Integrability and String Theory II

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Abstract

The aim of my lectures is to introduce the concepts and some techniques of integrable theories on the example of sine-Gordon and sinh-Gordon theory. I will try to draw a parallel between sG and AdS/CFT and use the developed finite size techniques to compute the anomalous dimension of the Konishi operator at four loop.

Introduction

Motivation: (why do we analyze 2D integrable theories?) The drawbacks of such models are obvious: they are in non-physical dimension, even more, they are so simple that cannot describe particle creation in scattering processes. In contrast, the advantages of these models are that they are in non-physical dimensions thus much simpler than their higher dimensional counterparts, moreover, due to their integrability we can solve them exactly as I will show in my lectures.

Then one can ask the question: What do we learn, if we solve such simple theories. First of all we can clarify and understand conceptual questions and problems as asymptotic freedom, dynamical mass generation by dimensional transmutation, analytic S-matrix theory, confinement etc. . Additionally exact solutions can be used to test approximation methods as perturbation theory, semiclassical approximation, truncated conformal space methods, Monte-Carlo simulation etc. Beyond this theoretical benefits two dimensional integrable theories can be directly applied in 2D stat phys / solid state system such as the 2D Coulomb gas, Kondo effect, Hubbard model etc. Their analysis turned out to be useful as it opened new mathematical research directions like infinite dimensional Lie algebras, Quantum groups, etc. Last but not least and most relevantly for the present school, string theory as a 2D world-sheet QFT on AdS backgrounds is proved to be integrable. The AdS/CFT correspondence provides a crucial link which makes possible the apply 2D integrable models in real 4D gauge theories.

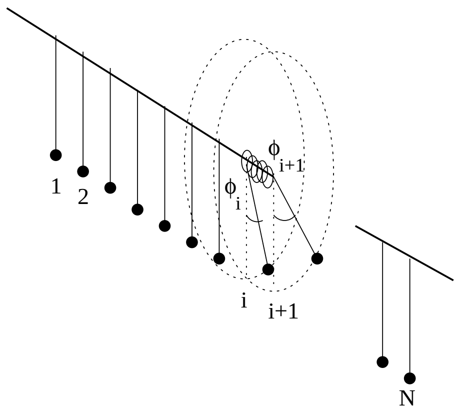
The plan of my talks is as follows: In my first lecture I introduce the sine-Gordon (sG) and sinh-Gordon (shG) theories. I analyze some simple solutions and show their integrability. The second lecture deals with the quantization of the models and give the foundation of asymptotic states and the scattering matrix together with its properties. Based on the first two lectures we axiomatise 2D integrable QFT-s and calculate the simplest scattering matrices such as that of the shG and sG ones in lecture 3. Having solved the shG and sG theories in infinite volume we turn to the description of their spectrum in finite volume in lecture 4. Lecture 5 deals with the leading exponential finite size correction of multiparticle states and its application in AdS/CFT correspondence.

1 Lecture: Classical sine-Gordon theory

In this lecture I introduce the sine-Gordon model as a continuum limit of a mechanical system. I show that the weakly coupled limit of the model is the Klein-Gordon theory. Then I turn to the construction of the finite energy solutions. I start with the static solutions then boost them up. I interpret them as moving particles. By means of the Bäcklund transformation I construct two particle solutions and introduce the concept of time delays. The integrability of the model is shown in the last part.

1.1 sine-Gordon theory as continuum limit of a mechanical system

Consider the following mechanical system composed of coupled pendula:



The angular coordinate of the i^{th} pendulum is ϕ_i, its length is l and mass is μ. The neighboring pendula are placed at distance a and are coupled via a harmonic potential of tension k. The Lagrangian is given by[1]

$$L = E_{kin} - E_{pot} = \sum_{i=1}^{N} \left[\frac{1}{2}\mu l^2 \dot{\phi}_i^2 - \frac{1}{2}k(\phi_i - \phi_{i+1})^2 \right] - \sum_{i=1}^{N} \mu g l (1 - \cos\phi_i)$$

We introduce continuum variables as

$$\varphi((i-1)a) = \frac{\phi_i}{\beta} \quad ; \qquad \frac{m^2}{\beta^2} = \frac{\mu g l}{a}$$

[1]We have to specify also the boundary conditions: we can take periodic, free, and fixed one.

If we take the $a \to 0$, $N \to \infty$ continuum limit, sending $k \to \infty$, $\mu l^2 \to 0$, while keeping $\mu l^2 \beta^{-2}/a = ka\beta^{-2} = 1$, we obtain the Lagrangian of the sine-Gordon theory

$$L = \int_0^L \mathcal{L}\, dx = \int_0^L \left[\frac{1}{2}(\partial_t \varphi)^2 - \frac{1}{2}(\partial_x \varphi)^2 - \frac{m^2}{\beta^2}(1 - \cos \beta\varphi)\right] dx$$

β is the coupling constant, and m determines the mass-scale. The speed of light is normalized to 1.

Let me emphasize that it is somehow a singular limit $k \to \infty$, $N \to \infty$ of a well-defined finite system so it should not surprise as if we encounter some singularity later.

1.1.1 sinh-Gordon theory

If we analytically continue in $\beta \to ib$ we obtain the Lagrangian of the sinh-Gordon theory

$$\mathcal{L} = \frac{1}{2}(\partial_t \varphi)^2 - \frac{1}{2}(\partial_x \varphi)^2 - \frac{m^2}{b^2}(\cosh b\varphi - 1)$$

In contrast to the sG theory where the field φ lives on a compact space (the circle) the shG field lives on the full line.

1.2 Klein-Gordon equation as the weakly coupled limit

Taking the $b \to 0$ or alternatively the $\beta \to 0$ limit we obtain

$$\mathcal{L} = \frac{1}{2}(\partial_t \varphi)^2 - \frac{1}{2}(\partial_x \varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{m^2 b^2}{4!}\varphi^4 - \dots$$

For $\beta = 0$ we obtain the Klein-Gordon equation and the relativistic wave equation as equation of motion

$$(\partial_t^2 - \partial_x^2 + m^2)\varphi = \Box_x \varphi = 0$$

It was invented by Schrödinger in order to describe the spectrum of the Hydrogen atom (and omitted by himself since it wrongly reproduced the fine-structure). Its relativistic invariance means

$$\Box_x = \Box_{x'} \quad ; \qquad x' = (x - vt)\gamma \quad , \quad t' = (t - vx)\gamma \quad , \quad \gamma^{-1} = \sqrt{1 - v^2}$$

Lorentz transformation in the rapidity parametrization

$$v = \tanh \Lambda \quad ; \qquad x' = x \cosh \Lambda - t \sinh \Lambda \quad , \quad t' = t \cosh \Lambda - x \sinh \Lambda$$

is a hyperbolic rotation. Light-cone coordinates diagonalize this transformation

$$x_\pm = \frac{1}{2}(t \pm x) \qquad ; \qquad x'_\pm = e^{\pm\Lambda} x_\pm$$

1.3 Static solutions

The sine-Gordon model

$$\int_0^L \left[\frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2 - V(\varphi)\right] dx \qquad ; \quad V(\varphi) = \frac{m^2}{\beta^2}(1-\cos\beta\varphi)$$

is also relativistically invariant. Energy and momentum are conserved[2]

$$E[\varphi] = \int_0^L \left[\frac{1}{2}(\partial_t\varphi)^2 + \frac{1}{2}(\partial_x\varphi)^2 + V(\varphi)\right] dx \qquad ; \quad P[\varphi] = \int_0^L \partial_x\varphi\partial_t\varphi$$

By exploiting the relativistic invariance we solve the static equation first and then boost the solutions.

