$$S[q(t)] = \int_{t_1}^{t_2} L(q(t), \dot q(t); t)\, dt$$

mozgásegyenletek származtatása: $S[q(t)]$ funkcionál deriváltja tűnjön el:

$$S[q(t) + \delta q(t)] = \delta S[q(t), \delta q(t)] + \mathcal{O}(\delta q^2)$$

$$\delta S[q(t), \delta q(t)] = \int_{t_1}^{t_2} \left[ \frac{\partial L(q,\dot q; t)}{\partial q}\, \delta q(t) + \frac{\partial L(q, \dot q, t)}{\partial \dot q}\, \delta \dot q \right] dt$$

$$= \underbrace{\left.\frac{\partial L}{\partial \dot q}\, \delta q \right|_{t_1}^{t_2}}_{0} + \int_{t_1}^{t_2} \underbrace{\left[ \frac{\partial L(q, \dot q, t)}{\partial q} - \frac{d}{dt} \frac{\partial L(q, \dot q, t)}{\partial \dot q} \right]}_{\text{funkcionálderivált}} \delta q\, dt$$

ennek eltűnése: Euler–Lagrange-egy.

$$\delta q(t_1) = \delta q(t_2) = 0$$

Ha $L(q, \dot q, t) = L(q, \dot q)$ (időtől nem függ), akkor

Beltrami-fv létezik, állandó

$$p = \frac{\partial L}{\partial \dot q} \qquad H = p \dot q - L \quad \Rightarrow \text{energiamegmaradás}$$

## Kanonikus mozgásegyenletek

$$H = p\dot q - L \qquad p = \frac{\partial L}{\partial \dot q} \qquad -\frac{d}{dt} \frac{\partial L}{\partial \dot q} + \frac{\partial L}{\partial q} = 0$$

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\frac{d}{dt} \frac{\partial L}{\partial \dot q} \quad \Rightarrow \quad \boxed{\dot p = -\frac{\partial L}{\partial q}}$$

$$\frac{\partial H}{\partial p} = \dot q + p \frac{\partial \dot q}{\partial p} - \underbrace{\frac{\partial L}{\partial \dot q}}_{p} \frac{\partial \dot q}{\partial p} = \dot q \quad \Rightarrow \quad \boxed{\dot q = \frac{\partial H}{\partial p}}$$

Problémamegoldás recept

1. Jó általános koordináták választása
   (minél több kényszer automatikusan teljesüljön)

2. Lagrange-fv. felírása (pl. $L = K - V$)

3. E-L egy. felírása

4. E-L. egy. mego

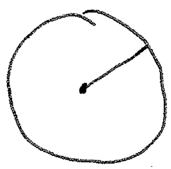
3! $p = \frac{\partial L}{\partial \dot q} \rightarrow H$ (pl. $H = K + V$)

4! Hamilton-egy. felírása

5! Hamilton-egyenletek megoldása

spec. jó koord. választása:

1. példa: körmozgás



$x, y$ $\quad x^2 + y^2 = R^2$ kényszer

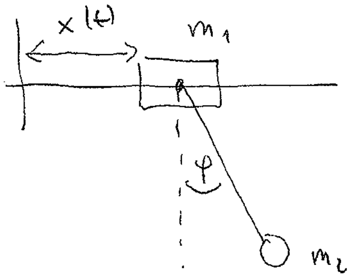
$$K = \frac{1}{2} m (\dot x^2 + \dot y^2)$$

jó koordináta: $\varphi$ $\quad x = R\cos\varphi \quad y = R\sin\varphi$

$\dot x = -R\sin\varphi\,\dot\varphi \quad \dot y = R\cos\varphi\,\dot\varphi \quad K = \frac{1}{2} m R^2 \dot\varphi^2$

