor corrections?

We will work in 1+1D integrable models as higher dimensional counterparts are conceptually the same

Motivation from AdS/CFT

IIB strings on $AdS_5 \times S^5$	Integrability $\longleftrightarrow$	$N = 4$ SYM
	finite volume energy levels	$O(x) = \mathrm{Tr}(\Phi(x)^J)$ J: length $\langle O(x)O(0)\rangle = x^{-2\Delta(\lambda)}$ scaling dimensions
	finite volume form factors	3pt functions $\langle O_1 O_2 O_3 \rangle = C_{123}(\lambda)$

How integrability works:



Observables: sinh-Gordon theory

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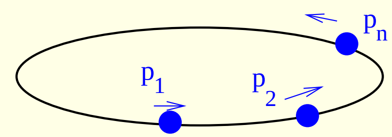
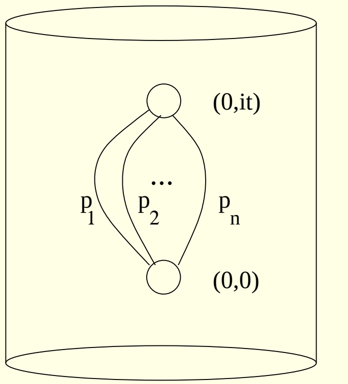
The simplest interacting QFT in 1+1 D: $\mathcal{L}=\frac{1}{2}(\partial_t\varphi)^2-\frac{1}{2}(\partial_x\varphi)^2-\frac{m^2}{b^2}(\cosh b\varphi-1)$

interesting observables: finite size spectrum,





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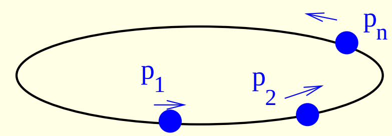

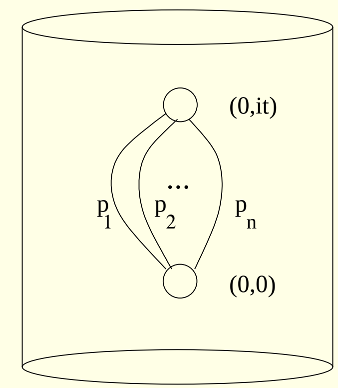

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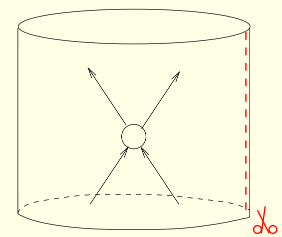
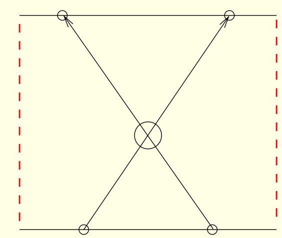
Infinite volume $\to$ LSZ reduction formula





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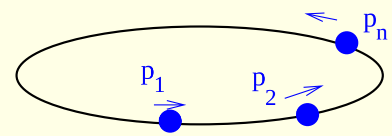

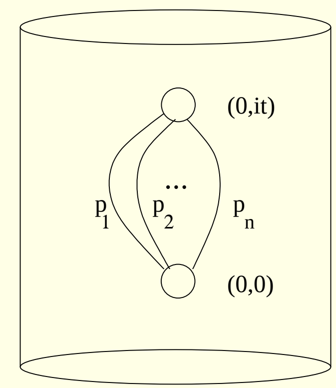

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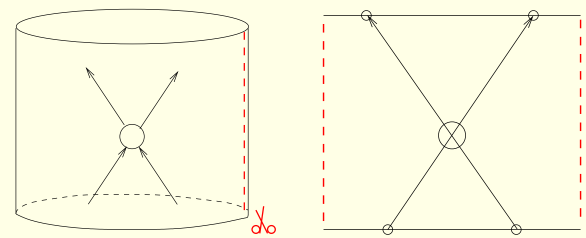

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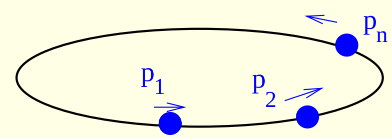

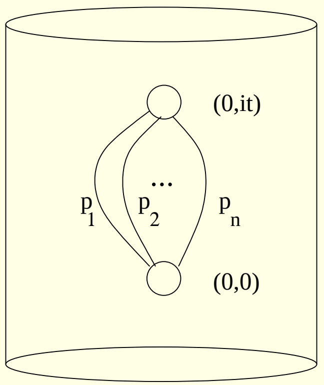

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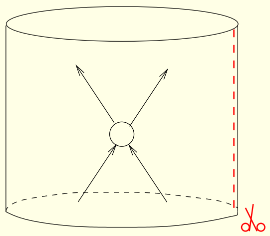

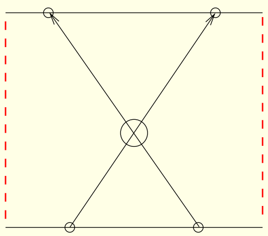

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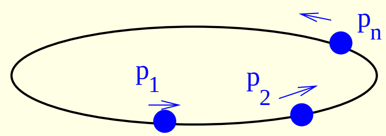

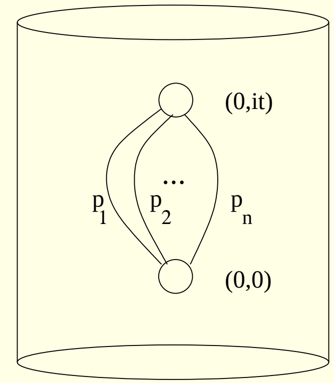

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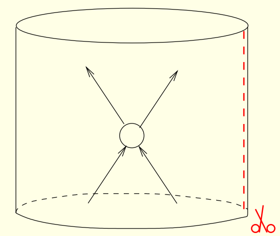

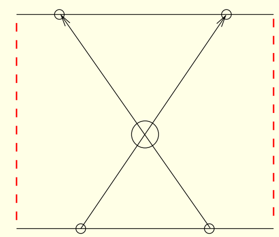

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Perturbative definition, calculational tool: [Arefyeva et al]

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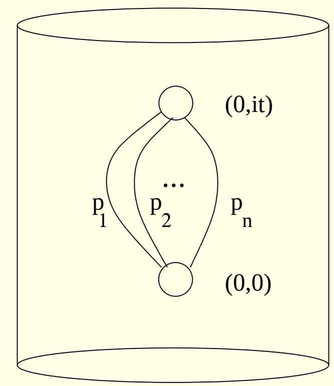

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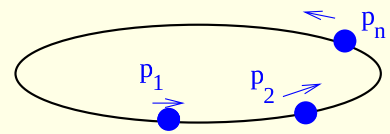

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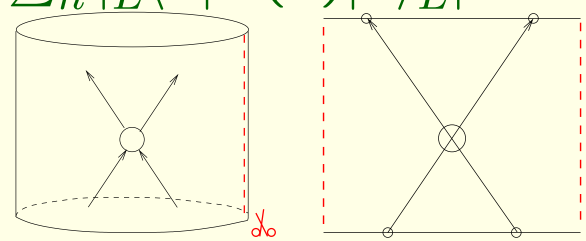

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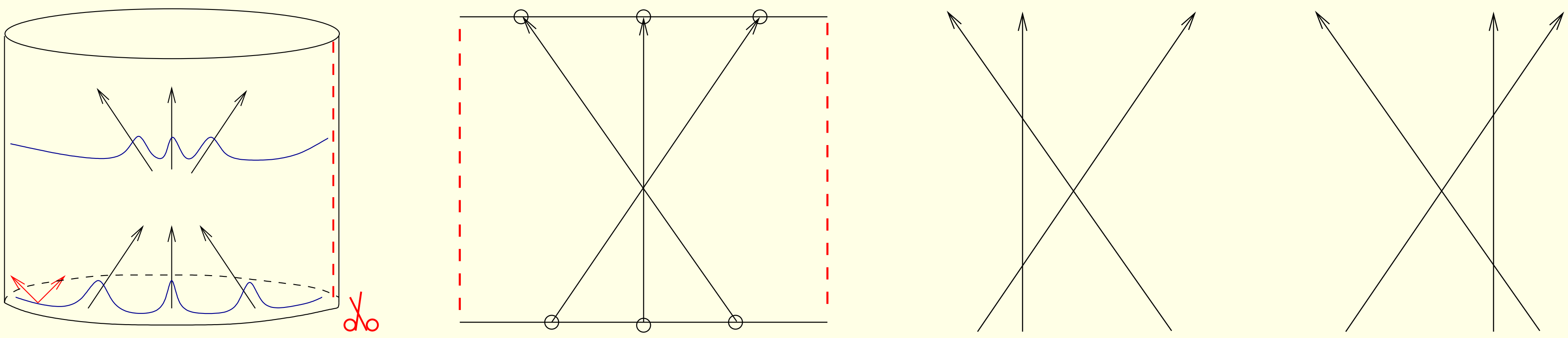
Analytical properties: unitarity, crossing $S(\theta)=S(-\theta)^{-1}=S(i\pi-\theta)$ FF

S-matrix bootstrap

S-matrix bootstrap

S-matrix bootstrap: Calculate the two particle S-matrix [Zamolodchikov[2]]



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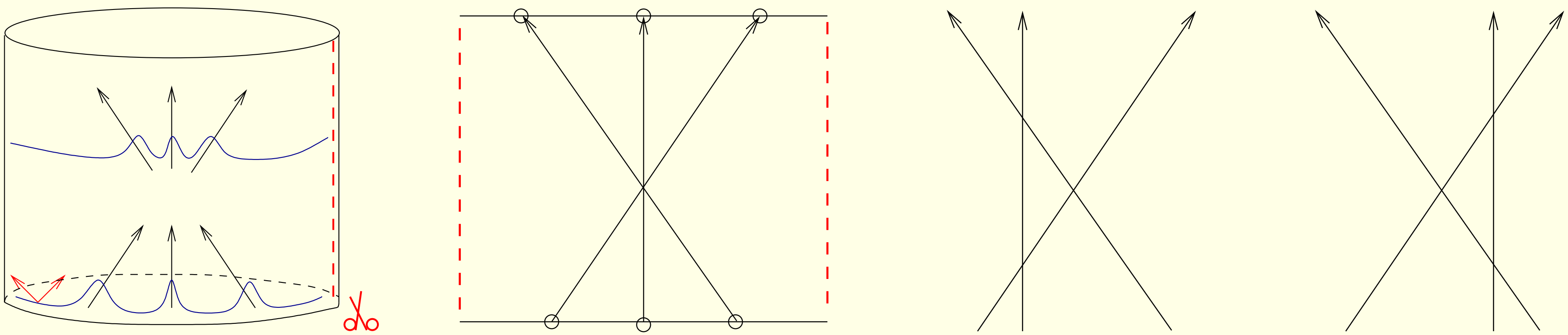
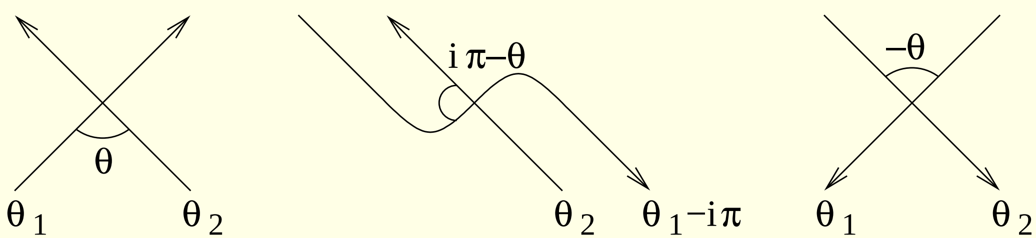


Infinite volume $\rightarrow$ crossing symmetry, $\theta \rightarrow i\pi - \theta$ in rapidity

$(E(\theta), p(\theta)) = m(\cosh\theta, \sinh\theta)$



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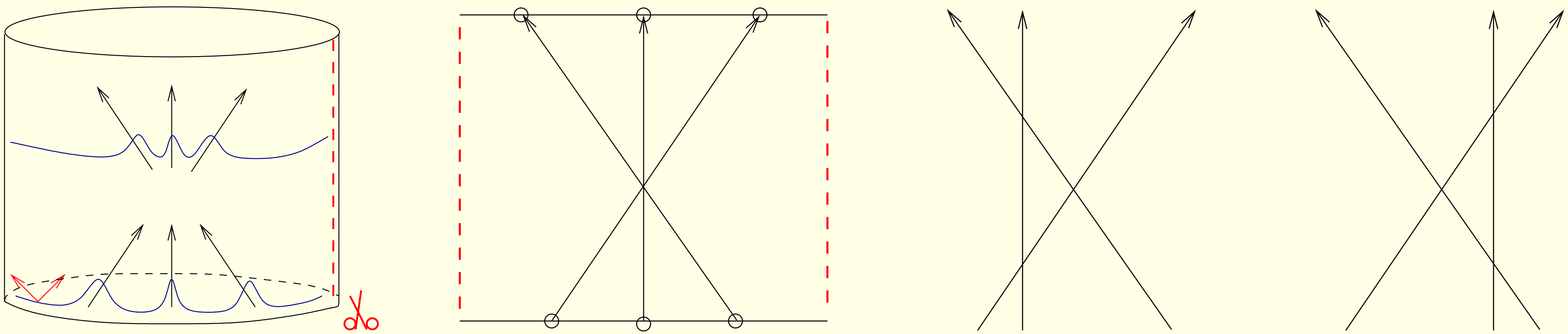