The static equation reads as

$$(\partial_t^2 - \partial_x^2) - \frac{m^2}{\beta}\sin\beta\phi = 0 \longrightarrow -\partial_x^2 - \frac{m^2}{\beta}\sin\beta\phi = 0$$

It is analogous to the Newton's equation of motion of a particle in the $-V(r)$ potential

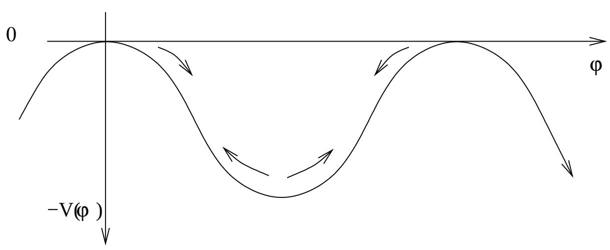
$$\frac{d^2r}{d\tau^2} = -V'(r) \longleftrightarrow \frac{d^2\varphi}{dx^2} = V'(\varphi)$$

We can integrate it using the conservation of classical energy $\epsilon = \frac{1}{2}(\frac{dr}{d\tau})^2 + V(r)$

$$\tau = \int \frac{dr}{\pm\sqrt{2(\epsilon - V(r))}} + \tau_0 \longleftrightarrow x = \int \frac{d\varphi}{\pm\sqrt{2(\epsilon + V(\varphi))}} + x_0$$

For the sine-Gordon potential we obtain elliptic integrals.

The infinite volume limit, however, simplifies considerably:



The finiteness of the energy of the field configuration requires: $\partial\varphi \to 0$, $V(\varphi) \to 0$ as $x \to \pm\infty$. Taking a look at the movement of the analogues particle in the $-V$ potential we can see that at negative and positive infinite times the particle has to be on the top of the potential. That is its energy has to vanish, $\epsilon = 0$, giving rise to

$$x - x_0 = \pm\int \frac{d\varphi}{\sqrt{2V(\varphi)}} = \pm\int \frac{d\frac{\beta\varphi}{2}}{m\sin\frac{\beta\phi}{2}} = \pm\log(\tan(\frac{\beta\varphi}{4}))$$

[2]Periodic BC. is understood

The two solutions $\pm$ are called soliton and anti-soliton, respectively:

$$\varphi(x) = \pm\frac{4}{\beta}\arctan(e^{-m(x-x_0)})$$

Both have energy $E = \frac{8m}{\beta^2} = M_0$ momentum $P = 0$ and look like two kinks. Observe that they are nonperturbative, ie. infinitely heavy in the weakly coupled limit: $\beta \to 0$. Their energy density is localized.

Exercise: calculate the energy density!

Once we have these solutions we can boost them up to obtain moving solutions:

$$\varphi(x,t) = \pm\frac{4}{\beta}\arctan(e^{-m\gamma(x-vt-x_0)})$$

The energy of these solutions is $E(v) = M_0\sqrt{1-v^2}$, while the momentum is $P(v) = M_0 v$. They are solitary waves: that is why they are called soliton and antisoliton. They have localized energy density and due to the nonlinearity of the sine-Gordon equation they are dispersionless solutions, keeping their form forever.

Exercise: Calculate their energy density and compare it with the standing particles one. Observe the relativistic contraction.

The difference between the soliton and the anti-soliton is in their topological charge:

$$Q_{top} = \frac{2\pi}{\beta}\int_{-\infty}^{\infty}\partial_x\varphi(x,t)dx = \frac{2\pi}{\beta}\left[\varphi(\infty) - \varphi(-\infty)\right]$$

The topological charge is conserved and takes ± 1 on the soliton/anti-soliton, respectively. Solitons and antisolitons are localized objects which we treat as particles. We analyze now their interaction.

1.4 Scattering solutions, time delays

1.4.1 Bäcklund transformation

Since the sine-Gordon equation is nonlinear we cannot just simply add to solitons. But Bäcklund transformation will do the job.

Exercise: Show that if φ_1 solves the sine-Gordon equation of motion

$$\partial_+\partial_-\varphi_1 = \frac{m^2}{\beta}\sin\beta\varphi_1 \quad ; \qquad \partial_\pm = \partial_t \pm \partial_x$$

and φ_2 satisfies

$$\begin{aligned}\partial_+\varphi_2 &= \partial_+\varphi_1 + \frac{2m\sigma}{\beta}\sin\left(\frac{\beta}{2}(\varphi_1+\varphi_2)\right)\\ \partial_-\varphi_2 &= -\partial_-\varphi_1 + \frac{2m}{\beta\sigma}\sin\left(\frac{\beta}{2}(\varphi_1-\varphi_2)\right)\end{aligned}$$

then φ_2 also solves the sine-Gordon equation as well.

Exercise: Use this transformation for the soliton/anti-soliton solution.

1.4.2 Two particle solutions

Thus if we plug into the Bäcklund transformation the one particle solution we obtain a two particle solution. Especially for two solitons we get:

$$\varphi_{ss}(x,t) = \frac{4}{\beta}\arctan\left(\frac{v\sinh mx\gamma}{\cosh mvt\gamma}\right)$$

Similarly the solition– anti-soliton solution reads as

$$\varphi_{s\bar{s}}(x,t) = \frac{4}{\beta}\arctan\left(\frac{\sinh mvt\gamma}{v\cosh mx\gamma}\right)$$

Observe that for asymptotically large times they do not interact

$$\varphi_{s\bar{s}}(x,t) = \frac{4}{\beta}\arctan\left(\frac{e^{mvt\gamma-\log v}-e^{-mvt\gamma-\log v}}{e^{mx\gamma}+e^{-mx\gamma}}\right)$$

For asymptotically large negative times ($t\to-\infty$ remote past)

$$\varphi_{s\bar{s}}(x,t) \approx \varphi_s((x+v(t+\frac{\Delta t}{2}))\gamma)+\varphi_{\bar{s}}((x-v(t-\frac{\Delta t}{2}))\gamma) \quad ; \qquad \Delta t = 2\frac{\log v}{mv\gamma}$$

while for asymptotically large positive times ($t\to\infty$ remote future)

$$\varphi_{s\bar{s}}(x,t) \approx \varphi_s((x+v(t-\frac{\Delta t}{2}))\gamma)+\varphi_{\bar{s}}((x-v(t+\frac{\Delta t}{2}))\gamma)$$

Thus we can interpret Δt as the time delay experienced by the soliton in the potential of the antisoliton. This interaction is attractive so we expect them to form a boundstate. Indeed putting $v\to iu$ we obtain the so called breather solution

$$\varphi_b(x,t) = \frac{4}{\beta}\arctan\left(\frac{\sin mut\gamma}{u\cosh mx\gamma}\right)$$

which is periodic in time and has one continuous parameter u.

Just for completeness I present the general finite energy solutions: they can be written into the Hirota form:

$$\varphi = \frac{4}{\beta}\arctan\frac{Im(\tau)}{Re(\tau)}$$

where

$$\tau = \sum_{\mu_j=0,1} e^{\frac{i\pi}{2}\sum_{j=1}^N \epsilon_j\mu_j}\, e^{-\sum_{j=1}^N \frac{\mu_j}{2}\left\{(k_j+k_j^{-1})x+(k_j-k_j^{-1})t-a_j\right\}+2\sum_{i<j}\mu_i\mu_j\log\frac{k_i-k_j}{k_i+k_j}}$$

In the remote past it is composed of well separated non-interacting particles of type soliton, antisoliton and breather and all have different velocities. They are ordered according to their velocity: the fastest is on the left. In the remote future the particle content is the same even with the same velocities, except that their ordering is just the opposite. The only difference compared to the free motion is in the acquired time delay. The time delay of any particle is the sum of the time delays they would acquire if they had scattered only with one particular constituent.

1.5 Integrability, conserved charges

The dispersionless nature of the solutions is a consequence of the integrability of the model. Integrability means the existence of infinite number of commuting conserved charges. This can be shown by defining the following gauge potentials:

$$A_x(\lambda) = i \begin{pmatrix} \lambda & \frac{\beta}{2}\partial_+\varphi \\ -\frac{\beta}{2}\partial_+\varphi & -\lambda \end{pmatrix} \quad ; \qquad A_t(\lambda) = \frac{1}{4i\lambda} \begin{pmatrix} \cos\beta\varphi & -i\sin\beta\varphi \\ i\sin\beta\varphi & -\cos\beta\varphi \end{pmatrix}$$

This is flat

$$F_{xt} = \partial_x A_t - \partial_t A_x + [A_x, A_t] = 0$$

provided φ satisfies the sine-Gordon equation of motion.

