1. Energiamegmaradás térelméletben

$$L = \int d^3x\, \mathcal{L}(\phi, \nabla\phi, \dot\phi, x, t) \qquad S = \int L\, dt = \int dt \int d^3x\, \mathcal{L}$$

$$\tilde{\mathcal{L}}(\underline{x}, t) = \mathcal{L}(\phi(\underline{x},t), \nabla\phi(\underline{x},t), \dot\phi(\underline{x},t), x, t)$$

ahol $\phi(\underline{x},t)$ a mozgásegyenletek egy megoldása

mozgásegyenletek:
$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\partial_t\phi} + \partial_j \frac{\partial\mathcal{L}}{\partial\partial_j\phi} = \frac{\partial\mathcal{L}}{\partial\phi}$$

nézzük meg, mi adódik $\frac{\partial\tilde{\mathcal{L}}}{\partial t}$-t $\mathcal{L}$ deriváltjaival kifejezve

(összetett függvény deriváltja) :

$$\frac{\partial\tilde{\mathcal{L}}}{\partial t} = \underbrace{\frac{\partial\mathcal{L}}{\partial\phi}\partial_t\phi} + \boxed{\frac{\partial\mathcal{L}}{\partial\partial_j\phi}\partial_t\partial_j\phi} + \frac{\partial\mathcal{L}}{\partial\partial_t\phi}\partial_t^2\phi + \frac{\partial\mathcal{L}}{\partial t}$$

átrendezve, figyelembe véve, hogy
$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\partial_t\phi} + \partial_j\frac{\partial\mathcal{L}}{\partial\partial_j\phi}$$

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\partial_t\phi}\partial_t\phi + \boxed{\partial_j\frac{\partial\mathcal{L}}{\partial\partial_j\phi}\partial_t\phi}$$

$$\frac{\partial\tilde{\mathcal{L}}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\partial_t\phi}\partial_t\phi\right) + \partial_j\left(\frac{\partial\mathcal{L}}{\partial\partial_j\phi}\partial_t\phi\right) + \frac{\partial\mathcal{L}}{\partial t}$$

azaz
$$\frac{\partial}{\partial t}\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\partial_t\phi}\partial_t\phi - \tilde{\mathcal{L}}\right)}_{\mathcal{E}} + \partial_j\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\partial_j\phi}\partial_t\phi\right)}_{S_j} = -\frac{\partial\mathcal{L}}{\partial t}$$

$\mathcal{E}$: energiasűrűség $\qquad$ S_j: −(energiaáram-sűrűség)

$$\partial_t \mathcal{E} = - \partial_j S_j$$

V térfogatra integrálva

$$E^V = \int_V \mathcal{E}\, dV$$

Gauss-tétel

$$\dot{E}^V = \int_V \dot{\mathcal{E}}\, dV = -\int_V \partial_j S_j \, dV \overset{\downarrow}{=} \underbrace{-\int_{\partial V} S_j \, dA_j}_{\text{felületen átáramló energia}}$$

Ha $r \to \infty$ -re $\mathcal{E}, S_j \to 0$

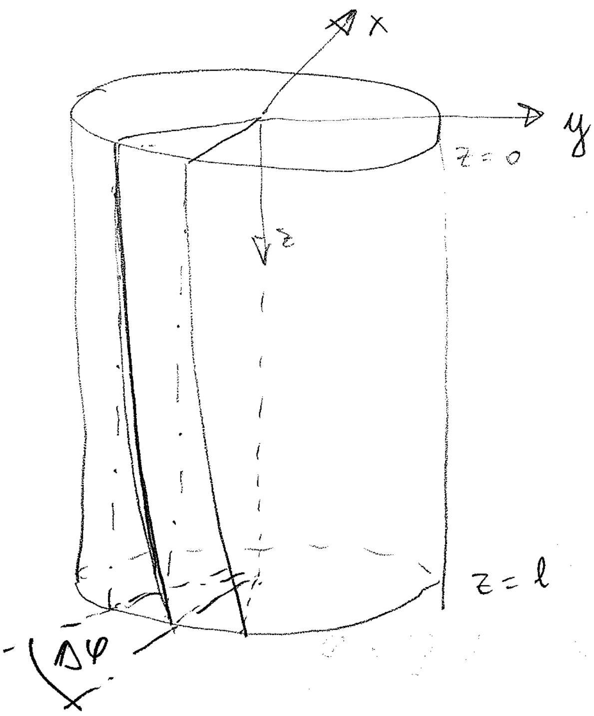
$$E = \lim_{V \to \infty} E^V \qquad \dot{E} = 0$$