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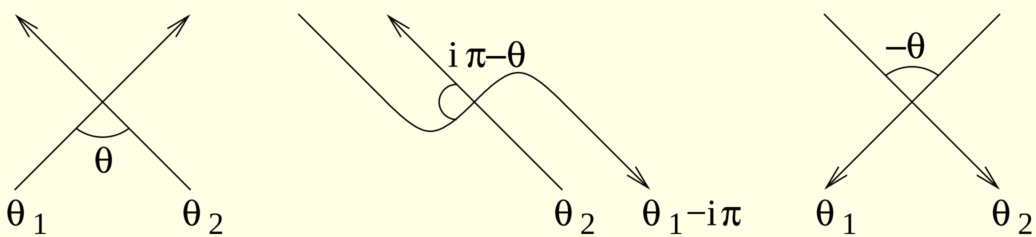


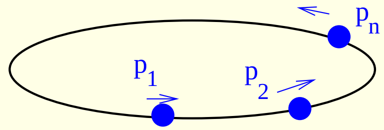
Simplest solution:
sinh-Gordon

$S(\theta) = \frac{\sinh\theta - i\sin a}{\sinh\theta + i\sin a}$ $\quad a = \frac{\pi b^2}{8\pi + b^2}$

$S(\theta_1 - \theta_2) = S(\theta) = S(i\pi - \theta) = S(-\theta)^{-1}$:

Spectral problem

Finite volume spectrum



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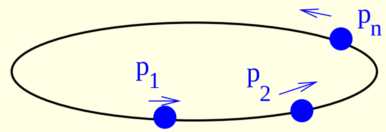

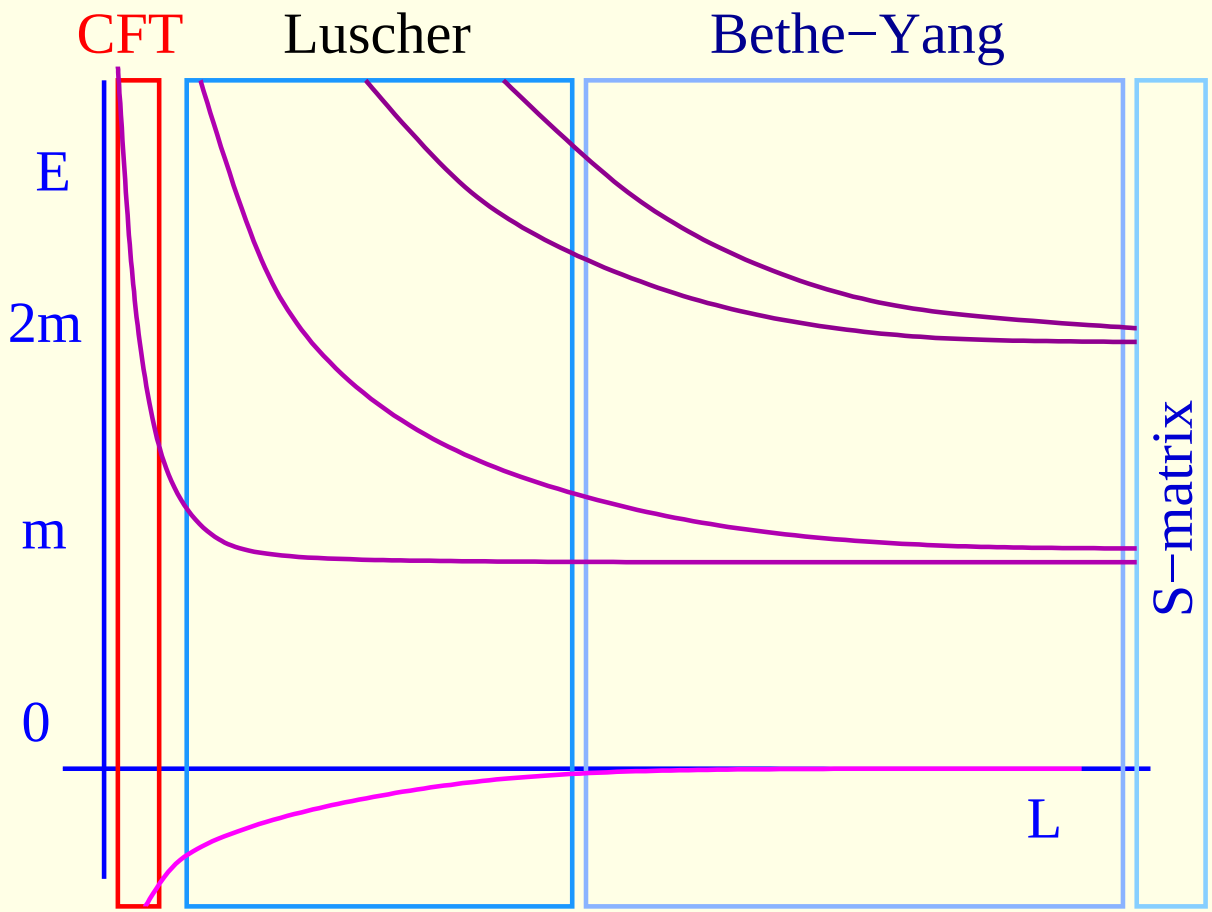

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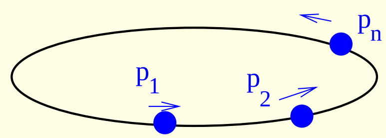
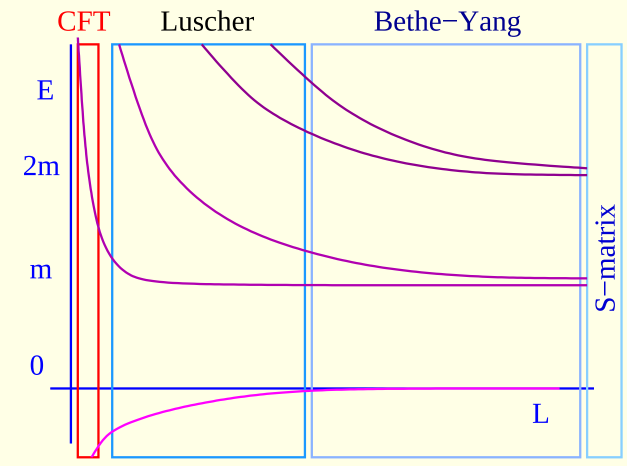


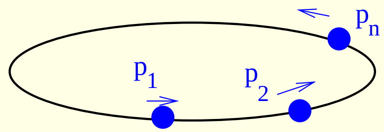


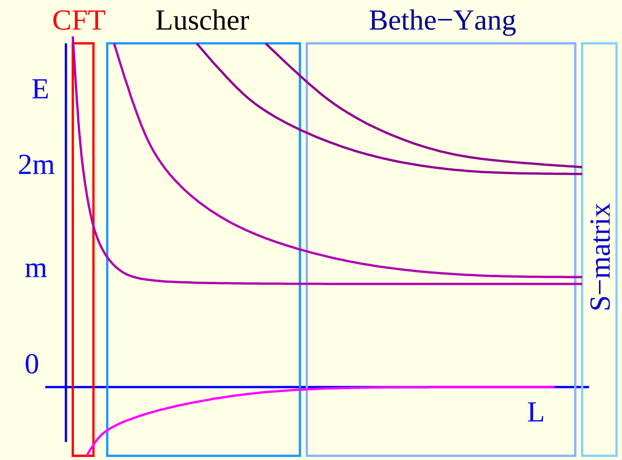
Polynomial volume corrections:

$E(\theta_1,\ldots,\theta_n) = \sum_i E(\theta_i)$

Spectral problem

Finite volume spectrum





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Bethe-Yang; p_i quantized: $e^{ip_jL}\prod_k S(\theta_j-\theta_k)=-1$

Spectral problem

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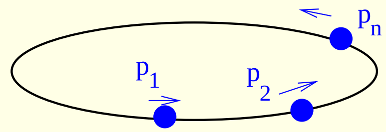

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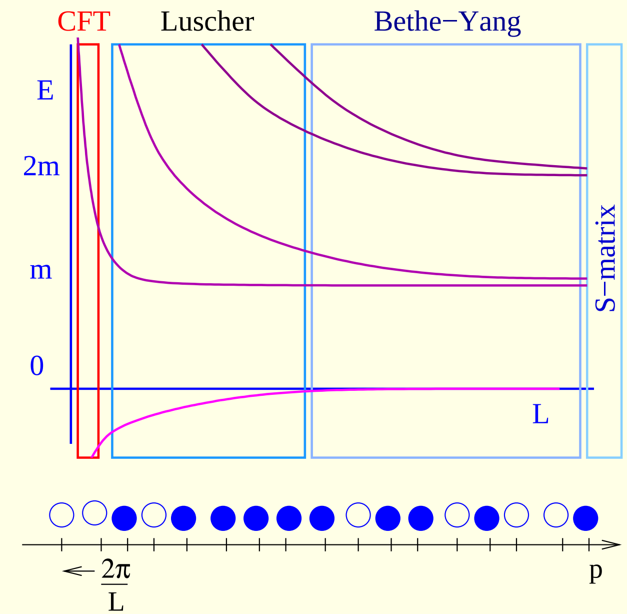
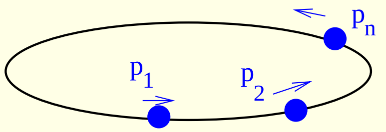
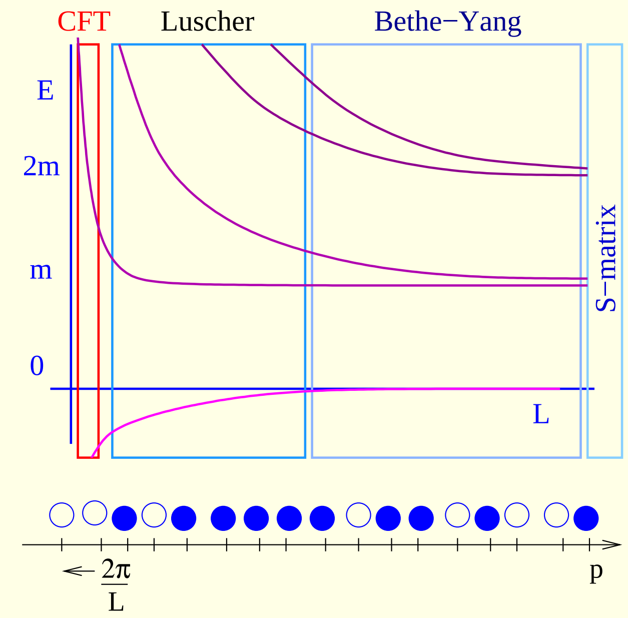


Spectral problem

Finite volume spectrum





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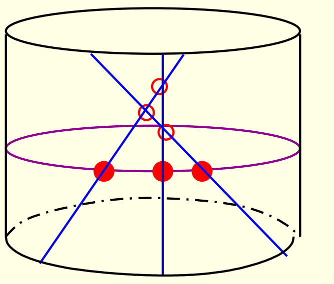
Lüscher-type corrections:

$E(\theta_1,\ldots,\theta_n)=\sum_i E(\theta_i)-\int\frac{d\theta}{2\pi}\prod_k S(\theta+i\frac{\pi}{2}-\theta_k)e^{-mL\cosh\theta}$