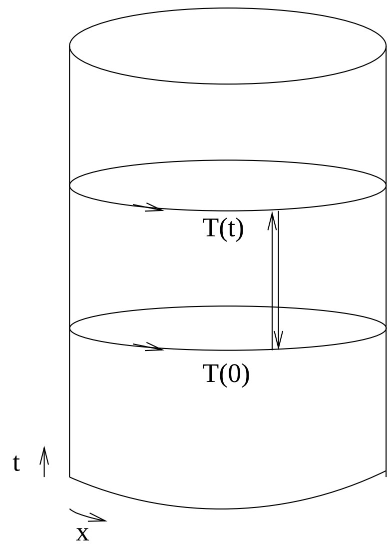
Exercise: Show this.

Thus we can define the Wilson loop

$$T(\lambda, 1, 2) = \mathcal{P}\exp\int_1^2 A(x)_\mu dx^\mu$$

for a curve from point 1 to point 2. It is invariant under continuous deformations of the path.

Let us consider the sG theory with periodic BC. We surround the space circle with the Wilson loop at fixed time $t = 0$. One can show that the same quantity evaluated at time t can be written as $T(\lambda, t) = GT(\lambda, 0)G^{-1}$. Thus by expanding the trace of $T(\lambda)$ in λ we can generate infinite number of local charges.



The first few read as

$$Q_{\pm 1}[\varphi] = E[\varphi] \pm P[\varphi] = \int \left\{ \frac{1}{2}(\partial_\pm\varphi)^2 + \frac{m^2}{\beta^2}(1 - \cos\beta\varphi) \right\} dx$$

$$Q_{\pm 3}[\varphi] = \int \left\{ \frac{1}{2}(\partial_\pm^2\varphi)^2 - \frac{1}{8}(\partial_\pm\varphi)^4 + \frac{m^2}{\beta^2}(\partial_\pm\varphi)^2(1 - \cos\beta\varphi) \right\} dx$$

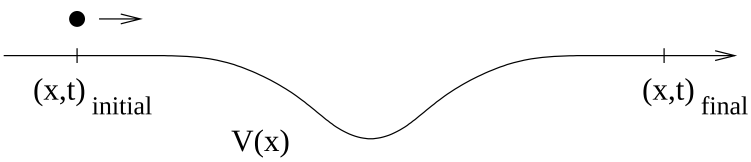
Further reading: see R.Rajaraman: "*Solitons and Instantons*", North-Holland Publishing Company, Amsterdam 1982, and references therein.

2 Lecture: Quantization

In this lecture I start with introducing the quantum analogue of the classical time delay: the scattering phase. I show how its semi-classical limit can be expressed in terms of the time-delay. This makes a link between the classical and quantum description. Then I turn to the quantization of the sinh-Gordon theory. I use the perturbative scheme in which the free Klein-Gordon part is quantized first and then the interaction is taken into account in the interaction picture. I introduce asymptotic states and their scattering matrix. Reduction formula links the S-matrix to the correlation functions and makes possible to derive its crossing symmetry.

2.1 Semiclassical considerations, phase shifts

Let me recall that we treat the soliton and the anti-soliton as particles. The soliton feels the anti-soliton as an attractive potential so when it passes by it experiences a time delay. Let us see how we can compute the time delay from the potential.



Suppose a particle moves in a potential with finite support. The time delay is defined as

$$\Delta t = (t_{final} - t_{initial})|_V - (t_{final} - t_{initial})|_{free}$$

and can be computed as follows:

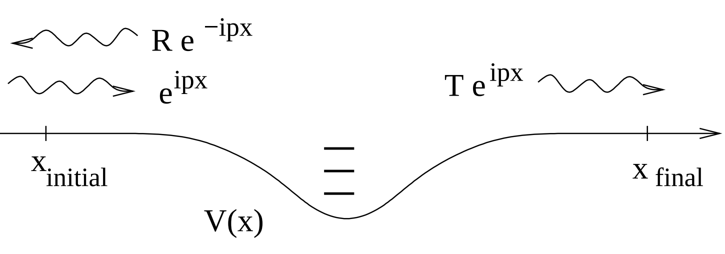
$$t_f - t_i = \int_{x_i}^{x_f} \frac{dx}{v(x)} = \int_{x_i}^{x_f} \frac{\partial p(x,E)}{\partial E} dx$$

where we used the Hamiltonian equation of motion $v(x) = \frac{\partial H}{\partial p}(x)$. Thus the time delay is

$$\Delta t(E) = \partial_E \int_{x_i}^{x_f} (p(x,E) - p(E))dx$$

It would be nice to reconstruct the potential from $\Delta t(E)$. This would be the classical analogue of a scattering experiment.

Suppose now that we analyze the quantum mechanics of the particle in the same potential. Since at large negative and positive x the potential vanishes we have free motion and the information on the potential contained in the reflection and transmission coefficients.



Observe that at the quantum level we may have reflection as well and the continuous spectrum of breathers has to turn into a discrete set of bound-states.

We calculate the transmission coefficient in the semiclassical approximation ($\hbar \to 0$). That is we solve the Schrodinger equation by separating the amplitude and the phase as

$$\hat{H}(\hat{p}, x)\Psi(x) = E\Psi(x) \quad ; \qquad \Psi(x,t) = A(x,t)e^{\frac{i}{\hbar}S(x,t)}$$

where it is supposed that $S(x,t) \gg \hbar$, so it oscillates fast. (We use $\hbar = 1$ from now on).

Exercise: Show that $S(x,t)$ is the classical action

$$S(x,t) = \int_{x_i}^{x} p(x', E)dx' + const$$

and the phase shift is simply

$$\delta(E) = \int_{x_i}^{x_f} (p(x,E) - p(E))dx$$

Comparing with the classical time delay we can conclude that in the semiclassical approximation

$$\partial_E \delta(E) = \Delta t(E)$$

or if we integrate this equation from the threshold energy we have

$$\delta(E) = \delta(E_{th}) + \int_{E_{th}}^{E} \Delta t(E')dE' = n_B \pi + \int_{E_{th}}^{E} \Delta t(E')dE'$$

where n_B is the number of bound-states.

Exercise: using Bohr-Sommerfeld quantization check the integration constant.

2.2 Quantization based on Klein-Gordon theory

Quantum field theory is more complicated then quantum mechanics as we have infinite number of degrees of freedom. The way we usually do the quantization is to choose a scheme we are working in. We adopt here a perturbative quantization scheme in which we separate the free Klein-Gordon part and quantize it first then we take into account the interaction perturbatively. By this we implicitly assume that the effect of the interaction is small, i.e. this can be valid only for the sinh-Gordon theory.

Thus in the shG theory we separate the free and the interaction part as:

$$\mathcal{L} = \frac{1}{2}(\partial_t \varphi)^2 - \frac{1}{2}(\partial_x \varphi)^2 - \frac{m^2}{2}\varphi^2 - b^2 U(\varphi) \quad ; \qquad U(\varphi) = \frac{m^2}{4!}\varphi^4 + \dots$$

For $b = 0$ we obtain the Klein-Gordon equation

$$(\partial_t^2 - \partial_x^2 + m^2)\varphi_0 = \Box_x \varphi_0 = 0$$

which we quantize first and define the sinh-Gordon QFT in the interaction picture.