megmaradó mennyiség

időeltolás-invariancia miatt marad meg

$\to$ energia

2. Henger csavarása



a rúd egyik lapja ($z=0$) rögzített

a másikat $\Delta\varphi$ szöggel elforgatjuk

- feltevés: a többi sík $\Rightarrow$ csak elfordul, $\varphi(z)$ szöggel

- $\varphi(0) = 0 \quad \varphi(l) = \Delta\varphi$

ekkor az elmozdulásvektor tiszta torzió esetén

$$\underline{u}(\underline{r}) = \vec{\varphi} \times \underline{r} \qquad \vec{\varphi} = \begin{pmatrix} 0 \\ 0 \\ \varphi(z) \end{pmatrix}$$

így $\quad \underline{u}(x,y,z) = \left(-\varphi(z)y \,,\, \varphi(z)x \,,\, 0\right)$

az $\underline{\underline{\varepsilon}}$ deformációtenzor:

$$\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

nemnulla deriváltak: $\quad \partial_y u_x \,,\, \partial_z u_x \,,\, \partial_x u_y \,,\, \partial_z u_y$

azaz

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \cancel{\frac{\partial u_z}{\partial y}}\right) = \frac{1}{2}x\varphi'(z)$$

$$\varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2}\left(\cancel{\frac{\partial u_z}{\partial x}} + \frac{\partial u_x}{\partial z}\right) = -\frac{1}{2}y\varphi'(z)$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}\left(-\varphi(z) + \varphi(z)\right) = 0$$

az összes többi komponens nulla

izotróp anyagra

$$\sigma_{ik} = 2\mu\,\varepsilon_{ik} + \lambda\,\delta_{ik}\,\varepsilon_{ll}$$

$$\underline{\underline{\sigma}} = 2\mu\,\underline{\underline{\varepsilon}} + \lambda\,\underline{\underline{1}}\,\mathrm{Tr}\,\underline{\underline{\varepsilon}}$$

a fenti esetben $\quad \mathrm{Tr}\,\underline{\underline{\varepsilon}} = \varepsilon_{ll} = 0$

így marad

$$\sigma_{yz} = \sigma_{zy} = 2\mu\,\varepsilon_{yz} = \mu x \varphi'(z)$$

$$\sigma_{zx} = \sigma_{xz} = 2\mu\,\varepsilon_{zx} = -\mu y \varphi'(z)$$

Mozgásegyenletek:

$$\varrho \ddot{\underline{u}} = \underline{f} + \nabla \underline{\underline{\sigma}}$$

egyensúlyt keresünk: $\quad \underline{u} = \text{áll.}, \quad \ddot{\underline{u}} = 0$

tömegerők nincsenek: $\quad \underline{f} = 0 \quad \Rightarrow \quad \boxed{\nabla \underline{\underline{\sigma}} = 0}$

határfeltétel a felületen: $\quad \underline{\underline{\sigma}}\,\underline{n} = 0$ (nincs felületi erő) a hengerpaláston

$\underline{\underline{\sigma}}\,\underline{n} = \underline{f}_{fel.}$ a lapokon

$\nabla \underline{\underline{\sigma}} = 0$ komponensei

$$\begin{array}{lll} \partial_x \sigma_{xx} + & \partial_y \sigma_{xy} + & \partial_z \sigma_{xz} = 0 \\ \quad\;\; yx & \quad\;\; yy & \quad\;\; yz \\ \quad\;\; zx & \quad\;\; zy & \quad\;\; zz \end{array}$$

az elsőből pl. σ_{xz} kivételével mind $0 \Rightarrow$

$$\partial_z \sigma_{xz} = 0 \Rightarrow \varphi''(z) = 0$$

$$\Rightarrow \varphi(z) = cz + d$$

$\varphi(0) = 0 \Rightarrow d = 0 \qquad \varphi(l) = \Delta\varphi \Rightarrow \varphi(z) = \dfrac{\Delta\varphi}{l} z$

$$\sigma_{yz} = \sigma_{zy} = \frac{\mu\Delta\varphi}{l} x \qquad \sigma_{zx} = \sigma_{xz} = -\frac{\mu\Delta\varphi}{l} y$$

határfeltételek :

$$\underline{\underline{\sigma}}\,\underline{n} = 0 \qquad \text{a paláston} \qquad \underline{n} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix} = \begin{pmatrix} x/R\\ y/R\\ 0\end{pmatrix}$$