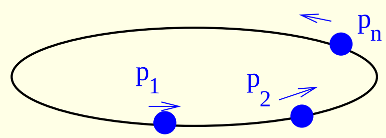


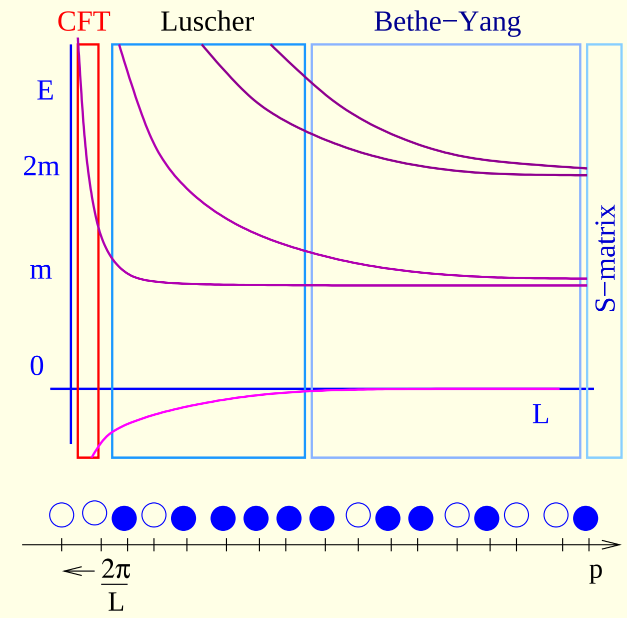
BY modified as $p_jL+\sum_k\frac{1}{i}\log S(\theta_j-\theta_k)+\delta=(2n+1)\pi$

where $\delta=i\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\log' S(\theta_j-\theta')\prod_k S(i\frac{\pi}{2}+\theta_k-\theta')e^{-mL\cosh\theta'}$

Spectral problem

Finite volume spectrum





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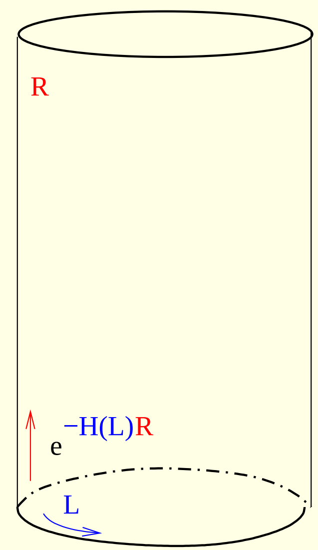


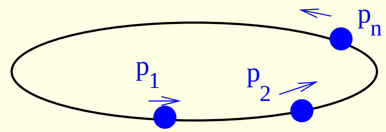
Ground-state energy from

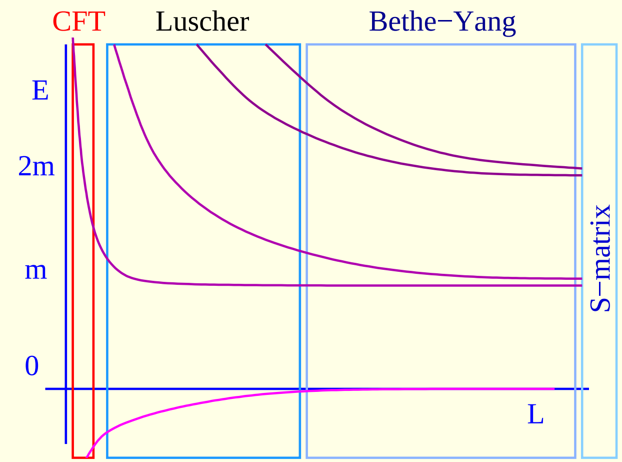
Euclidean partition function:

$Z(L,R) =_{R\to\infty} \mathrm{Tr}(e^{-H(L)R}) = e^{-E_0(L)R} + \ldots$

Spectral problem

Finite volume spectrum





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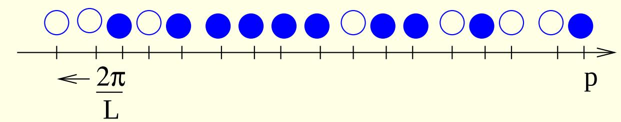

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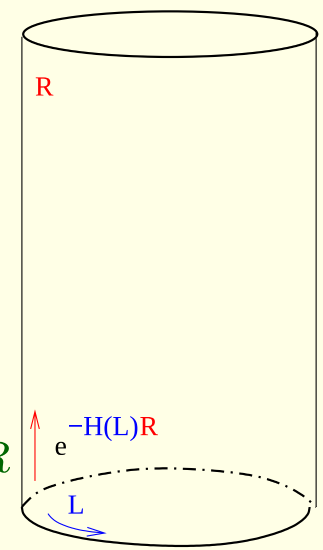

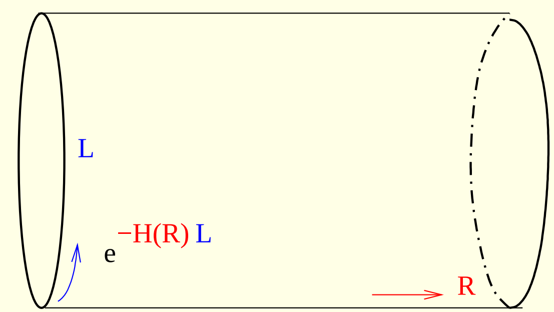

Spectral problem

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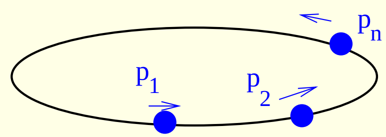

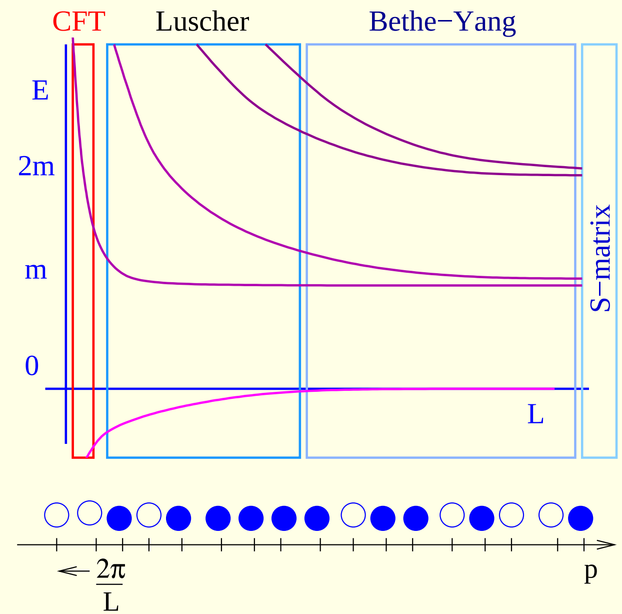

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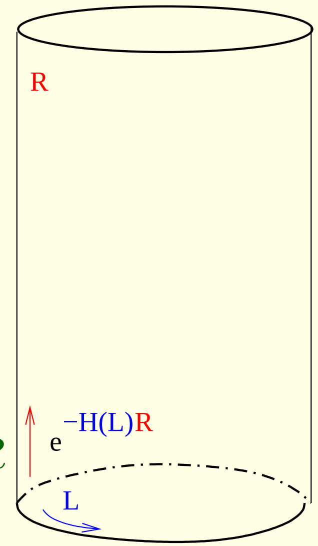
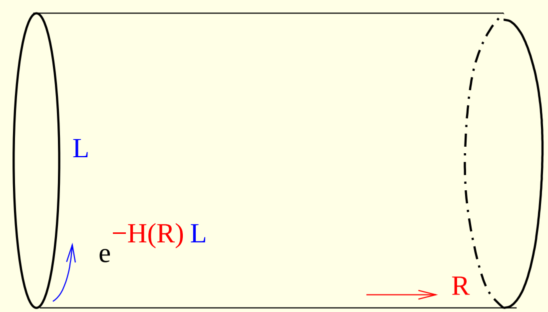
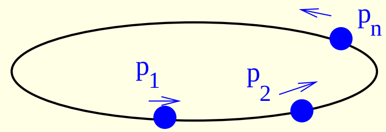


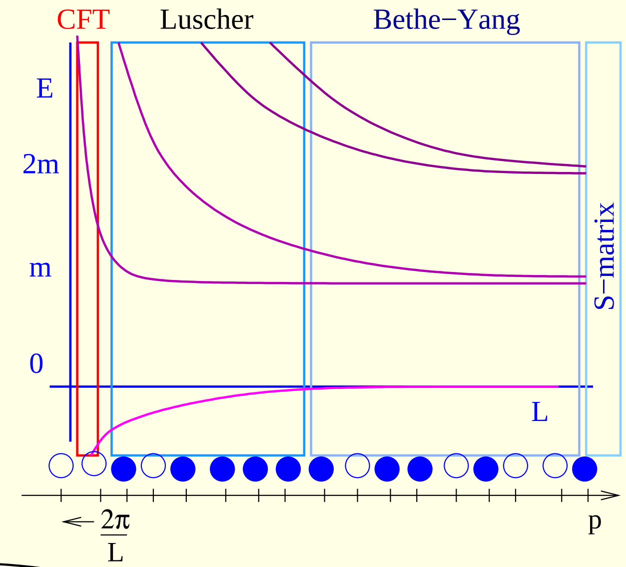


Main contribution:
finite density ρ, ρ_h

Spectral problem

Finite volume spectrum





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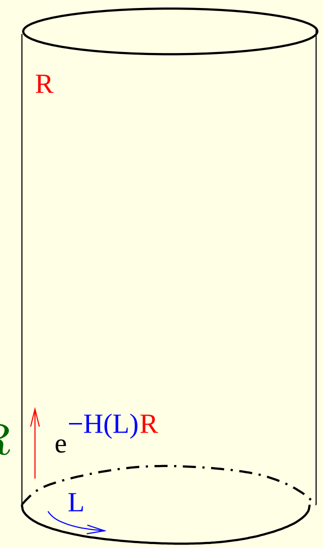
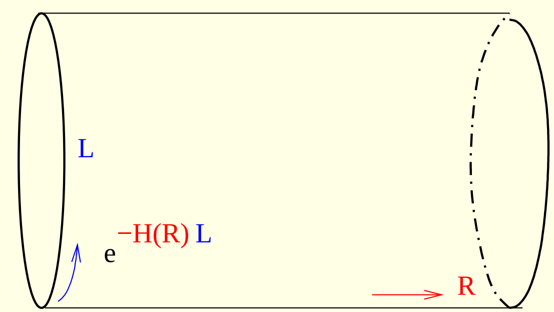




Main contribution: finite density ρ,ρ_h

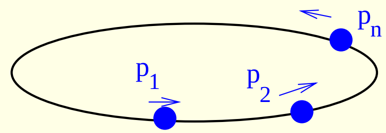
Large volume: Bethe-Yang can be used

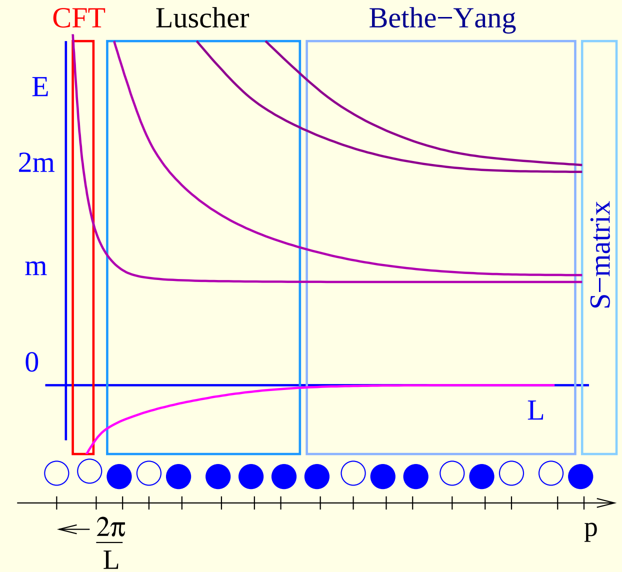
$p_jR+\sum_k \frac{1}{i}\log S(\theta_j-\theta_k)=(2n+1)\pi \quad\longrightarrow\quad R+\int(-id_p\log S(p,p'))\rho(p')dp'=2\pi(\rho+\rho_h)$

$Z(L,R)=\int d[\rho,\rho_h]e^{-LE(R)-\int((\rho+\rho_h)\ln(\rho+\rho_h)-\rho\ln\rho-\rho_h\ln\rho_h)dp}$

Spectral problem

Finite volume spectrum





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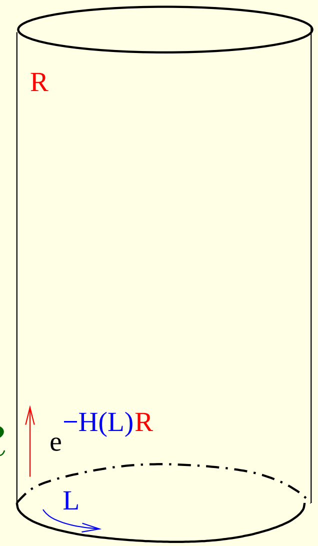