2.2.1 Solving the free part

The free massive boson is easily quantized:

Exercise: Quantize the free massive boson in 1+1 dimensions and show that

$$\varphi_0(x,t)=\int_{-\infty}^{\infty}\frac{dk}{2\pi 2\omega(k)}(a(k)e^{-i\omega(k)t+ikx}+a^+(k)e^{i\omega(k)t-ikx})$$

where the creation and annihilation operators satisfy

$$[a(k),a^+(k^{'})]=2\pi 2\omega(k)\delta(k-k^{'})\quad ;\qquad \omega(k)=\sqrt{k^2+m^2}$$

They generate the Fock Hilbert space of the model:

$$a^+(k_1)\dots a^+(k_n)|0\rangle=|k_1,\dots k_n\rangle\quad ;\qquad a(k)|0\rangle=0$$

Each state is an eigenstate of the energy and the momentum

$$H_0=\int_{-\infty}^{\infty}:\left[\frac{1}{2}(\partial_t\varphi_0)^2+\frac{1}{2}(\partial_x\varphi_0)^2+\frac{m^2}{2}\varphi_0^2\right]:dx\qquad ;\quad P=\int_{-\infty}^{\infty}:\partial_x\varphi_0\partial_t\varphi_0:$$

with eigenvalue

$$H_0|k_1,\dots,k_n\rangle=\sum_i\omega(k_i)|k_1,\dots,k_n\rangle\qquad ;\qquad P|k_1,\dots,k_n\rangle=\sum_i k_i|k_1,\dots,k_n\rangle$$

The free Feynman propagator is

$$\langle 0|T(\varphi_0(x,t)\varphi_0(x^{'},t^{'})|0\rangle=\int\frac{d^2k}{(2\pi)^2}\frac{i}{k^2-m^2+i\epsilon}e^{-ik_0(t-t^{'})+ik_1(x-x^{'})}$$

2.2.2 Perturbative definition

The interacting theory is defined in the interaction picture:

$$H=H_0+H_I=H_0+b^2\int dx:U(\varphi_0):$$

Normal ordering can change the parameters of U (renormalization).

Time evolution of operators and vectors in the Heisenberg picture is given by:

$$\varphi(x,t)=e^{iHt}\varphi(x,0)e^{-iHt}\qquad ;\qquad |k_1,\dots k_n;t\rangle=|k_1,\dots,k_n;0\rangle$$

This is manifestly Lorentz covariant, since state vectors are time independent. Since we understand the free time evolution we can separate it from the complete time evolution. This means that we evolve the operators by the free time evolution (generated by H_0) and the vectors by the so called evolution operator:

$$\varphi_0(x,t)=e^{iH_0t}\varphi_0(x,0)e^{-iH_0t}\qquad ;\qquad |k_1,\dots k_n;t\rangle=U(t,0)|k_1,\dots,k_n;0\rangle$$

This is the so called interaction picture.

Exercise: show that $U(t,0) = e^{iH_0 t} e^{-iHt}$ and derive its time derivative. By solving its time evolution show that

$$U(t,0) = T \exp \left\{ -i \int_0^t H_I(\varphi_0(t^{'}))dt^{'} \right\}$$

The Heisenberg picture and the interaction picture coincides at a reference time. It is very natural to choose this reference time to be $-\infty$. It is motivated by the fact that for asymptotically large negative time we have no interaction: finite energy solutions are well separated. This is valid for massive excitations only, which are localized. Taking into account the canonical normalization of the fields we basically suppose that

$$\lim_{t\to-\infty} \varphi(x,t) \approx \lim_{t\to-\infty} Z^{\frac{1}{2}} \varphi_0^{in}(x,t)$$

where $0 < Z < 1$ is called the wave function renormalization constant.

In the quantized theory we are interested in the correlation functions of Heisenberg operators.

Exercise: show that they can be written in the interaction picture as

$$\langle 0|T(\varphi(x_1,t_1)\dots\varphi(x_n,t_n)|0\rangle = \frac{\langle 0|T(\varphi_0(x_1,t_1)\dots\varphi_0(x_n,t_n)\exp\left\{i\int d^2x\mathcal{L}_I(\varphi_0(x))\right\})|0\rangle}{\langle 0|T(\exp\left\{i\int d^2x\mathcal{L}_I(\varphi_0(x))\right\})|0\rangle}$$

By expanding the exponential we can define the quantum field theory perturbatively. The Feynman rules summarize the way one can systematically compute the correlation functions.

Exercise: derive the following Feynman rules:

- Draw all topologically distinct diagrams
- associate a propagator $\frac{i}{k^2-m^2+i\epsilon}$ for each inner line
- introduce im^2b^{2n-2} for each vertex of $2n$legs
- integrate for inner momenta not fixed by momentum conservations
- divide by the symmetry factor of the graph

Once these rules are derived the model is defined in principle, so we can try to compute the correlation functions order by order.

The simplest problem one can try to calculate is the propagator. Immediately at one loop we face with a divergent integral, which can be regularized and compensated by a counter term. Calculating other correlators at one loop we arrive at the same divergence and the induced counter term Lagrangian has exactly the same form as the original one. Thus the divergences can be absorbed into the renormalization of the mass term:

$$: V(\varphi) :=: \frac{m^2}{b^2}(\cosh b\varphi - 1) := V(\varphi) - V_{CT}(\varphi) = \frac{m^2 - \delta m^2}{b^2}(\cosh b\varphi - 1)$$

which at one loop reads as:

$$\delta m^2 = -m^2 b^2 \int_0^\Lambda \frac{dp}{\sqrt{p^2+m^2}}$$

Then basically everything is well defined and finite so one can calculate the remaining finite mass and finite wave function renormalization order by order.

2.2.3 Scattering matrix, reduction formulae

Recall that classically at asymptotically large times the particle type excitations are well separated and non-interacting. This motivates us to adiabatically switch off the interaction for large times. We suppose that

$$\lim_{t\to\mp\infty} \varphi(x,t) \approx \lim_{t\to\mp\infty} Z^{\frac{1}{2}} \varphi_0^{in/out}(x,t)$$

where Z takes care of the canonical normalization of the fields and the limit is understood in the week sense. Asymptotic annihilation/creation operators are defined as

$$a^{as}(k) = i\int dx\, e^{i\omega(k)t-ikx} \overleftrightarrow{\partial_t} \varphi_0^{as}(x,t) \qquad ; \quad a^{as}(k)^+ = -i\int dx\, e^{-i\omega(k)t+ikx} \overleftrightarrow{\partial_t} \varphi_0^{as}(x,t)$$

They create asymptotic states

$$|k_1,\dots,k_n\rangle^{as} = a^{as}(k_1)^+ \dots a^{as}(k_1)^+|0\rangle$$

Asymptotic completeness means that both the initial and final states form a complete set. Thus, they can be expressed in terms of each other by the so called scattering matrix

$$S_{fi} = \langle final|initial\rangle$$

By definition

$$S = U(\infty,-\infty) = T\exp\left\{-i\int_{-\infty}^{\infty} H_I(\varphi_0(t'))dt'\right\} = T\exp\left\{i\int d^2x \mathcal{L}_I(\varphi_0)\right\}$$

Clearly it is unitary and commutes with the symmetries.

The simplest nontrivial S-matrix element is

$${}_{out}\langle k_3,k_4|k_1,k_2\rangle_{in} = S(k_1,k_2|k_3,k_4)(2\pi)^2 2\omega(k_1)2\omega(k_2)\delta(k_1-k_3)\delta(k_2-k_4)$$

In a Lorentz invariant theory the Smatrix depends only on the Mandelstam variables $s=(k_1+k_2)^2$. Actually the other two $t=(k_1-k_3)^2$ and $u=(k_1-k_4)^2$ in two dimensions are not independent.

Exercise: show that the connected part of scattering matrix can be expressed in terms of the correlator. This is the so called reduction formula and can be written as

$$\begin{aligned} {}_{out}\langle k_3,k_4|k_1,k_2\rangle_{in} \;=\; & \text{disconnected} + Z^{-2}\int d^2x_1 e^{-i\omega(k_1)t_1+ik_1x_1}\Box_1 \int d^2x_2 e^{-i\omega(k_2)t_2+ik_2x_2}\Box_2 \\ & \int d^2x_3 e^{i\omega(k_3)t_3-ik_3x_3}\Box_3 \int d^2x_4 e^{i\omega(k_1)t_1-ik_1x_1}\Box_4 \langle 0|T(\varphi(1)\varphi(2)\varphi(3)\varphi(4))|0\rangle \end{aligned}$$

where $\varphi(i)$ means $\varphi(x_i, t_i)$ and $\Box_i = -\partial_{t_i}^2 + \partial_{x_i}^2 - m^2$.