2. megjegyzés: ciklikus koordináta is nagy előny

## 1. Példa a Lagrange- (Hamilton-...) függvény alkalmazására 




sín mentén szabadon elcsúszó test

a testre felfüggesztett inga

① általános koordináták választása

$\underline{x, \varphi}$

$$x_1 = x(t)$$
$$z_1 = 0$$
$$x_2 = x + l\sin\varphi$$
$$z_2 = -l\cos\varphi$$

② L-fgv felírása:

$$\dot{x}_1 = \dot{x}$$
$$\dot{z}_1 = 0$$
$$\dot{x}_2 = \dot{x} + l\cos\varphi\,\dot{\varphi}$$
$$\dot{z}_2 = l\sin\varphi\,\dot{\varphi}$$

$$K = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{z}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{z}_2^2) =$$
$$= \frac{1}{2}m_1(\dot{x}^2 + 0)$$
$$+ \frac{1}{2}m_2\left((\dot{x} + l\cos\varphi\,\dot{\varphi})^2 + (l\sin\varphi\,\dot{\varphi}^2)\right)$$
$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\varphi}^2$$
$$+ \cancel{\frac{1}{2}}\cdot\cancel{2}\,m_2\,l\cos\varphi\,\dot{\varphi}\dot{x}$$

$$V = m_1 g z_1 + m_2 g z_2 = -m_2 g l\cos\varphi$$

$$L = K - V = \frac{m_1+m_2}{2}\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\varphi}^2 + m_2 l\cos\varphi\,\dot{\varphi}\dot{x} + m_2 l g\cos\varphi$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 l\cos\varphi\cdot\dot{\varphi}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m_2 l^2\dot{\varphi} + m_2 l\cos\varphi\,\dot{x}$$

$$\frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad x \text{ ciklikus koordináta}, \; p_x = \text{áll.}$$

$p_x$ = összimpulzus! $\qquad \underline{p_x = (m_1+m_2)\dot{x} + m_2 l \dot{\varphi}\cos\varphi}$

$$\frac{d}{dt} p_x = \frac{\partial L}{\partial x} = 0 \qquad p_x = \text{áll.} \qquad \dot{p}_x = (m_1+m_2)\ddot{x} + m_2 l \ddot{\varphi}\cos\varphi - m_2 l \dot{\varphi}^2 \sin\varphi$$

$$\frac{\partial L}{\partial \varphi} = - m_2 l \sin\varphi \, \dot{\varphi}\dot{x} - m_2 l g \sin\varphi$$

$$\dot{p}_x = 0$$

$$\dot{p}_\varphi = m_2 l^2 \ddot{\varphi} + m_2 l \cos\varphi \, \ddot{x} - \cancel{m_2 l \sin\varphi \, \dot{\varphi}\dot{x}}$$

$$\overset{!}{=} -\cancel{m_2 l \sin\varphi \, \dot{\varphi}\dot{x}} - m_2 l g \sin\varphi$$

rendezve:

$$\underbrace{\begin{pmatrix} m_1+m_2 & m_2 l\cos\varphi \\ m_2 l \cos\varphi & m_2 l^2 \end{pmatrix}}_{B} \begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} = \begin{pmatrix} m_2 l \dot{\varphi}^2 \sin\varphi \\ - m_2 l g \sin\varphi \end{pmatrix}$$

ki tudjuk-e fejezni $\ddot{x}$-ot, $\ddot{\varphi}$-ot?

$$\det B = \underbrace{(m_1+m_2) m_2 l^2}_{m_1 m_2 l^2 + \underbrace{m_2^2 l^2}_{m_2^2 l^2(\sin^2\varphi + \cos^2\varphi)}} - m_2^2 l^2 \cos^2\varphi \quad > 0 \Rightarrow \text{igen, kifejezhető}$$

$$= m_1 m_2 l^2 + m_2^2 l^2 \sin^2\varphi$$

$$B^{-1} = \frac{1}{\det B} \begin{pmatrix} m_2 l^2 & -m_2 l \cos\varphi \\ -m_2 l\cos\varphi & m_1+m_2 \end{pmatrix}$$

$$\begin{pmatrix}\ddot{x}\\ \ddot{\varphi}\end{pmatrix} = \frac{1}{m_1 m_2 \ell^2 + m_2^2 \ell^2 \sin^2\varphi} \begin{pmatrix} m_2 \ell^2 & -m_2 \ell \cos\varphi \\ -m_2 \ell \cos\varphi & m_1 + m_2 \end{pmatrix} \begin{pmatrix} \dot{\varphi}^2 \\ -g \end{pmatrix} m_2 \ell \sin\varphi$$