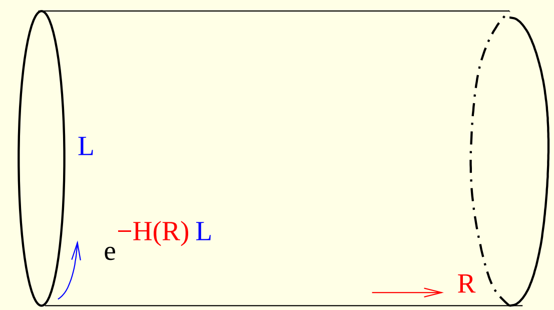

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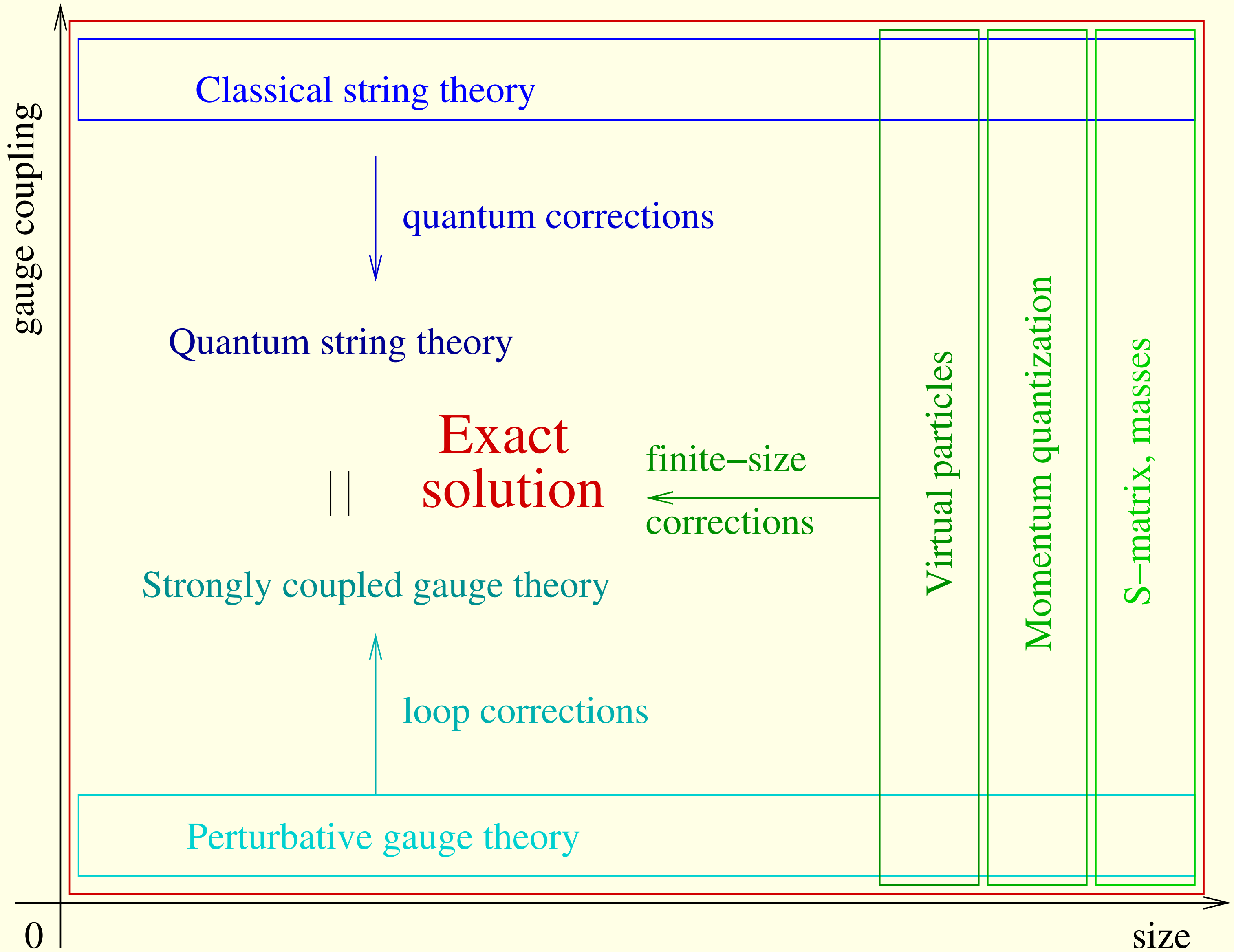
Main contribution: finite density ρ, ρ_h

Saddle point : $\epsilon(p) = \ln \frac{\rho_h(p)}{\rho(p)}$ $\boxed{\epsilon(p) = E(p)L + \int \frac{dp}{2\pi} i\partial_p \log S(p', p) \log(1 + e^{-\epsilon(p')})}$

Exact ground state energy: $\boxed{E_0(L) = -\int \frac{dp}{2\pi} \log(1 + e^{-\epsilon(p)})}$ [Zamolodchikov]

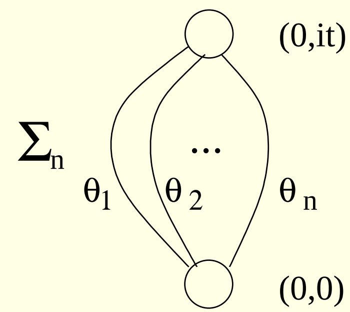
How integrability works:



Form factor bootstrap

Form factor bootstrap

Correlation functions: [Smirnov, Karowszki] $\langle 0|\mathcal{O}(it)\mathcal{O}(0)|0\rangle =$



$$\sum_n \frac{1}{n!}\int\frac{d\theta_1}{2\pi}\cdots\int\frac{d\theta_n}{2\pi}|\langle 0|\mathcal{O}(0)|\theta_1,\ldots,\theta_n\rangle|^2 e^{-m(\sum_i \cosh\theta_i)t}$$

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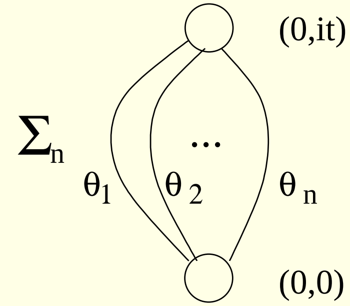

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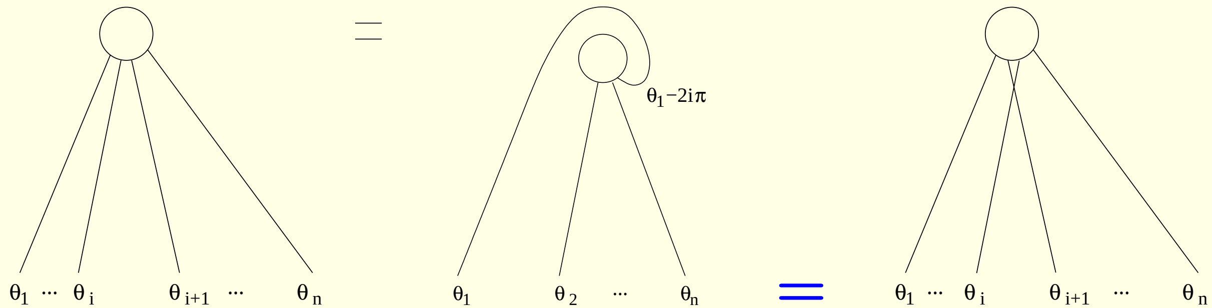

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$F(\theta_1,\ldots,\theta_n) = \quad F(\theta_2,\ldots,\theta_n,\theta_1-2i\pi\rangle \quad = S(\theta_i-\theta_{i+1})F(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots,\theta_n)$

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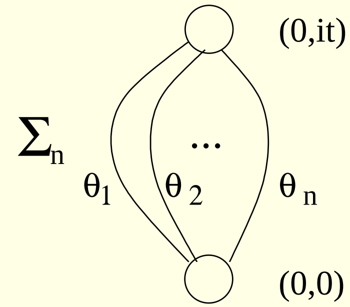

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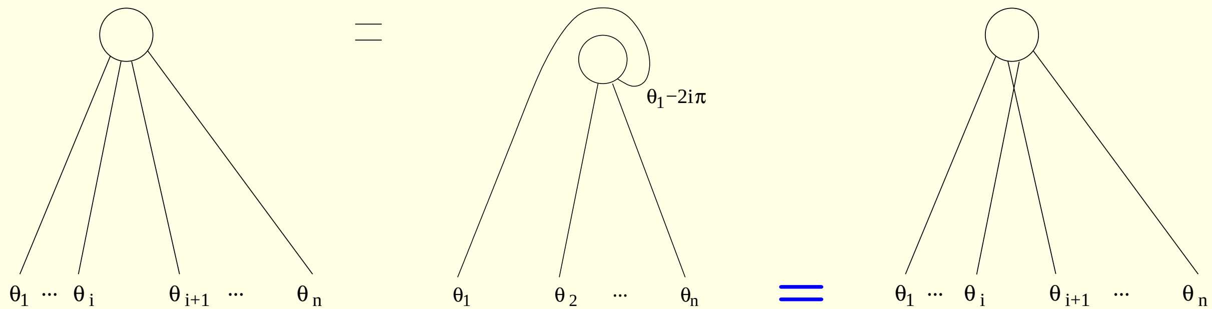

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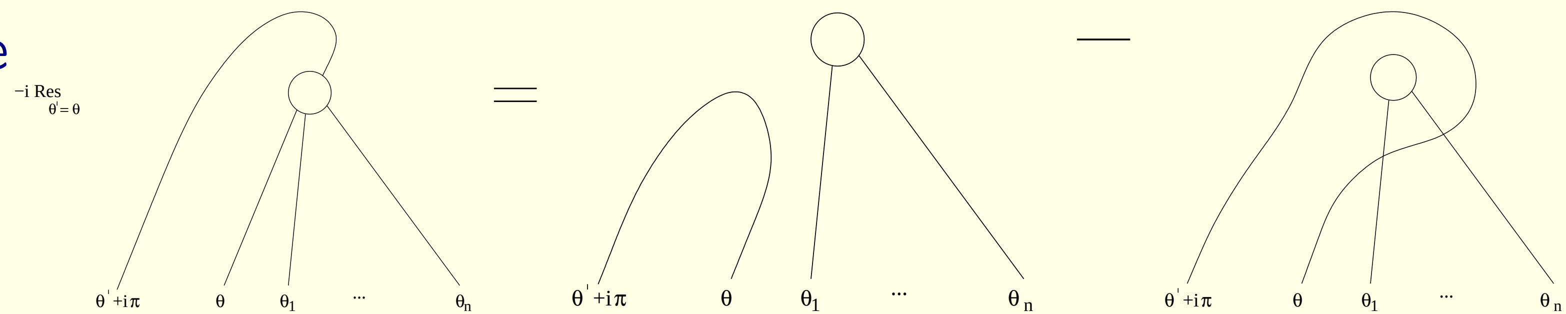
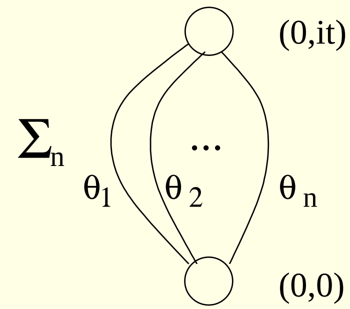
Singularity stucture



$$-i\mathrm{Res}_{\theta'=\theta}F(\theta'+i\pi,\theta,\theta_1\ldots,\theta_n) = (1-\prod_i S(\theta-\theta_i))F(\theta_1,\ldots,\theta_n)$$

Form factor bootstrap



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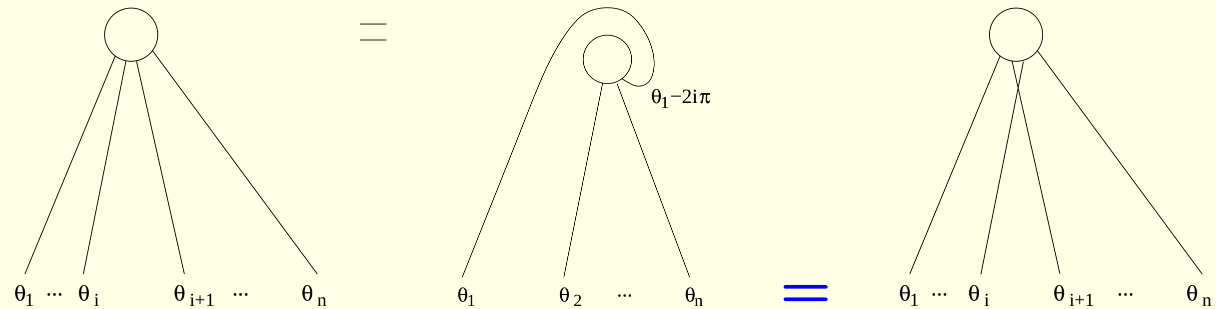