Observing the only difference between the initial and final states being in the sign of the (ω, k) term we can see the crossing symmetry of the scattering matrix:

$$S(k_1, k_2|k_3, k_4) = S(k_1, \bar{k}_3|\bar{k}_2, k_4)$$

where for the antiparticle we change the sign of the energy and momentum: $\bar{k} \to (-\omega(k), -k)$.

Now we can try to calculate explicitly the scattering matrix of the sinh-Gordon theory.

Exercise: Calculate it at tree level. Pay attention to the energy momentum conservation laws.

Further reading: see C Itzykson, JB Zuber, *Quantum field theory*, McGraw-Hill, 1980 and references therein.

3 Lecture: Quantum sine-Gordon theory: the bootstrap

We learnt from the analysis of classical solutions that the finite energy solutions behave as relativistic particles, which are well separated for asymptotically large times, without any interactions. The differences between an initial and finite states are the time-delays. The quantum correspondent of this picture is the scattering matrix which connects asymptotic states and satisfies unitarity and crossing symmetry. Having collected all this knowledge we axiomatize them in the present lecture.

3.1 Asymptotic states, scattering matrix

We specify first the Hilbert space. It consists of all particles which can appear as asymptotic states. For simplicity we start with just one single relativistic particle with mass m. The dispersion relation can be written as

$$E(p) = \omega(p) = \sqrt{p^2 + m^2} \quad ; \qquad E(p)^2 - p^2 = m^2$$

Using the rapidity parametrization we can write

$$E(\theta) = \omega(\theta) = m \cosh\theta \quad ; \qquad p(\theta) = m \sinh(\theta)$$

Light-cone components behave as

$$(E \pm p)(\theta) = Q_{\pm 1}(\theta) = m e^{\pm\theta}$$

As it follows from integrability they are just a first member of an infinite family of conserved charges with spin s: $Q_s(\theta) = q_s e^{s\theta}$.

We suppose asymptotic completeness that is the asymptotic multiparticle states span the Hilbert space:

Initial state

$$A_{in}^{+}(\theta_1) \dots A_{in}^{+}(\theta_n)|0\rangle = |\theta_1, \dots \theta_n\rangle_{in} \quad ; \qquad \theta_1 > \dots > \theta_n$$

Final state

$$A^+_{out}(\theta_1)\dots A^+_{out}(\theta_m)|0\rangle = |\theta_1,\dots\theta_m\rangle_{out} \quad ; \qquad \theta_m > \dots > \theta_1$$

They also diagonalize the action of the infinite conserved charges

$$Q_s|\theta_1,\dots\theta_n\rangle_{in} = \sum_{i=1}^{n} q_s e^{s\theta_i}|\theta_1,\dots\theta_n\rangle_{in}$$

The scattering matrix connects the two basis for the Hilbert space, thus they relate initial and final states:

$$S_{n\to m} =_{out} \langle\theta_1^{'},\dots\theta_m^{'}|\theta_1,\dots\theta_n\rangle_{in}$$

and describe the probability with which the initial state evolves into the final state. Let us formulate the requirements the S-matrix has to satisfy.

3.2 S-matrix properties

The S-matrix has to satisfy the following requirements:

- Symmetry

Since the S-matrix is the time-evolution operator it commutes with the conserved charges Q_s. Thus if we evaluate them before and after the scattering they have to coincide:

$$\sum_{i=1}^{n} q_s e^{s\theta_i} = \sum_{i=1}^{m} q_s e^{s\theta_i^{'}}$$

These are functionally independent equations for infinite different values of the spin s. They can be satisfied for a finite number of $\{\theta_i\}$ and $\{\theta_j^{'}\}$ only if the two sets completely agree: $\{\theta_i\} = \{\theta_j^{'}\}$. In particular, this means that the number of particles in the initial and final states coincides $n = m$, that is there is no particle creation in quantum integrable theories.

Conserved charges generate symmetry transformations: H generates uniform shift in time, while P generates uniform shift in space. Higher spin charges, however, generate momentum dependent shifts in space-time. By acting a higher spin charge we can separate the particle interactions, thus multiparticle scattering amplitudes factorize into the product of two particle scatterings:

$$S_{n\to n}(\theta_1,\dots,\theta_n) = \prod_{i,j} S_{2\to 2}(\theta_i,\theta_j)$$

This enables us to determine $S_{2\to 2}(\theta_1,\theta_2)$ only. Lorentz invariance acts on the rapidity as $\theta \to \theta + \Lambda$ and as it is a symmetry we can write

$$S_{2\to 2}(\theta_1,\theta_2) = S(\theta_1 - \theta_2)$$

- Unitarity

Unitarity of the scattering matrix translates into:

$$S(\theta)S(-\theta) = 1$$

- Crossing symmetry

The particle $\leftrightarrow$ anti-particle transformation at the language of the rapidity parameter reads as $(\omega(\theta), p(\theta)) \to (-\omega(\theta), p(\theta)) = (\omega(i\pi - \theta), p(i\pi - \theta))$. Thus the crossing symmetry of the scattering matrix means:

$$S(\theta) = S(i\pi - \theta)$$

- Maximal analyticity:

The scattering matrix $S(\theta)$ is a meromorphic function of the rapidity variable on the physical strip $0 < \Im m(\theta) < \pi$ having poles at the imaginary axis. Each pole must correspond either to a boundstate or to an anomalous threshold. The physical value of the scattering matrix is given by $\lim_{\epsilon\to 0} S(\theta + i\epsilon)$ for $\Re e(\theta) > 0$ and for the crossed process by $\lim_{\epsilon\to 0} S(\theta + i(\pi - \epsilon)$ for $\Re e(\theta) < 0$.

3.2.1 Zamolodchikov-Fateev algebra

We can encode some of the properties of the scattering matrix into the ZF algebra. We introduce abstract creation-annihilation operators which satisfy

$$A^+(\theta_1)A^+(\theta_2) = S(\theta_1 - \theta_2)A^+(\theta_2)A^+(\theta_1)$$

Exercise: derive unitarity!

An initial state can be written as

$$A^+(\theta_1)A^+(\theta_2)\dots A^+(\theta_n)|0\rangle$$

if $\theta_1 > \theta_2 > \dots > \theta_n$. The final state is simply

$$A^+(\theta_n)\dots A^+(\theta_2)A^+(\theta_1)|0\rangle$$

Exercise: show that they are connected by the multiparticle scattering matrix $S_{n\to n}$.

3.3 Solving the simplest models by bootstrap

Now we can try to find the scattering matrices which satisfies the requirements above.

The simplest solution is just

$$S(\theta) = 1$$

This corresponds to the free boson that is to the Klein Gordon theory.

The next simplest is

$$S(\theta)=\frac{\sinh\theta-i\sin\alpha}{\sinh\theta+i\sin\alpha}\quad;\qquad\alpha>0$$

We choose $\alpha>0$ since in this case we do not have any singularity in the physical strip. The corresponding QFT is a quantum integrable model in which we have one particle type only. We claim that the scattering matrix corresponds to the sinh-Gordon theory if

$$\alpha=\frac{\pi b^2}{8\pi+b^2}$$

This has been checked in perturbation theory upto 3 loops.