$$\underline{\underline{\sigma}}\,\underline{n} = \frac{\mu\Delta\varphi}{l}\begin{pmatrix} \sigma_{xx}\frac{x}{R} + \sigma_{xy}\frac{y}{R} \\ \sigma_{yx}\frac{x}{R} + \sigma_{yy}\frac{y}{R} \\ \sigma_{zx}\frac{x}{R} + \sigma_{zy}\frac{y}{R}\end{pmatrix} = \frac{\mu\Delta\varphi}{R}\begin{pmatrix} 0 \\ 0 \\ -y\frac{x}{R} + x\frac{y}{R}\end{pmatrix} = 0$$

teljesül

a henger alján a felület erők: $\underline{n} = \begin{pmatrix}0\\0\\-1\end{pmatrix}$

$$\underline{f}_{\text{felület}} = \underline{\underline{\sigma}}\,\underline{n} = \begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{pmatrix} = \frac{\mu\Delta\varphi}{l}\begin{pmatrix}-y\\ x\\ 0\end{pmatrix}$$

forgatónyomaték : $\quad |f| = \frac{\mu\Delta\varphi}{l}\, r$

$$dM = \underline{r} \times \underline{f}_{fel.}\, dA = \frac{\mu\Delta\varphi}{l}\, r^2 \underbrace{2\pi r\, dr}_{dA}$$

$$M = \int dM = \underbrace{\mu}_{G} \frac{\pi R^4}{2l}\,\Delta\varphi$$

G torziómodulus $\quad G = \mu$

$M = D\cdot\Delta\varphi \qquad$ D : direkciós nyomaték

$$D = \frac{\mu\pi}{2}\,\frac{R^4}{l}$$

Megjegyzések

– tisztán csavarás (torzió) felületi erők nélkül: csak körhengerre

3. Gömb deformációja saját gravitációs tere hatására

$$\varrho(\underline{r}) = \begin{cases} 0 & r > R \\ \varrho & r \le R \end{cases} \qquad M(r) = \frac{4\pi}{3}\varrho r^3$$

$$g(r) = \frac{GM(r)}{r^2} = \frac{G\frac{4\pi}{3}\varrho r^3}{r^2} = \frac{4\pi}{3} G\varrho r$$

nyugalmas egyenletek:

$$\partial_l \sigma_{kl} + f_k = 0 \qquad f_k = \varrho \cdot g(r) \frac{x_k}{r}$$

feltesszük, hogy $u_r = u(r)$ és a többi komponens 0

$$\Rightarrow \quad \mathrm{rot}\, \underline{u} = 0$$

nyugalmas egyenletek:

$$\varepsilon_{kl} = \frac{1}{2}(\partial_k u_l + \partial_l u_k) \qquad \varepsilon_{rr} = \partial_r u_r$$

$$\sigma_{kl} = 2\mu\, \varepsilon_{kl} + \lambda\, \delta_{kl}\, \varepsilon_{rr}$$

$$(\mathrm{div}\, \underline{\underline{\sigma}})_k = \partial_l \sigma_{kl} = 2\mu\, \partial_l \varepsilon_{kl} + \lambda \partial_k \varepsilon_{ll}$$

$$\partial_l \varepsilon_{kl} = \frac{1}{2}\partial_l(\partial_k u_l + \partial_l u_k) = \frac{1}{2}\partial_k\partial_l u_l + \frac{1}{2}\partial_l^2 u_k$$

így $\partial_l \sigma_{kl} = (\lambda + \mu)\, \partial_k \partial_l u_l + \mu\, \partial_l^2 u_k$

$$\mathrm{div}\, \underline{\underline{\sigma}} = (\lambda + \mu)\, \mathrm{grad}\, \mathrm{div}\, \underline{u} + \mu \Delta u$$

$$\mathrm{rot}\, \mathrm{rot}\, \underline{u} = \mathrm{grad}\, \mathrm{div}\, \underline{u} - \Delta \underline{u}$$

$$\mathrm{div}\, \underline{\underline{\sigma}} = (\lambda + 2\mu)\, \mathrm{grad}\, \mathrm{div}\, \underline{u} - \mu\, \mathrm{rot}\, \mathrm{rot}\, u$$

most $\mathrm{rot}\, \underline{u} = 0 \Rightarrow \quad \mathrm{div}\, \underline{\underline{\sigma}} = (\lambda + 2\mu)\, \mathrm{grad}\, \mathrm{div}\, \underline{u}$