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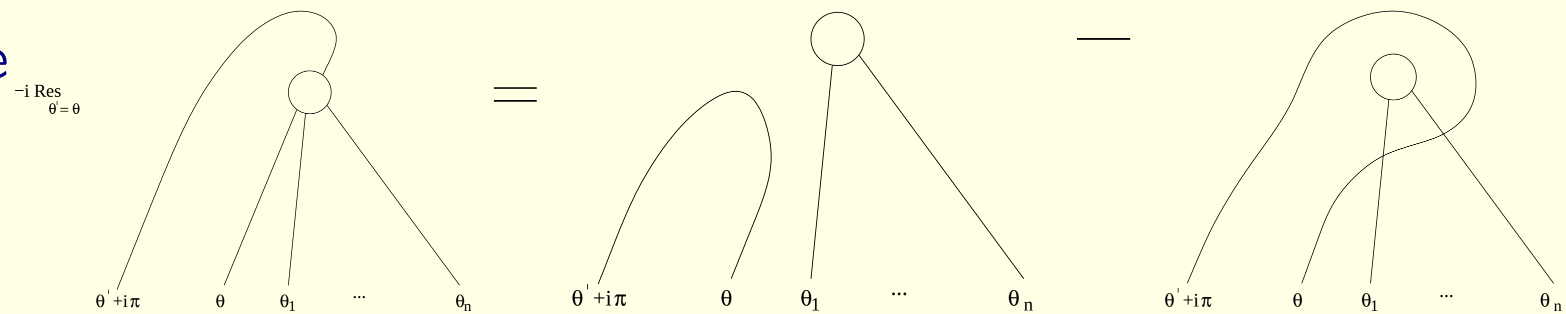

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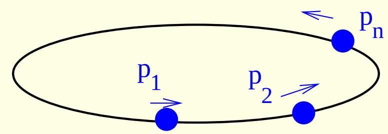
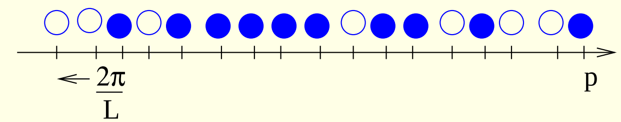
Solution for sinh-Gordon: $f(\theta_1-\theta_2) = e^{(D+D^{-1})^{-1}\log S}$; $Df(\theta) = f(\theta+i\pi)$

[Fring, Mussardo, Simonetti]

Finite volume form factors

Finite volume state $|\{\theta_i\}\rangle_L \equiv |\{n_i\}\rangle$ 



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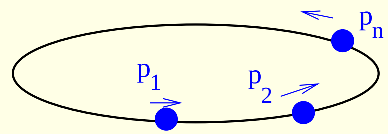

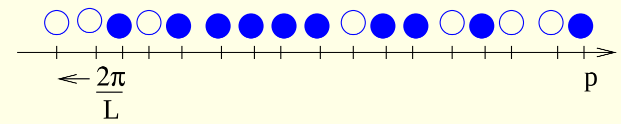

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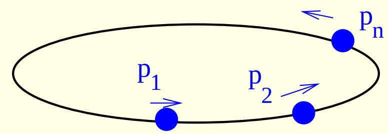

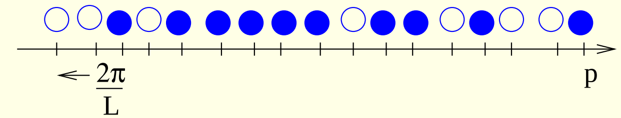

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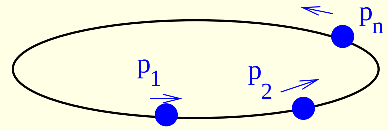
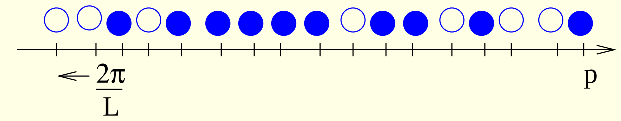
Nondiagonal FF: $\langle\theta'_1,\dots,\theta'_m|\mathcal{O}|\theta_n,\dots,\theta_1\rangle_L = \frac{F_{n+m}(\bar\theta'_1,\dots,\bar\theta'_m,\theta_n,\dots,\theta_1)}{\sqrt{\rho_n\rho'_m}} + O(e^{-mL})$

[proved Pozsgay, Takacs] crossing $\bar\theta = \theta + i\pi$

Finite volume form factors

Finite volume state $|\{\theta_i\}\rangle_L \equiv |\{n_i\}\rangle$





Polynomial volume corrections: $Q_j = p(\theta_j)L + \sum_{k\neq j} \frac{1}{i} \log S(\theta_j - \theta_k) = 2n_j\pi$

Normalization of states: $|\{n_i\}\rangle = \frac{|\{\theta_i\}\rangle}{\sqrt{\rho_n(\{\theta_i\})}}$ where $\rho_n = \det|\partial_i Q_j|$

Nondiagonal FF: $\langle\theta'_1,\dots,\theta'_m|\mathcal{O}|\theta_n,\dots,\theta_1\rangle_L = \frac{F_{n+m}(\bar{\theta}'_1,\dots,\bar{\theta}'_m,\theta_n,\dots,\theta_1)}{\sqrt{\rho_n\rho'_m}} + O(e^{-mL})$

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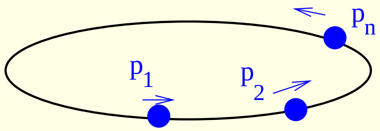

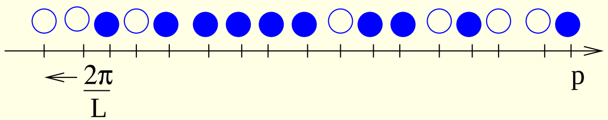

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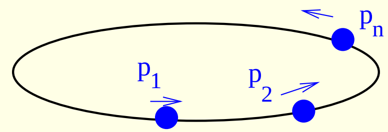

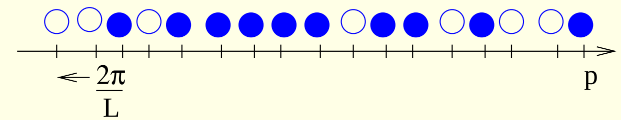

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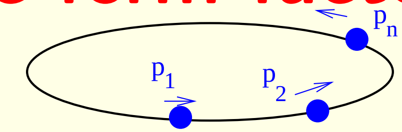

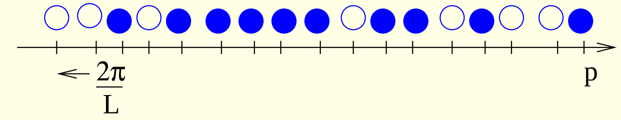

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$$\rho_2(\theta_1,\theta_2) = \begin{vmatrix} E_1L+\phi & -\phi \\ -\phi & E_2L+\phi \end{vmatrix} = E_1E_2L^2 + \phi(E_1+E_2)L \qquad \phi(\theta) = -i\partial_\theta \log S(\theta)$$

and $\rho_1(\theta_1) = E_1L + \phi \quad ; \qquad \rho_1(\theta_2) = E_2L + \phi$

Connected form factors

We need $F(\bar{\theta}_1 + \epsilon_1, \ldots, \bar{\theta}_n + \epsilon_n, \theta_n, \ldots, \theta_1)$

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Graphical representation: $F(\bar{\theta}_1+\epsilon_1,\dots,\bar{\theta}_n+\epsilon_n,\theta_n,\dots,\theta_1)=\sum_{\text{graphs}}F_{\text{graphs}}$

[Pozsgay, Takacs]

graphs: oriented, tree-like, at each vertex only at most one outgoing edge

contributions: $(i_1,\dots,i_k)$ with no outgoing edges $F^c(\theta_{i_1},\dots,\theta_{i_k})$,
for each edge from i to j : factor $\frac{\epsilon_j}{\epsilon_i}\phi(\theta_i-\theta_j)$,

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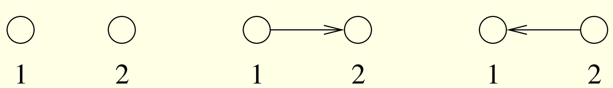

which gives $F_4(\bar{\theta}_1+\epsilon_1,\bar{\theta}_2+\epsilon_2,\theta_2,\theta_1)=F_4^c(\theta_1,\theta_2)+\frac{\epsilon_1}{\epsilon_2}\phi_{12}F_2^c(\theta_1)+\frac{\epsilon_2}{\epsilon_1}\phi_{21}F_2^c(\theta_2)$

Exact finite volume 1-point function

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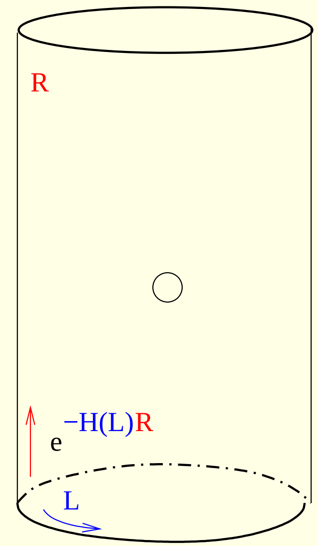
LeClair-Mussardo formula

from thermal evaluation:

$$\langle 0|\mathcal{O}|0\rangle_L =_{R\to\infty} \mathrm{Tr}(\mathcal{O}e^{-H(L)R})/Z(L,R) + \ldots$$



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$$_{R\to\infty}\mathrm{Tr}(\mathcal{O}e^{-H(L)R})/Z =_{R\to\infty} \mathrm{Tr}(e^{-H(R)L})/Z$$

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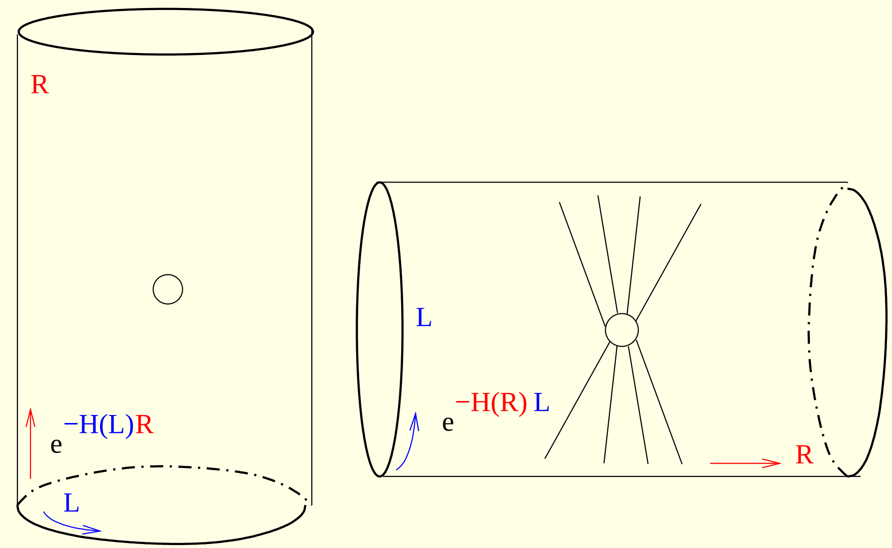

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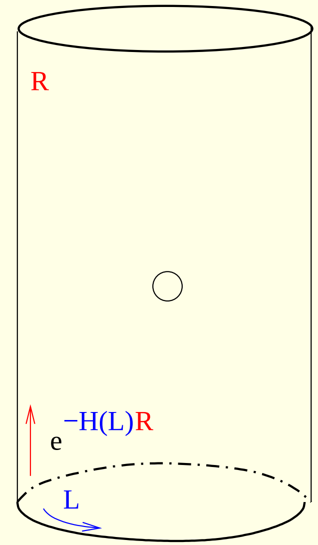

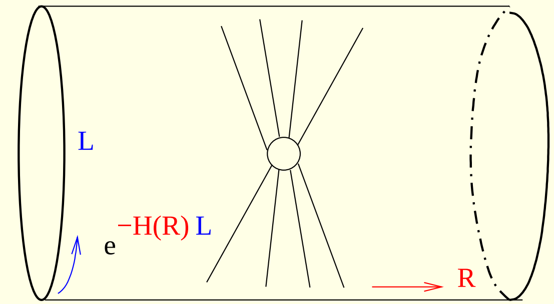

Main contribution:
finite density ρ, ρ_h

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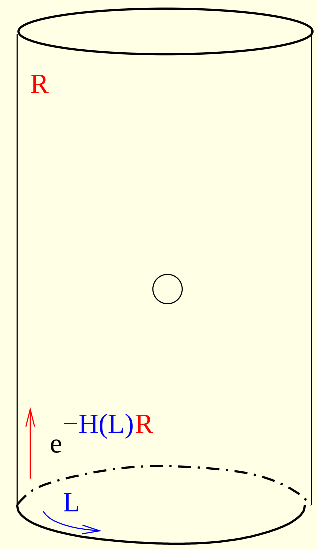