$$= \frac{\sin\varphi}{m_1 \cancel{\ell} + m_2 \cancel{\ell} \sin\varphi} \begin{pmatrix} m_2 \ell^{\cancel{2}} \dot{\varphi}^2 + m_2 \cancel{\ell} \cos\varphi\, g \\ -m_2 \cancel{\ell} \cos\varphi\, \dot{\varphi}^2 - \frac{1}{\ell} g (m_1 + m_2) \end{pmatrix}$$

$$\ddot{x} = \frac{m_2 \sin\varphi}{m_1 + m_2 \sin\varphi} \left( \dot{\varphi}^2 + g \cos\varphi \right)$$

$$\ddot{\varphi} = \frac{-\sin\varphi}{m_1 + m_2 \sin\varphi} \left( m_2 \cos\varphi\, \dot{\varphi}^2 + g \frac{m_1 + m_2}{\ell} \right)$$

Hamilton-fv:

$$\cancel{H = p_x \dot{x} + p_\varphi \varphi - L}$$

$$\cancel{\begin{pmatrix} p_x \\ p_\varphi \end{pmatrix} = \dots}$$

~~itt nem érdemes kiszámolni, hosszú~~

Egyensúly körüli kis rezgések $\quad \varphi = 0 \quad$ a potenciál minimuma

ekörül sorfejtés $\quad \cos\varphi \approx 1 - \frac{\varphi^2}{2}$

$$L \approx \frac{m_1 + m_2}{2} \dot{x}^2 + \frac{1}{2} m_2 \ell^2 \dot{\varphi}^2 + m_2 \ell \dot{\varphi} \dot{x} + m_2 \ell g \left(1 - \frac{\varphi^2}{2}\right)$$

EOM: $\quad p_x \approx (m_1 + m_2)\dot{x} + m_2 \ell \dot{\varphi} \qquad \dot{p}_x \approx (m_1 + m_2)\ddot{x} + m_2 \ell \ddot{\varphi}$

$\quad p_\varphi \approx m_2 \ell^2 \dot{\varphi} + m_2 \ell \dot{x}$

$$\underbrace{\begin{pmatrix} m_1+m_2 & m_2 l \\ m_2 l & m_2 l^2 \end{pmatrix}}_{\underline{\underline{M}}} \begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ -m_2 l g \varphi \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & -m_2 l g \end{pmatrix}}_{-\underline{\underline{K}}} \begin{pmatrix} x \\ \varphi \end{pmatrix}$$

~~sötét vá~~

$$\begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} = -\underline{\underline{M}}^{-1}\underline{\underline{K}} \begin{pmatrix} x \\ \varphi \end{pmatrix} \qquad \underline{\underline{M}}^{-1} = \frac{1}{\det M} \operatorname{adj} M$$

$$\det M = (m_1+m_2) m_2 l^2 - m_2^2 l^2 = m_1 m_2 l^2$$

$$\operatorname{adj} M = \frac{1}{m_1 m_2 l^2} \begin{pmatrix} m_2 l^2 & -m_2 l \\ -m_2 l & m_1+m_2 \end{pmatrix}$$

$$\ddot{x} = 0$$

$$\ddot{\varphi} = -\frac{1}{m_1 \cancel{m_2} \cancel{l^2}} (m_1+m_2) \cancel{m_2} \cancel{l} g \varphi$$

$$\omega_1^2 = 0$$

$$\omega_2^2 = \frac{m_1+m_2}{m_1} \frac{g}{l}$$ a másik rezgési frekv.