If we analytically continue in the coupling $b\to i\beta$ we should describe the fundamental field excitation of the sine-Gordon theory. This should correspond to the breather solutions: let us call it the first breather and denote its generator in the ZF algebra by $B_1^+(\theta)$. But since $\alpha<0$ we have a pole in the physical strip on the imaginary axis at $\theta=-i\alpha$. We interpret it as a boundstate and associate a new particle in the spectrum. Let us call it the second breather and denote its ZF generator by $B_2(\theta)$. It is an asymptotic state thus we have to include it in the Hilbert space of the theory. Since it is composed of two B_1-s

$$B_1^+(\theta-i\frac{\alpha}{2})B_1^+(\theta+i\frac{\alpha}{2})|0\rangle=B_2^+(\theta)|0\rangle$$

we can compute the spin s charge of the state

$$[Q_s,B_2^+(\theta)]=[Q_s,B_1^+(\theta-i\frac{\alpha}{2})B_1^+(\theta+i\frac{\alpha}{2})]=2q_se^{s\theta}\cos\frac{s\alpha}{2}B_2^+(\theta)$$

In particular $m_{B_2}=2m_{B_1}\cos\frac{\alpha}{2}$. The scattering matrix of B_2can be calculated as well:

$$S_{B_1B_2}(\theta_1-\theta_2)=S_{B_1B_1}(\theta_1-\theta_2+i\frac{\alpha}{2})S_{B_1B_1}(\theta_1-\theta_2-i\frac{\alpha}{2})$$

Exercise: show this relation together with

$$S_{B_2B_2}(\theta_1-\theta_2)=S_{B_2B_1}(\theta_1-\theta_2+i\frac{\alpha}{2})S_{B_2B_1}(\theta_1-\theta_2-i\frac{\alpha}{2})$$

One can ask the question whether this is really a new particle? Since if $S_{B_1B_2}=S_{B_1B_1}$ then there is no need to introduce B_2 we can simply identify with B_1. Actually it happens for $\alpha=-\frac{2\pi}{3}$. This theory is also consistent: all poles of the scattering matrix corresponds to bound-states. This model is called the scaling Lee-Yang model.

If we are not at this particular coupling $\alpha\neq-\frac{2\pi}{3}$, then we have to consider B_2 as a new particle and we have to include it in the spectrum of the theory. Then we have to analyze the pole structure of $S_{B_1B_2}$ and $S_{B_2B_2}$ and for each pole we have to associate either a boundstate or an anomalous threshold. Once we managed to find all particles of the theory in this procedure such that in all scattering matrix all singularity is explained then we consider the theory to be solved. This procedure is called the S-matrix bootstrap.

3.4 The sine-Gordon S-matrix

Now we would like to push forward the same procedure for the sine-Gordon theory. We feed in what we learnt from the classical theory, that we have in the spectrum two particles with the same mass, namely the soliton and the anti-soliton. They carry different topological charges and in the quantum theory we will consider them as a doublet $A_i \quad i = \pm$. Let us introduce the scattering via the ZF algebra. The generator which creates the doublet state is denoted by $A_i^+(\theta)$ and the scattering matrix can be written as

$$A_i^+(\theta_1)A_j^+(\theta_2) = S_{ij}^{lk}(\theta_1 - \theta_2)A_k^+(\theta_2)A_l^+(\theta_1)$$

It is a four by four matrix. Thus the unitarity equation, which can be read off from the ZF algebra reads as:

$$S_{ij}^{kl}(\theta_1 - \theta_2)S_{lk}^{mn}(\theta_2 - \theta_1) = \delta_i^m \delta_j^l$$

In the case of the crossing symmetry we have to take into account that the soliton is the anti-particle of the anti-soliton (as the name suggests):

$$S_{ij}^{kl}(\theta_1 - \theta_2) = S_{j\bar{l}}^{\bar{i}k}(i\pi - \theta_1 + \theta_2)$$

Now we use the symmetry to restrict the scattering matrix. We suppose that the topological charge $Q_{top}|\pm\rangle = \pm|\pm\rangle$ acting on the ZF operators

$$[Q_{top}, A_i^+(\theta)] = iA_i^+(\theta)$$

is conserved. By acting first the charge and then scatter, or alternatively by scattering first and acting with the charge second we obtain the relation:

$$\begin{aligned}
[Q_{top}, A_i^+(\theta_1)A_j^+(\theta_2)] &= (i+j)A_i^+(\theta_1)A_j^+(\theta_2) \\
&= S_{ij}^{lk}(\theta_1, \theta_2)[Q_{top}, A_k^+(\theta_2)A_l^+(\theta_1)] = (k+l)S_{ij}^{lk}(\theta_1, \theta_2)A_k^+(\theta_2)A_l^+(\theta_1) \\
&= (k+l)A_i^+(\theta_1)A_j^+(\theta_2)
\end{aligned}$$

That is $(i+j) = (k+l)$.

Exercise: Use the parity $|\pm\rangle \leftrightarrow |\mp\rangle$ symmetry to the restrict further the scattering matrix into the form:

$$S(\theta) = \rho(\theta) \begin{pmatrix} a(\theta) & 0 & 0 & 0 \\ 0 & b(\theta) & c(\theta) & 0 \\ 0 & c(\theta) & b(\theta) & 0 \\ 0 & 0 & 0 & a(\theta) \end{pmatrix}$$

Exercise: Derive the following Yang-Baxter equation from the associativity of the ZF algebra ($A_i^+(\theta_1)A_j^+(\theta_2)A_k^+(\theta_3)$ can be reordered in two different ways):

$$S_{ij}^{pr}(\theta_1 - \theta_2)S_{pk}^{lq}(\theta_1 - \theta_3)S_{rq}^{mn}(\theta_2 - \theta_3) = S_{jk}^{pr}(\theta_2 - \theta_3)S_{ir}^{qn}(\theta_1 - \theta_3)S_{qp}^{lm}(\theta_1 - \theta_2)$$

This is an over-determined system of equations which has the following solution

$$a(\theta) = 1 \quad ; \quad b(\theta) = -\frac{\sin(i\lambda\theta)}{\sin\lambda(\pi + i\theta)} \quad ; \quad b(\theta) = \frac{\sin(\lambda\pi)}{\sin\lambda(\pi + i\theta)}$$

The scalar coefficient can be fixed from the unitary and crossing symmetry requirement to be

$$\rho(\theta) = \prod_{l=1}^{\infty} \left[\frac{\Gamma(\lambda(2l-2) + \frac{i\lambda\theta}{\pi})\Gamma(1 + \lambda 2l + \frac{i\lambda\theta}{\pi})}{\Gamma(\lambda(2l-1) + \frac{i\lambda\theta}{\pi})\Gamma(1 + \lambda(2l-1) + \frac{i\lambda\theta}{\pi})} / (\theta \to -\theta) \right]$$

The soliton antisoliton scatterings ($S_{+-}^{+-}(\theta) = \rho(\theta)b(\theta)$ and $S_{+-}^{-+}(\theta) = \rho(\theta)c(\theta)$) have poles at $i\pi(1 - \frac{n}{\lambda})$, they are in the physical strip for $n = 1, \dots, [\lambda]$. Thus for $\lambda > 1$ the soliton and the anti-soliton can form bound-states. The lightest one should correspond to the fundamental breather. If we calculate its scattering matrix with itself we recognize that it is indeed in the same form as the sinh-Gordon scattering matrix. Moreover, it is the analytical continuation of the sinh-Gordon S-matrix if

$$\lambda = \frac{8\pi}{\beta^2} - 1$$

One can perform an additional check of the sine-Gordon scattering matrix. It is possible to calculate the semiclassical limit ($\beta \to 0$) of the soliton-soliton scattering phase and compare it with the classical time delay.

Exercise: Check that the relation we derived for quantum mechanical systems is valid also for this scattering.

Further reading: see Patrick Dorey, *Exact S-matrices*, hep-th/9810026 and references therein.

4 Lecture: Quantum field theories in finite volume

In the last lecture we determined the particle content of 2D integrable quantum field theories using the S-matrix bootstrap approach. We also determined their scattering matrix. In the present lecture we will use these quantities to calculate their spectrum in finite size.

We start with a simple system having one particle type only. This can be either the sinh-Gordon or the Lee-Yang model. The spectrum in infinite volume is very boring: $E(\theta_1, \dots, \theta_n) = \sum_i m \cosh \theta_i$. We see that above m we have a continuum of energy levels. This continuous spectrum is replaced with discrete energy levels in finite volume. We analyze first how the energy levels are discretized for a free system then we turn to the investigation of the effect of the interaction.