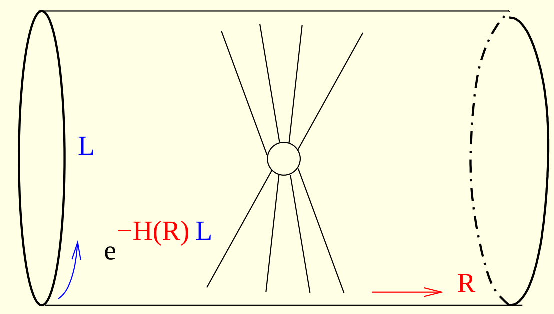

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we need $\langle \rho, \rho_n|\mathcal{O}|\rho, \rho_n\rangle$ in a highly excited Bethe state [Pozsgay]

Large volume: asymptotic formula $\frac{\sum_{\alpha\cup\bar{\alpha}} F^c_\alpha \rho_{\bar{\alpha}}}{\rho_n}$ can be used as

$\frac{\sum_{\alpha\cup\bar{\alpha}} F^c_\alpha \rho_{\bar{\alpha}}}{\rho_n} = F_0 + \lim_{n\to\infty} \int \frac{d\theta}{2\pi} F^c(\theta)\frac{\rho_{n-1}}{\rho_n} + \dots$ giving $F_0 + \int \frac{d\theta}{2\pi} F^c(\theta)\frac{e^{-\epsilon(\theta)}}{1+e^{-\epsilon(\theta)}} + \dots$

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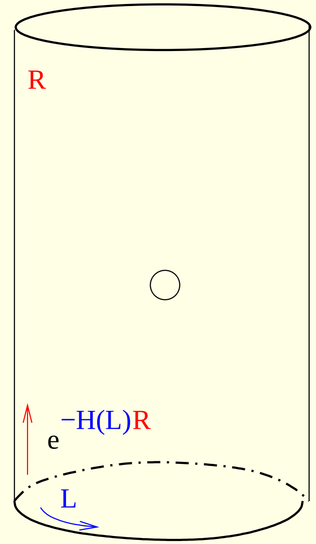

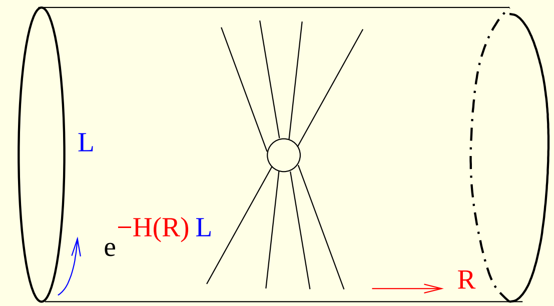

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Saddle point : $\epsilon(p) = \ln\frac{\rho_h(p)}{\rho(p)}$ $\boxed{\epsilon(\theta) = E(\theta)L - \int \frac{d\theta}{2\pi}\phi(\theta-\theta')\log(1+e^{-\epsilon(\theta')})}$

Finite volume expectation value: $\boxed{\langle \mathcal{O}\rangle_L = \sum_n \frac{1}{n!}\prod_{j=1}^n \int \frac{d\theta_i}{2\pi}\frac{e^{-\epsilon(\theta)}}{1+e^{-\epsilon(\theta)}} F^c(\theta_1,\ldots,\theta_n)}$

[LeClair-Mussardo] diagonal FF: excited states [Pozsgay]
What about non-diagonal?

Lüscher correction for nondiagonal FFs: the method

Finite volume 2-point function: $\langle \mathcal{O}(x,t)\mathcal{O}\rangle_L = \frac{\int[\mathcal{D}\phi]\mathcal{O}(x,t)\mathcal{O}(0,0)e^{-S[\phi]}}{\int[\mathcal{D}\phi]e^{-S[\phi]}}$

in Fourier space: $\Gamma(\omega,q) = \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{i(\omega t+qx)}\langle \mathcal{O}(x,t)\mathcal{O}\rangle_L$

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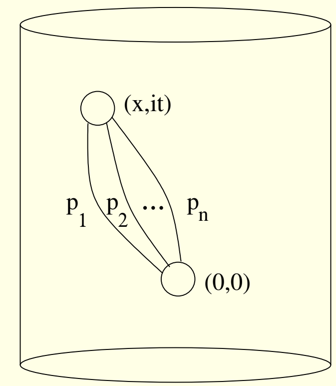

evaluating in the finite volume channel

$\Gamma(\omega,q) = \sum_N |\langle 0|\mathcal{O}|\theta_1,\dots,\theta_N\rangle_L|^2 \left\{\frac{\delta_{q-P_N(L)}}{E_N(L)-i\omega} + \frac{\delta_{q+P_N(L)}}{E_N(L)+i\omega}\right\}$

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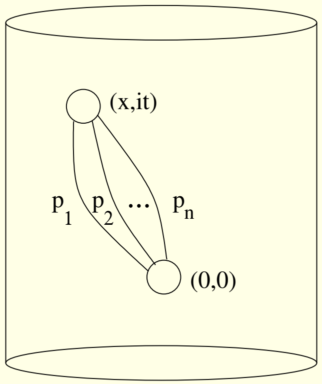

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exact finite volume energy levels: poles, form factors: residues

Finite volume LSZ: $\lim_{\omega\to iE_N(L)}(E_N(L)+i\omega)\Gamma(\omega,P_N(L)) = |\langle 0|\mathcal{O}|\theta_1,\dots,\theta_N\rangle_L|^2$

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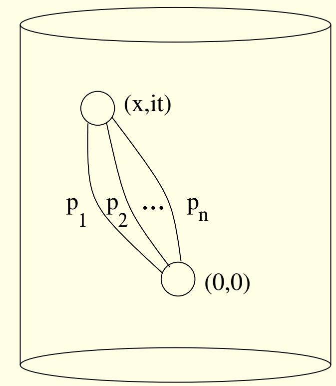

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Finite volume LSZ: $\lim_{\omega\to iE_N(L)}(E_N(L)+i\omega)\Gamma(\omega,P_N(L)) = |\langle 0|\mathcal{O}|\theta_1,\dots,\theta_N\rangle_L|^2$

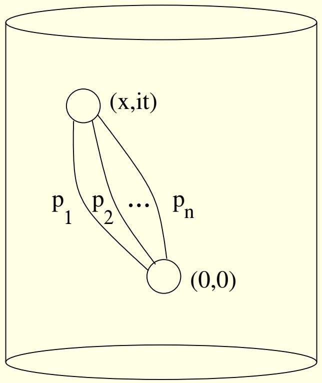
evaluating in the finite temperature channel

$$\langle \mathcal{O}(x,t)\mathcal{O}\rangle_L = \Theta(x)\frac{\mathrm{Tr}[\mathcal{O}(0,t)e^{-Hx}\mathcal{O}e^{-H(L-x)}]}{\mathrm{Tr}[e^{-HL}]} + \Theta(-x)\frac{\mathrm{Tr}[\mathcal{O}e^{Hx}\mathcal{O}(0,t)e^{-H(L+x)}]}{\mathrm{Tr}[e^{-HL}]}$$

Lüscher correction for nondiagonal FFs: the method

Finite volume 2-point function: $\langle\mathcal{O}(x,t)\mathcal{O}\rangle_L = \frac{\int[\mathcal{D}\phi]\mathcal{O}(x,t)\mathcal{O}(0,0)e^{-S[\phi]}}{\int[\mathcal{D}\phi]e^{-S[\phi]}}$

in Fourier space: $\Gamma(\omega,q) = \frac{1}{L}\int_{-L/2}^{L/2}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{i(\omega t+qx)}\langle\mathcal{O}(x,t)\mathcal{O}\rangle_L$

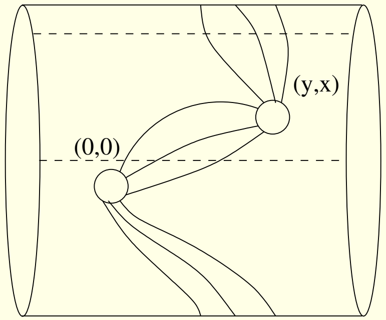


evaluating in the finite volume channel

$\Gamma(\omega,q) = \sum_N |\langle 0|\mathcal{O}|\theta_1,\dots,\theta_N\rangle_L|^2\left\{\frac{\delta_{q-P_N(L)}}{E_N(L)-i\omega}+\frac{\delta_{q+P_N(L)}}{E_N(L)+i\omega}\right\}$

exact finite volume energy levels: poles, form factors: residues

Finite volume LSZ: $\lim_{\omega\to iE_N(L)}(E_N(L)+i\omega)\Gamma(\omega,P_N(L)) = |\langle 0|\mathcal{O}|\theta_1,\dots,\theta_N\rangle_L|^2$



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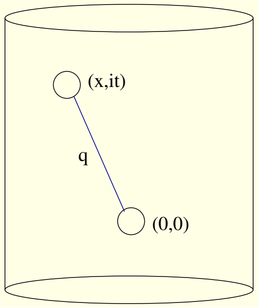
Insert two complete systems of states:

$$Z\Gamma(\omega,q) = \frac{2\pi}{L}\sum_{\mu,\nu}|\langle\nu|\mathcal{O}|\mu\rangle|^2 e^{-E_\nu L}\delta(P_\mu-P_\nu+\omega)\left\{\frac{1}{E_\mu-E_\nu-iq}+\frac{1}{E_\mu-E_\nu+iq}\right\}$$

Use asymptotic expressions. Do analytical continuation as $\omega\to iE_N(L)$

1-particle energy and form factor from 2-pt function

Specify to a 1-particle pole $\Gamma(\omega, q) = \frac{\mathcal{F}(q)^2}{E(q)+i\omega} + \ldots$



Exact 1-particle energy: $E(q)$, form factor: $\mathcal{F}(q) = \langle 0|\mathcal{O}|q\rangle$

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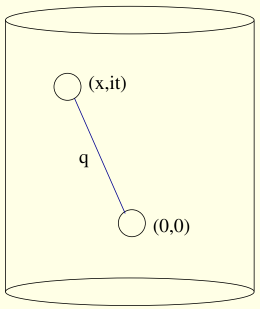

Exact 1-particle energy: $E(q)$, form factor: $\mathcal{F}(q) = \langle 0|\mathcal{O}|q\rangle$

Lüscher correction: Expand around the BY pole $\mathcal{E}(q) = \sqrt{q^2 + m^2}$

$$\Gamma(\omega, q) = \frac{2\pi F_1^2(q)}{L\mathcal{E}(q)} \frac{-i}{\omega - i\mathcal{E}(q)} + \frac{\mathcal{L}_0(q)}{(\omega - i\mathcal{E}(q))^2} + \frac{\mathcal{L}_1(q)}{\omega - i\mathcal{E}(q)} + \text{regular}$$

Energy correction: $E(q) = \mathcal{E}(q)\left\{1 + \frac{L}{2\pi F_1^2}\mathcal{L}_0(q) + \dots\right\}$

FF correction $\mathcal{F}(q) = \frac{\sqrt{2\pi}F_1}{\sqrt{L\mathcal{E}(q)}}\left\{1 + \frac{iL\mathcal{E}(q)}{4\pi F_1^2}\mathcal{L}_1(q) + \dots\right\}$

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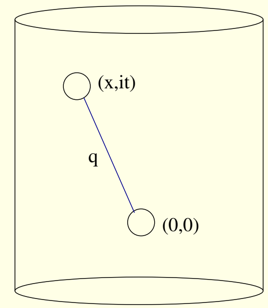

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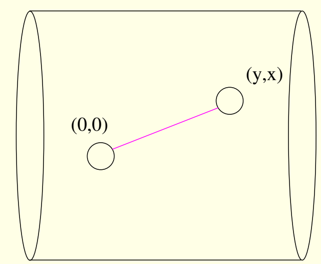
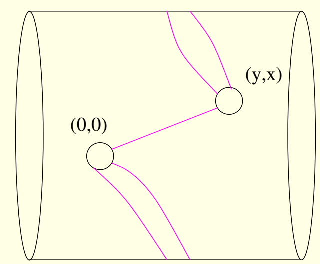
Leading Lüscher correction from ν 1-particle state,
relevant pole: μ vacuum or 2-particle state