## 2. Példa Hamilton- egyenletekre

térbeli oszcillátor $\quad L = K - V$

$$K = \frac{1}{2} m \dot{\underline{r}}^2 \qquad V = \frac{1}{2} k \underline{r}^2$$

$$L = K - V = \frac{1}{2} m \dot{\underline{r}}^2 - \frac{1}{2} k \underline{r}^2$$

Euler-Lagrange: $\quad \dfrac{\partial L}{\partial r_i} = -\dfrac{1}{2} k \cdot 2 r_i = -k r_i$

$$p_i = \frac{\partial L}{\partial \dot{r}_i} = m \dot{r}_i$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} - \frac{\partial L}{\partial r_i} = 0 \rightarrow m \ddot{r}_i + k r_i = 0 \qquad \omega^2 = k/m\text{-el}$$

$$\ddot{r}_i + \omega^2 r_i = 0$$

$$H = \underline{p}\,\underline{\dot{r}} - L \qquad \underline{\dot{r}} = \frac{\underline{p}}{m}$$

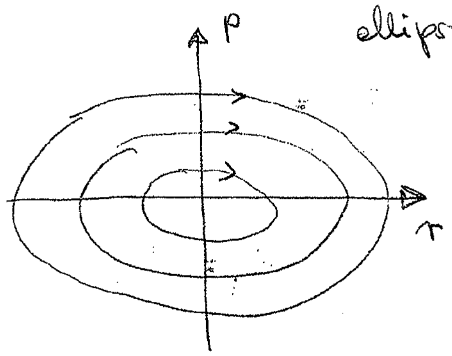
$$H = \frac{p^2}{m} - \left(\frac{1}{2} m \left(\frac{p}{m}\right)^2 - \frac{1}{2} k\, \underline{r}^2\right) = \frac{p^2}{2m} + \frac{1}{2} k r^2$$

$$= \underline{\frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2}$$

fázistér - trajektóriák:

H = const

ellipszisek




$$\underline{\dot{r}} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\underline{\dot{p}} = -\frac{\partial H}{\partial \underline{r}} = -k\,\underline{r}$$

## 3. Centrális mozgás Lagrange - függvényes tárgyalása

$$L = K - V$$

$$K = \frac{1}{2} m \underline{\dot{r}}^2$$

- síkmozgás: $\underline{r} = r \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$

$$\underline{\dot{r}} = r\dot{\varphi} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} + \dot{r} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \qquad \underline{\dot{r}}^2 = r^2 \dot{\varphi}^2 + \dot{r}^2 + \dot{r} r \dot{\varphi} \underbrace{(-\sin\varphi\cos\varphi + \cos\varphi\sin\varphi)}_{0}$$

- ezzel $K = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2)$

- a teljes Lagrange - függvény

$$L = K - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) - V(r)$$

Euler - Lagrange - egyenletek

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{d}{dt} p_r = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = m r \dot{\varphi}^2 - V'(r)$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} \qquad \frac{d}{dt} p_\varphi = 2m\dot{r}\dot{\varphi} + m r^2 \ddot{\varphi}$$

$$\frac{\partial L}{\partial \varphi} = 0 \quad \to \varphi \quad \text{ciklikus koordináta}$$

$$\boxed{p_\varphi = N_z = \text{áll.}}$$

$$\dot{\varphi} = \frac{p_\varphi}{m r^2} \qquad \dot{\varphi}^2 = \frac{p_\varphi^2}{m^2 r^4}$$

$$\to \quad \frac{\partial L}{\partial r} = m r \frac{p_\varphi^2}{m^2 r^4} - V'(r) = \frac{p_\varphi^2}{m r^3} - V'(r) = -V_{eff}'(r)$$

$$V_{eff} = V(r) + \frac{p_\varphi^2}{2 m r^2}$$

visszakaptuk, amit már egyszer levezettünk, csak sokkal egyszerűbben

Félig áttérünk Hamiltoni formalizmusba: Routh-fv.

$$R = p_\varphi \dot{\varphi} - L = \frac{p_\varphi^2}{m r^2} - \left( \frac{p_\varphi^2}{2 m r^2} + \frac{1}{2} m \dot{r}^2 - V(r) \right)$$

$$p_\varphi = m r^2 \dot{\varphi}$$

$$\dot{\varphi} = \frac{p_\varphi}{m r^2} \qquad = \frac{p_\varphi^2}{2 m r^2} - \frac{1}{2} m \dot{r}^2 + V(r)$$