4.1 Free spectrum in finite volume

Suppose now that a free quantum field theory is defined in a finite volume L with periodic boundary condition. This means that all field are periodic $\phi(x + L) = \phi(x)$. As the conserved momentum P generates the space translation $e^{iPL}\phi(x,t)e^{-iPL} = \phi(x + L, t)$ its spectrum has to be quantized $e^{iPL} = 1$. This means for a one particle state with momentum p that

$$e^{ipL} = e^{imL \sinh \theta} = 1 \quad ; \qquad p_n = \frac{2\pi}{L} n$$

Then the spectrum of a free multiparticle state is simply

$$E(n_1, \dots, n_k) = \sum_i \sqrt{m^2 + (\frac{2\pi}{L} n_i)^2}$$

Clearly the spectrum is discrete. It is denser and denser for $L \to \infty$ and gives the continuous spectrum in the infinite volume limit. Let us see now how we can take into account the interaction among the particles.

4.2 Bethe-Yang equations

If we analyze a quantum mechanical particle in finite volume in the presence of a localized potential then the wave function additionally to the free phase e^{ipL} picks up the transmission phase $T(p)$ when passing by the potential. The periodicity requirement is then modified to $e^{ipL}T(p) = 1$. We use this observation for the particle like excitation of the quantum field.

4.2.1 Sinh-Gordon theory

Recall that we treat the excitation of the quantum field as particles, but this is correct so far as its size (m^{-1}) is much smaller than the volume L. When this particle picture is correct then we use the quantum mechanical argumentation above. This is valid since in integrable theories there is no particle production, the only interaction is the phase that a particle acquires when it scatters on the others. This also shows that a particle number is a sensible quantity.

The one particle quantization condition is not changed since there is nobody to scatter on:

$$e^{im \sinh \theta L} = 1 \leftrightarrow m \sinh \theta_n = \frac{2\pi}{L} n$$

The energy is given by

$$E(n) = m \cosh \theta_n$$

In the case of two particles in finite volume with momentum p_1, p_2 we have to take into account their scattering interactions. Thus when we move around the circle with the first particle we not only pick up e^{ip_1L}but also scatter on the other:

$$e^{imL \sinh \theta_1} S(\theta_1 - \theta_2) = e^{imL \sinh \theta_1} e^{i\delta(\theta_1 - \theta_2)} = 1$$

where we introduced the phase of the scattering matrix

$$S(\theta_1 - \theta_2) = e^{i\delta(\theta_1 - \theta_2)}$$

We can calculate the momentum/rapidity by solving the equation

$$mL \sinh \theta_1 + \delta(\theta_1 - \theta_2) = 2\pi n_1$$

The correction compared to free particle quantization is

$$p_n = p_n^{free} - \frac{\delta(p_1, p_2)}{L}$$

which is of order $O(L^{-1})$ and vanishes as $L \to \infty$. An analogous way we can write for the second particle

$$mL \sinh \theta_2 + \delta(\theta_2 - \theta_1) = 2\pi n_2$$

Since the unitarity property $S(\theta)S(-\theta)$ translates into the phase as $\delta(\theta) + \delta(-\theta) = 0$ we can write:

$$mL(p_1 + p_2) = 2\pi(n_1 + n_2)$$

That is the total momentum is quantized in the units of $2\pi/L$. This is expected as the momentum is conserved and generates space translation: $e^{iPL} = 1$. The energy of the state is given by

$$E(n_1, n_2) = m \cosh \theta_1(n_1, n_2) + m \cosh \theta_2(n_1, n_2)$$

Exercize: develop a method to measure the scattering matrix from the finite size spectrum.

In the case of an N-particle state the quantization condition is

$$e^{imL \sinh \theta_1} S(\theta_1 - \theta_2) S(\theta_1 - \theta_3) \dots S(\theta_1 - \theta_n) = 1$$

and similar equations for each θ_i. These are called the Bethe-Yang equations and determine the finite volume energy levels via the formula

$$E(\theta_1, \dots, \theta_n) = \sum_i m \cosh \theta_i$$

This approximation is valid whenever $m^{-1} \ll L$. We will show later that the next order correction are of order e^{-mL}. But before turning to this problem let us extend our findings to non-diagonal scattering theories.

4.2.2 Sine-Gordon theory, connection to XXZ

We analyze the sine Gordon system in the $\lambda < 1$ region when there is no boundstates only the soliton and antisoliton doublet $|\pm\rangle$.

The one particle states are not part of the spectrum with periodic boundary condition. If one takes the $L \to \infty$ limit of the periodic theory then only the $Q = 0$ topological charge sector can be reached. If we would like to describe the charge Q states we have to demand a boundary condition in which the field φ jumps $Q\frac{2\pi}{\beta}$ when we go around the circle: $\varphi(x + L, t) = \varphi(x, t) + \frac{2\pi Q}{\beta}$.

The $Q = \pm 1$ one particle states can be described then as free states

$$e^{iML \sinh \theta} = 1 \leftrightarrow ML \sinh \theta = \frac{2\pi}{L} n$$

Here we denote the mass of the doublet by M.

In the case of two particle states with charge $Q=2$ we have two solitons on the circle. Since the scattering is diagonal $S_{++}^{++}(\theta)$ we can use the same equations as before

$$e^{iML\sinh\theta_1}S_{++}^{++}(\theta_1-\theta_2)=1$$

By parity symmetry we have the same equation for $Q=-2$. The $Q=0$ sector is more complicated as the scattering mixes up solitons and antisolitons and we have to find such a state which is invariant under the scatterings. Since parity is conserved the even and the odd combinations of the states $A_+^+(\theta_1)A_-^+(\theta_2)\pm A_-^+(\theta_1)A_+^+(\theta_2)$ diagonalizes the scattering matrix with eigenvalues $S^{sym/asym}(\theta)$. The quantization condition then reads as

$$e^{iML\sinh\theta_1}S^{sym/asym}(\theta_1-\theta_2)=1$$

If we consider a theory in a finite volume L which contains all topological charges then the twoparticle BY equation can be formulated as

$$e^{iML\sinh\theta_1}S_{ij}^{kl}(\theta_1-\theta_2)\Psi_{kl}=\Psi_{ij}$$

where Ψ_{ij} is the coefficient matrix of the two particle states which diagonalizes the scatterings: $\Psi_{ij}A_i^+(\theta_1)A_j^+(\theta_2)$.

In the case of general n-particle state the matrix we have to diagonalize can be obtained by bringing the doublet with $\theta_1$around all the others:

$$e^{iML\sinh\theta_1 L}S_{i_1i_2}^{k_1j_2}(\theta_1,\theta_2)S_{k_1i_3}^{k_2j_3}(\theta_1,\theta_3)\dots S_{k_{n-1}i_n}^{j_1j_n}(\theta_1,\theta_n)\Psi_{j_1\dots j_n}=\Psi_{i_1\dots i_n}$$

Basically we have to diagonalize the following matrix

$$T_1(\theta_1;\theta_2,\dots,\theta_n)_{i_1i_2\dots i_n}^{j_1j_2\dots j_n}=S_{i_1i_2}^{k_1j_2}(\theta_1,\theta_2)S_{k_1i_3}^{k_2j_3}(\theta_1,\theta_3)\dots S_{k_{n-1}i_n}^{j_1j_n}(\theta_1,\theta_n)$$

matrix. If we bring around the circle the i^{th}particle we need to diagonalize

$$T_l(\theta_1,\theta_2,\dots;\theta_l;\dots,\theta_n)_{i_1i_2\dots i_n}^{j_1j_2\dots j_n}=S_{i_li_{l+1}}^{k_lj_{l+1}}(\theta_l,\theta_{l+1})S_{k_li_{l+2}}^{k_{l+1}j_{l+2}}(\theta_l,\theta_{l+2})\dots S_{k_{l-1}i_{l-1}}^{i_lj_{l-1}}(\theta_l,\theta_{l-1})$$

But can we diagonalize them simultaneously? The answer is yes and can be shown as follows:

Exercise: define the monodromy matrix to be

$$T(\theta|\theta_1,\dots,\theta_n)_{i_1,i_2,\dots i_n}^{j_1,j_2,\dots j_n}=S_{i\,i_1}^{k_1j_1}(\theta,\theta_1)S_{k_1i_2}^{k_2j_2}(\theta,\theta_2)\dots S_{k_{n-1}i_n}^{i\,j_n}(\theta,\theta_n)$$

and show that T reduces to T_l for $\theta=\theta_l$. Let us note that this monodromy matrix is proportional to the monodromy matrix which was introduced by Changrim to describe the inhomogeneous XXZ model. In his lecture the commutativity of this matrix with different spectral parameter has been shown, thus the BY matrix equations are consistent.