gömbi koordinátarendszerben fogunk számolni

$$\text{grad div } \underline{u} = -\frac{1}{\lambda+2\mu}\underline{f} = \frac{g}{\lambda+2\mu}\frac{g_0}{R}\underline{r} = \frac{\underline{r}}{a^2}$$

ahol $\quad g_0 = g(R) = \frac{4\pi}{3} G g R \qquad \text{grad div } \underline{u} = \frac{\underline{r}}{a^2}$

gömbi koordinátarendszer : $\underline{u}$ -nak csak $u_r = u(r)$ komp. $\neq 0$

$$\text{div } \underline{u} = u' + \frac{2}{r}u$$

grad div $\underline{u}$ -nak is csak radiális komponense van :

$$(\text{grad div } \underline{u})_r = \left(u' + \frac{2}{r}u\right)' = \frac{r}{a^2} \quad (= f_r)$$

ezt integrálva

$$\underbrace{u' + \frac{2}{r}u}_{\frac{1}{r^2}(r^2u)'} = \frac{r^2}{2a^2} + \underbrace{3b}_{\text{integrálási áll.}}$$

$$\frac{1}{r^2}(r^2u)' = \frac{r^2}{2a^2} + 3b$$

$$(r^2u)' = \frac{r^4}{2a^2} + 3br^2 \qquad \Big/ \int dr$$

$$r^2u = \frac{r^5}{10a^2} + br^3 + c \qquad \Big/ \frac{1}{r^2}$$

$$u(r) = \frac{r^3}{10a^2} + br + \frac{c}{r^2}$$

$r=0$ -ban ne legyen szinguláris $\Rightarrow$ $\boxed{c=0}$

$$u(r) = \frac{r^3}{10a^2} + br$$

nyúlástenzor:

$$\varepsilon_{rr} = \partial_r u_r = \frac{3r^2}{10a^2} + b$$

$$\varepsilon_{\vartheta\vartheta} = \varepsilon_{\varphi\varphi} = \frac{u}{r} = \frac{r^2}{10a^2} + b$$

$$\mathrm{Tr}\,\underline{\underline{\varepsilon}} = \varepsilon_{ll} = \frac{r^2}{2a^2} + 3b$$

$$\sigma_{rr} = 2\mu\varepsilon_{rr} + \lambda\,\mathrm{Tr}\,\underline{\underline{\varepsilon}} = 2\mu\left(\frac{3r^2}{10a^2} + b\right) + \lambda\left(\frac{r^2}{2a^2} + 3b\right)$$

$$= \frac{r^2}{a^2}\left(\frac{3\mu}{5} + \frac{\lambda}{2}\right) + b\,\underbrace{(2\mu + 3\lambda)}_{3K}$$

$r=R$-nél $\sigma_{rr} = 0$ kell legyen

$$\frac{R^2}{a^2}\left(\frac{3}{5}\mu + \frac{\lambda}{2}\right) + (2\mu + 3\lambda)b = 0$$

$$b = -\frac{6\mu + 5\lambda}{2\mu + 3\lambda}\,\frac{R^2}{10a^2}$$

így

$$u(r) = \frac{1}{10a^2}\left(r^3 - \frac{6\mu + 5\lambda}{2\mu + 3\lambda}R^2 r\right)$$

$$u(R) = \frac{R^3}{10a^2}\left(1 - \frac{6\mu + 5\lambda}{2\mu + 3\lambda}\right) = -\frac{R^3}{10a^2}\left(\frac{2(2\mu + \lambda)}{2\mu + 3\lambda}\right)$$

$$a^2 = \frac{R}{g_0}\,\frac{2\mu + \lambda}{\varrho}$$

$$u(R) = -\frac{R^3}{10}\,\frac{g_0\varrho}{R}\,\frac{2}{3K} = -\frac{R^2 g_0 \varrho}{15K}$$

$$\varepsilon_{rr} = u' = \frac{1}{10a^2}\left(3r^2 - \frac{6\mu + 5\lambda}{2\mu + 3\lambda}R^2\right) = \frac{3}{10a^2}\left(r^2 - \frac{6\mu + 5\lambda}{6\mu + 9\lambda}R^2\right)$$

ennek nullahelye:

$$r_0 = R\sqrt{\frac{6\mu + 5\lambda}{6\mu + 9\lambda}} < R$$

ezen belül $\varepsilon_{rr}(r) < 0$ összenyomódik

kívül < 0 kitágul / megnyúlik