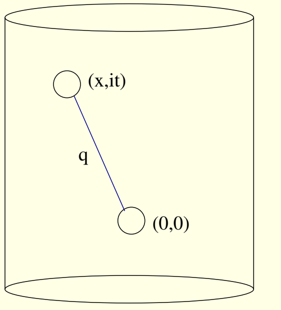


$$Z\Gamma(\omega,q)=\frac{2\pi}{L}\sum_{\mu,\nu}|\langle\nu|\mathcal{O}|\mu\rangle|^2e^{-E_\nu L}\delta(P_\mu-P_\nu+\omega)\left\{\frac{1}{E_\mu-E_\nu-iq}+\frac{1}{E_\mu-E_\nu+iq}\right\}$$

1-particle energy and form factor from 2-pt function

Specify to a 1-particle pole $\Gamma(\omega, q) = \frac{\mathcal{F}(q)^2}{E(q)+i\omega} + ...$

Exact 1-particle energy: $E(q)$, form factor: $\mathcal{F}(q) = \langle 0|\mathcal{O}|q\rangle$



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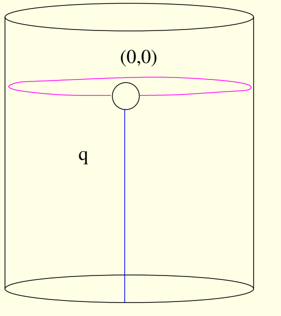


The main result: energy correction reproduced, form factor

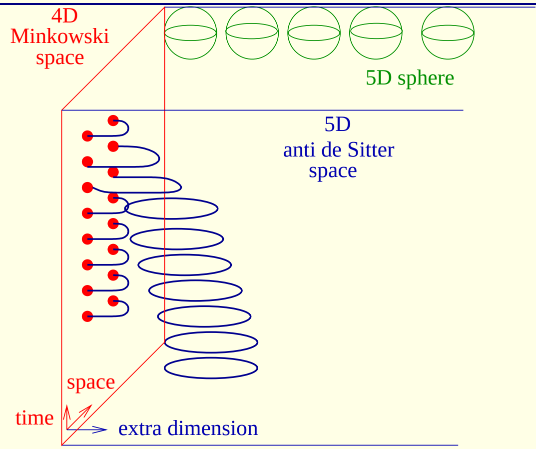
$$\boxed{\mathcal{F}(q) = \frac{\sqrt{2\pi}}{\sqrt{\rho_1^{(1)}}} \left\{ F_1 + \int_{-\infty}^{\infty} d\theta\, F_3^{\text{reg}}(\theta + i\pi, \theta, \theta_1 - i\tfrac{\pi}{2}) e^{-mL\cosh\theta} + \ldots \right\}}$$

$$F_3^{\text{reg}}(\theta, \theta_1, \theta_2) = F_3(\theta, \theta_1, \theta_2) - \frac{iF_1}{\theta - \theta_1 - i\pi} \left[1 - S(\theta_1 - \theta_2)\right] + i\frac{F_1}{2} S'(\theta_1 - \theta_2)$$

density of states at Lüscher order: $\rho_1^{(1)}$ from Lüscher quantization

AdS/CFT correspondence (Maldacena 1998)

II_B superstring on $AdS_5 \times S^5$



$\sum_1^6 Y_i^2 = R^2 \qquad -++++- = -R^2$

$\frac{R^2}{\alpha'} \int \frac{d\tau d\sigma}{4\pi} \left(\partial_a X^M \partial^a X_M + \partial_a Y^M \partial^a Y_M\right) + \dots$

$\equiv$

$\mathcal{N} = 4$ D=4 $SU(N)$ SYM

$$\frac{2}{g_{YM}^2} \int d^4x \,\mathrm{Tr} \left[-\frac{1}{4}F^2 - \frac{1}{2}(D\Phi)^2 + i\overline{\Psi}\not{D}\Psi + V\right]$$

$$V(\Phi, \Psi) = \frac{1}{4}[\Phi, \Phi]^2 + \overline{\Psi}[\Phi, \Psi]$$

$\beta = 0$ superconformal $\frac{PSU(2,2|4)}{SO(5)\times SO(1,4)}$

gaugeinvariants: $\mathcal{O} = \mathrm{Tr}(\Phi^2), \det(\ )$

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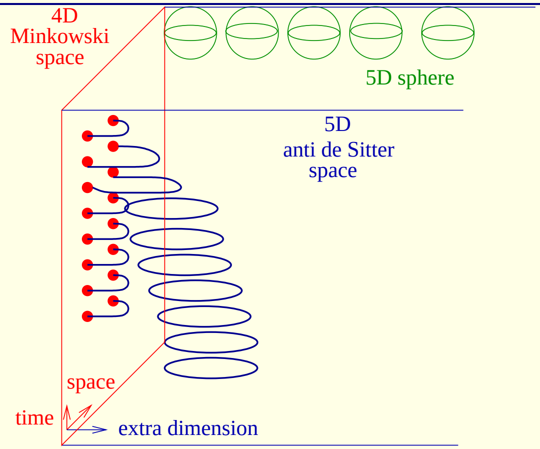

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Dictionary

Coupl.: $\sqrt{\lambda} = \frac{R^2}{\alpha'}$, $g_s = \frac{\lambda}{N} \to 0$

2D QFT

String energy levels: $E(\lambda)$

$E(\lambda) = E(\infty) + \frac{E_1}{\sqrt{\lambda}} + \frac{E_2}{\lambda} + \ldots$

strong$\leftrightarrow$weak

$\Downarrow$

$\lambda = g_{YM}^2 N$, $N \to \infty$ planar

$\langle \mathcal{O}_n(x)\mathcal{O}_m(0)\rangle = \frac{\delta_{nm}}{|x|^{2\Delta_n(\lambda)}}$

Anomalous dim $\Delta(\lambda)$

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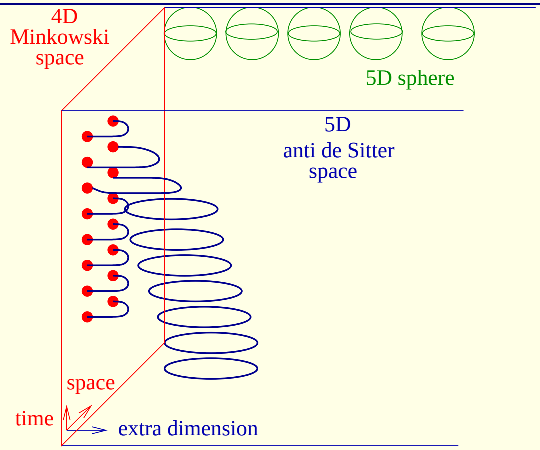

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2D integrable QFT

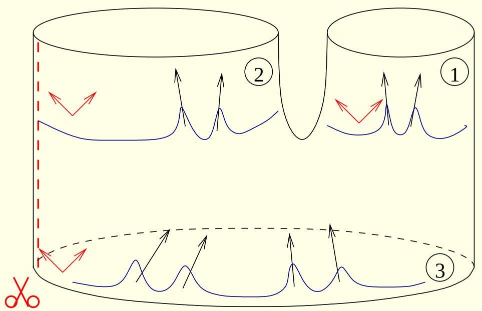
spectrum: $Q = 1, 2, \ldots, \infty$ dispersion: $\epsilon_Q(p) = \sqrt{Q^2 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}$

Exact scattering matrix: $S_{Q_1Q_2}(p_1, p_2, \lambda)$

Decompactification limit of the string vertex

Decompactification limit of the string vertex

Decompactify string 2 & 3:



Decompactification limit of the string vertex

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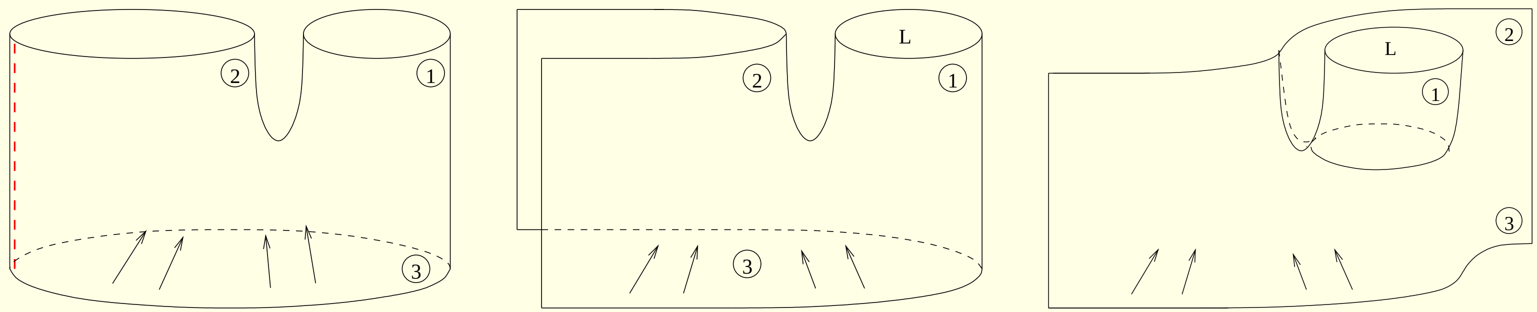

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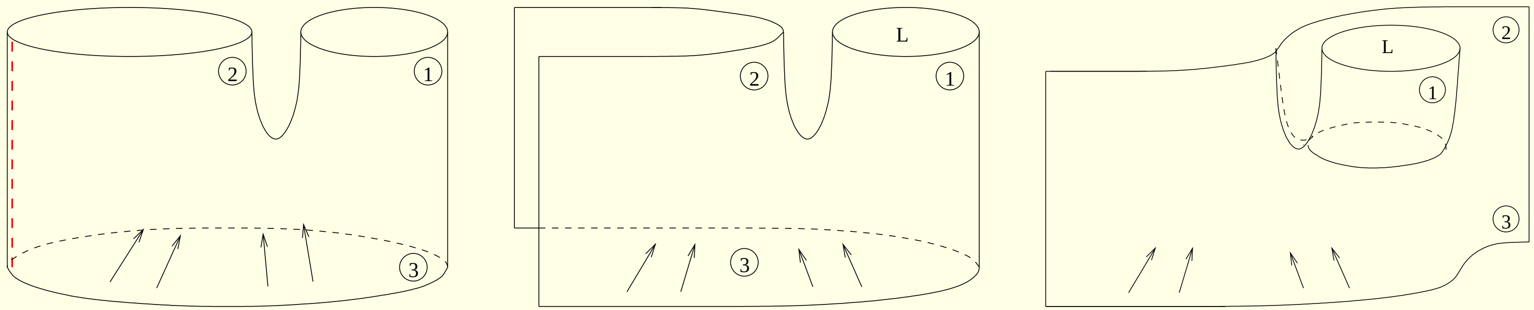
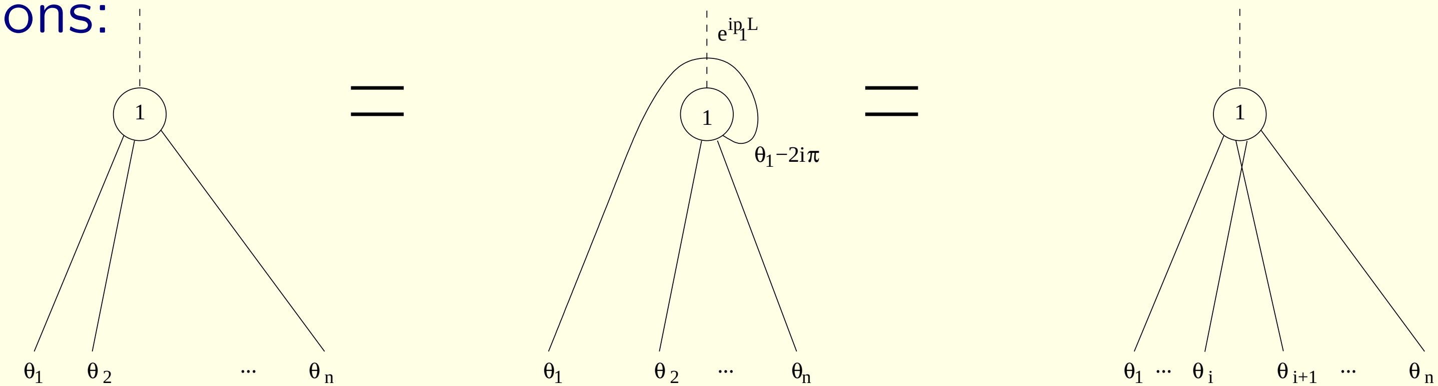