We can see now that the calculation of the finite size energy spectrum of a non-diagonal quantum field theory is reduced to the solution of an inhomogeneous spin chain.

As we mentioned before our calculation of the multi-particle energy levels were asymptotically correct only, and it is false when the size of the particle is comparable with the volume when vacuum polarization effects start to play a role. We calculate now the leading correction to the vacuum.

4.3 Lüscher correction for the ground-state

Let us consider again the sinh-Gordon theory for simplicity where we have only one particle type in the theory. We would like to calculate the groundstate energy $E_0(L)$ of the system in a volume L. Lets denote its Hamiltonian by $H(L)$, it generates evolution in time. We calculate its lowest eigenvalue by evolving in Euclidean time an asymptotically large amount R:

$$E_0(L) = -\lim_{R\to\infty} \frac{1}{R} \log e^{-E_0(L)R} = -\lim_{R\to\infty} \frac{1}{R} Tr\, e^{-H(L)R}$$

Here we supposed periodic boundary condition in Euclidean time. Now we can observe the appearance of the Euclidean partition function. But this partition function can be calculated by an alternative way by taking the coordinate L as Euclidean time and R as space. In this case the former momentum will be the Hamiltonian which we denote by $\tilde{H}(R)$:

$$E_0(L) = -\lim_{R\to\infty} \frac{1}{R} Z(L,R) = -\lim_{R\to\infty} \frac{1}{R} Tr\, e^{-\tilde{H}(R)L}$$

We calculate the trace by introducing a complete normalized system of states:

$$Tr\, e^{-\tilde{H}(R)L} = \sum_{|n\rangle\in\tilde{\mathcal{H}}} \langle n|e^{-\tilde{H}(R)L}|n\rangle = \sum_{|n\rangle\in\tilde{\mathcal{H}}} e^{-\tilde{\epsilon}_n(R)L}$$

If L is also large, then we can keep the leading terms

$$\sum_{|n\rangle\in\tilde{\mathcal{H}}} e^{-\tilde{\epsilon}(R)L} = 1 + \sum_{\tilde{p}_n(R)} e^{-\tilde{\epsilon}(\tilde{p}_n(R))L} + \sum_{\tilde{p}_n,\tilde{p}_m} e^{-(\tilde{\epsilon}(\tilde{p}_n(R))+\tilde{\epsilon}(\tilde{p}_n(R)))L} + \dots$$

For leading order it is enough the keep the second term. Recall that the one-particle spectrum in large volume R is

$$\tilde{p}_n(R) = \frac{2\pi}{R} n$$

We change the sum into integration

$$\sum_n \to \frac{R}{2\pi} \int d\tilde{p}$$

Then using the definition above we obtain the leading groundstate energy correction as

$$E_0(L) = -\int \frac{d\tilde{p}}{2\pi}\, e^{-\tilde{\epsilon}(\tilde{p})L}$$

The dispersion relation in a relativistically invariant theory is $\tilde{\epsilon}(\tilde{p}) = \sqrt{m^2+\tilde{p}^2} > m$, thus the order of the correction is e^{-mL}. The next order correction we neglected is of order $O(e^{-2mL})$.

The physical interpretation of this term is as follows: a virtual (mirror) particle-antiparticle pairs appear from the vacuum, travels around the world (circle) and annihilates on the other side. Clearly virtual particle creation and annihilation exist in infinite volume as well giving an energy correction to the bare vacuum. Here, however, we calculated only the finite energy correction compared to the infinite volume limit groundstate energy.

Further reading: Alyosha Zamolodchikov, *Thermodynamic Bethe Ansatz In Relativistic Models. Scaling Three State Potts And Lee-Yang Models.* Nucl.Phys.**B342**:695-720,1990. and references therein.

5 Lecture: Lüscher correction for multiparticle states and the anomalous dimension of the Konishi operator

In my last lecture I naively generalize the physical picture we obtained for the vacuum finite size energy correction to excited states and apply it to calculate the leading weak coupling expansion of the anomalous dimension of the Konishi operator.

5.1 Lüscher correction for multi-particle states

We conjecture the process which changed the energy of the vacuum in finite volume to change the finite size energy levels of a one particle state with momentum p, too. For a standing particle this will be the mass correction. So we calculate the amplitude of the process in which a virtual (mirror) particle antiparticle pair appears from the vacuum. They travel all over the world, one of them scatters with the particle, then they annihilate on the other side . The resulting finite size energy correction (when we subtract also the vacuum energy correction) takes the form:

$$E_1(p, L) - E(L) = -\int \frac{d\tilde{p}}{2\pi} (S(\tilde{p}, p) - 1) e^{-\tilde{\epsilon}(\tilde{p})L}$$

For $p = 0$ this is the integral term of the famous Lüscher formula. The residue term can be easily obtained by shifting the contour and picking up the residue of the scattering matrix. We note, however that there is another correction of the same order for relativistic theories, but subleading for AdS theories, which affects the quantization of the momentum p.

The generalization for multiparticle states in the case of diagonal theories is straightforward: suppose we solved the Bethe-Yang equations for a multiparticle state and the momenta are given by $p_1, \dots, p_n$. The correction from the vacuum polarization is given by:

$$E_n(p_1, \dots, p_n, L) - E_0(L) = -\int \frac{d\tilde{p}}{2\pi} (S(\tilde{p}, p_1) \dots S(\tilde{p}, p_n) - 1)\, e^{-\tilde{\epsilon}(\tilde{p})L}$$

In the case of nondiagonal theories we have to take into account the all possible scattering of the virtual particle when it travels around the world, moreover it has to come back on the other side with the same state it started in. Additionally, we have to sum over all possible particles. The final result is then:

$$E_n^{\psi}(p_1, \dots, p_n, L) - E_0(L) = -\sum_{a \in \bar{\mathcal{H}}} \int \frac{d\tilde{p}}{2\pi} (S_{i\,i_1}^{k_1 j_1}(\tilde{p}, p_1) S_{k_1 i_2}^{k_2 j_2}(\tilde{p}, p_2) \dots S_{k_{n-1} i_n}^{i\, j_n}(\tilde{p}, p_n) - 1)\, e^{-\tilde{\epsilon}_a(\tilde{p})L}$$

Thus the quantity which appears in the finite size correction is basically the transfer matrix. As a consequence it preserves the scattering states $\Psi_{i_1\dots i_n}$. This formula has been checked in the sine-Gordon theory for two particle states.

5.2 Application to AdS/CFT: Wrapping correction for the Konishi operator

We would like to apply the same formula in the AdS/CFT correspondence:

$$E_0(L) = -\sum_a (-1)^F \int \frac{d\tilde{p}}{2\pi} (S_{i\,i_1}^{k_1 j_1}(\tilde{p}, p_1) S_{k_1 i_2}^{k_2 j_2}(\tilde{p}, p_2) \dots S_{k_{n-1} i_n}^{i\,j_n}(\tilde{p}, p_n) - 1)\, e^{-\tilde{\epsilon}_a(\tilde{p})L}$$

We need

- to specify the state we are interested in
- the knowledge of the complete mirror spectrum
- the scattering matrix between particles and mirror particles
- the mirror dispersion relation

The details of the calculation can be found in

Zoltan Bajnok, Romuald A. Janik: *Four-loop perturbative Konishi from strings and finite size effects for multiparticle states.*, Nucl.Phys.**B807**:625-650,2009. e-Print: arXiv:0807.0399.