Form factor equations:



$$N_L(\theta_1,\dots,\theta_n\rangle = e^{-ip_1L}N_L(\theta_2,\dots,\theta_n,\theta_1-2i\pi\rangle = S(\theta_i-\theta_{i+1})N_L(\dots,\theta_{i+1},\theta_i,\dots)$$

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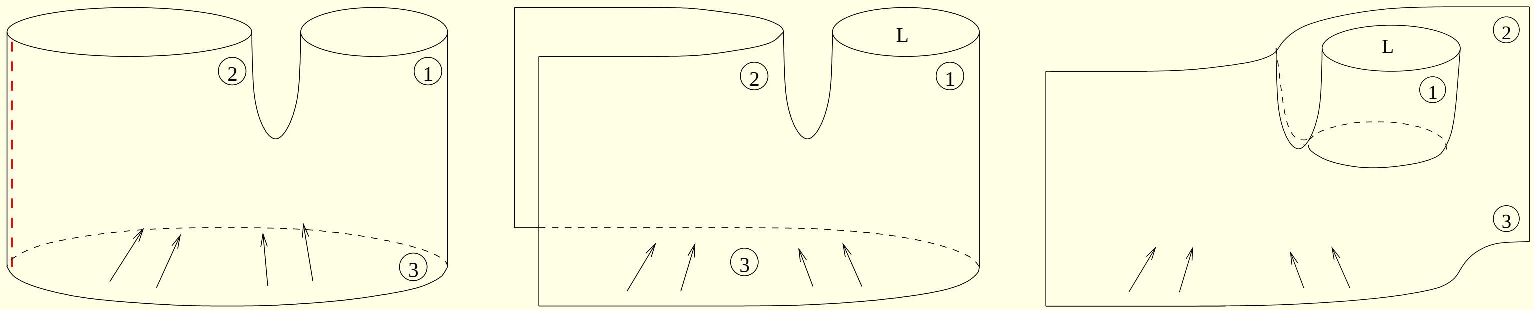

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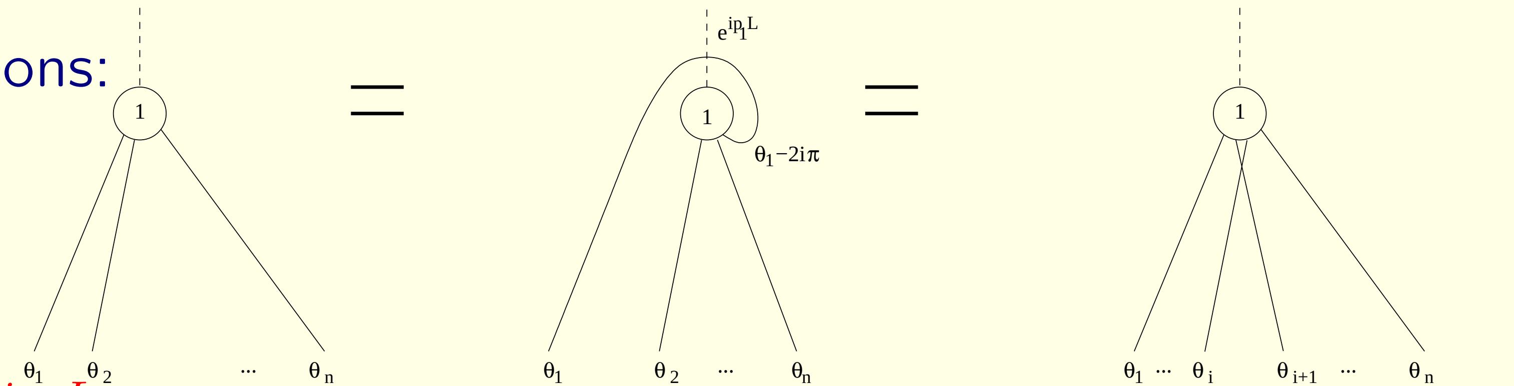

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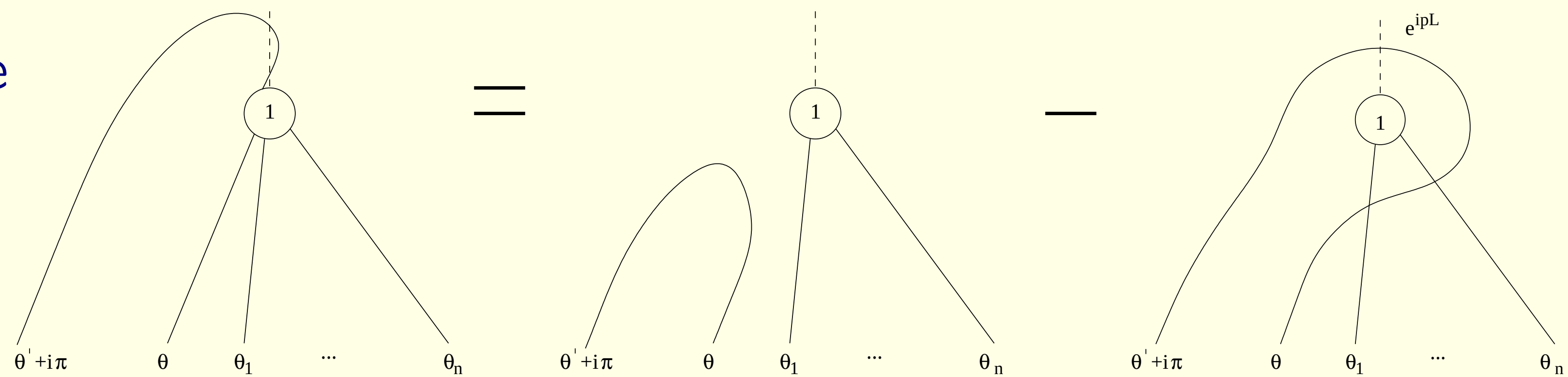
Singularity stucture

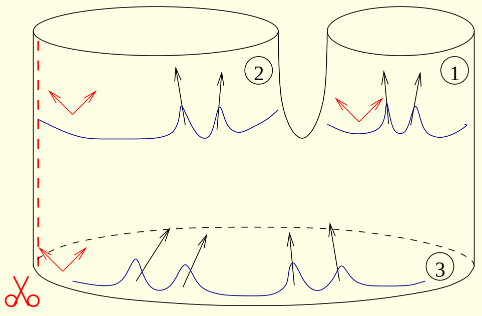


$$-i\text{Res}_{\theta'=\theta}N_L(\theta'+i\pi,\theta,\theta_1\dots,\theta_n) = (1-e^{ipL}\prod_i S(\theta-\theta_i))N_L(\theta_1,\dots,\theta_n)$$

The string vertex for $L_1 = 0$: diagonal form factor

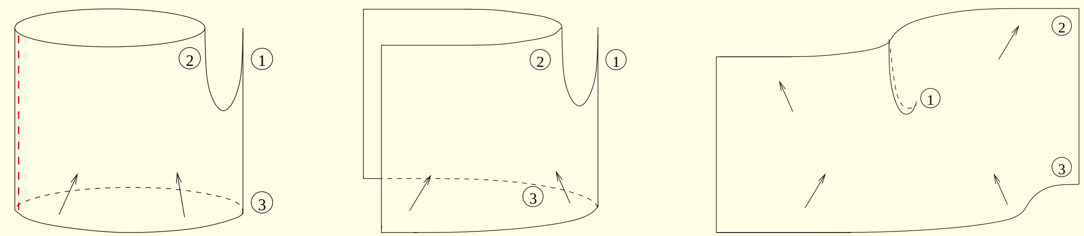
The string vertex for $L_1 = 0$: diagonal form factor

Decompactify string 2 & 3 but $L_1 = 0$:



The string vertex for $L_1 = 0$: diagonal form factor

Decompactify string 2 & 3



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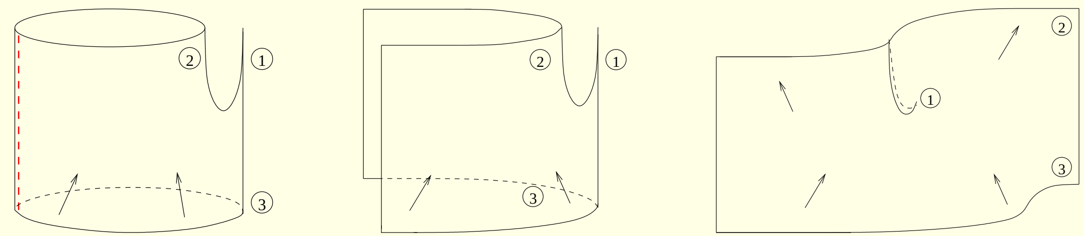
Decompactify string 2 & 3



Local operator form factor equations:

$$N_0(\theta_1, \ldots, \theta_n\rangle = N_0(\theta_2, \ldots, \theta_n, \theta_1 - 2i\pi\rangle = S(\theta_i - \theta_{i+1})N_0(\ldots, \theta_{i+1}, \theta_i, \ldots)$$

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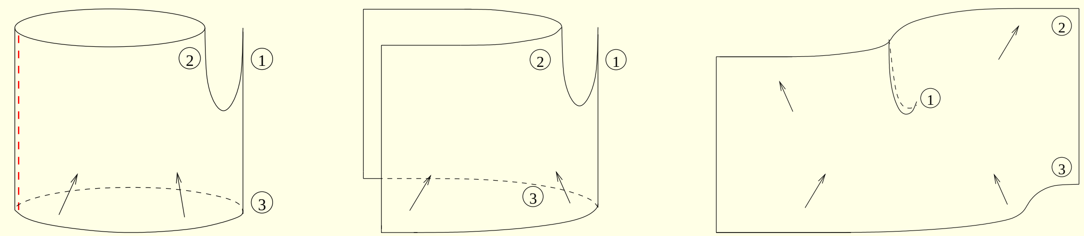

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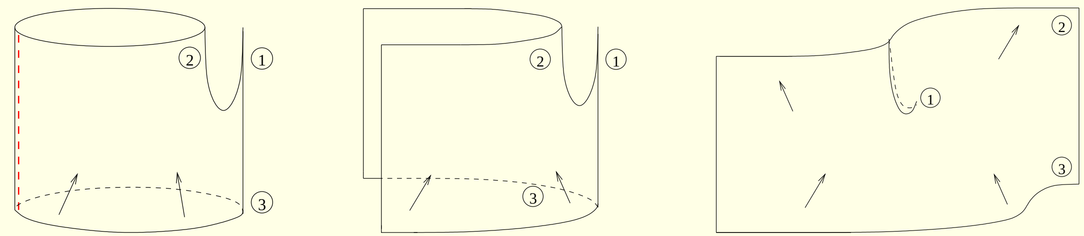
HeavyHeavyLight 3pt function strong coupling prescription

[Costa et al., Zarembo]: $C_{HHL} = \int_{\text{world sheet}} \mathcal{V}(X[\text{heavy solution}(\sigma, \tau)])d^2\sigma$

The string vertex for $L_1 = 0$: diagonal form factor

Decompactify string 2 & 3



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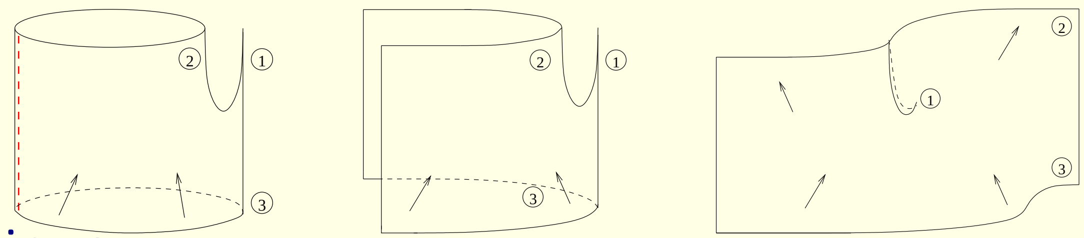
for multiparticle state: $C_{HHL} = \int_{\text{moduli space}\,\{y_i\}} \mathcal{V}(X[\text{heavy solution}(\mathrm{y_i})])d^n y$

The string vertex for $L_1 = 0$: diagonal form factor

Decompactify string 2 & 3



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classical diagonal form factors:

$${}_L\langle\theta_2,\theta_1|\mathcal{V}|\theta_1,\theta_2\rangle_L = \frac{F_2^s(\theta_1,\theta_2)+\rho_1(\theta_1)F_1^s(\theta_2)+\rho_1(\theta_2)F_1^s(\theta_1)}{\rho_2(\theta_1,\theta_2)}$$

Explicitly checked at weak coupling [Hollo, Jiang, Petrovskii],
checked from hexagon [Basso, Komatsu, Vieira] by [Jiang]