$\varphi$ mozgásegyenlete Hamilton-féle

$$\dot{p}_\varphi = -\frac{\partial R}{\partial \varphi} = 0$$

$$\dot{\varphi} = \frac{\partial R}{\partial p_\varphi} = \frac{p_\varphi}{mr^2} \quad \text{valóban visszakaptuk}$$

$r$ mozgásegyenlete Lagrange-féle

$$-p_r = \frac{\partial R}{\partial \dot{r}} = -\frac{\partial L}{\partial \dot{r}} = -m\dot{r}$$

$$p_r = m\dot{r} \qquad \dot{p}_r = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = -\frac{\partial R}{\partial r} = \frac{p_\varphi^2}{mr^3} - V'(r)$$

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{r}} = \frac{\partial R}{\partial r} \qquad m\ddot{r} = \frac{p_\varphi^2}{mr^3} - V'(r)$$

<u>Hamilton-függvény</u>

$$H = \dot{r}p_r + \dot{\varphi}p_\varphi - L = \frac{p_r}{m}p_r + \frac{p_\varphi}{mr^2}p_\varphi$$

$$-\left(\frac{1}{2}m\left(\frac{p_r}{m}\right)^2 + \frac{1}{2}m\,r^2\left(\frac{p_\varphi}{mr^2}\right)^2 - V(r)\right) =$$

$$= \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + V(r) \quad = \text{a teljes energia}$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2} \qquad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \underbrace{\frac{p_\varphi^2}{mr^3} - V'(r)}_{V'_{eff}(r)}$$

tehát a Lagrange- és a Hamilton-formalizmusból is visszakapjuk a jól ismert eredményeket, de sokkal rövidebben.

## 4. Poisson-zárójelek

Hamiltoni rendszer: $q_i, p_i$ $(i=1,\dots,N)$ $H = H(q,p,t)$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

tekintsünk egy általános $f = f(q,p,t)$ függvényt!

$$\dot{f} = \frac{df}{dt} = \frac{\partial f(q,p,t)}{\partial t} + \sum_{i=1}^{N}\left[\frac{\partial f(q,p,t)}{\partial q_i}\dot{q}_i + \frac{\partial f(q,p,t)}{\partial p_i}\dot{p}_i\right]$$

$$= \frac{\partial f}{\partial t} + \sum_{i=1}^{N}\left[\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right]$$

$$= \frac{\partial f}{\partial t} + \{f, H\}$$

ahol

$$\underbrace{\{f,g\} = \sum_{i=1}^{N}\left[\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right]}_{\text{Poisson-zárójel}}$$

(Goldstein, Landauban fordítva)

A PZ-ek tulajdonságai:

1. linearitás $\{f+g, h\} = \{f, h\} + \{g, h\}$

$c \in \mathbb{R}$ $\{cf, h\} = c\{f, h\}$

2. Leibniz-szabály

$$\{f \cdot g, h\} = f\{g, h\} + \{f, h\}g$$

ez a differenciálás Leibniz-szabályából triviálisan adódik.

3. $\{f, g\} = -\{g, f\}$

4. Jacobi-azonosság

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

ehhez: $\{f_1, f_2\} = \sum_{i=1}^{N} \left( \frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q_i} \right) =$

$$= \underbrace{\sum_{i=1}^{N} \left( \frac{\partial f_2}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f_2}{\partial q_i} \frac{\partial}{\partial p_i} \right)}_{D_2} f_1$$

$$\{\{f, g\}, h\} + \underbrace{\{\{h, f\}, g\}}_{-\{\{f, h\}, g\}} = D_h D_g f - D_g D_h f$$

$$D_h = \sum_k \xi_k \frac{\partial}{\partial x_k} \qquad \xi_k = \left( \frac{\partial h}{\partial p_i}, -\frac{\partial h}{\partial q_i} \right)$$

$$D_g = \sum_k \eta_k \frac{\partial}{\partial x_k} \qquad \frac{\partial}{\partial x_k} = \left( \frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i} \right)$$

$D_h D_g f - D_g D_h f$ másodrendű deriváltak kiesnek

$$D_h D_g f - D_g D_h f = \sum_{k,l=1}^{2N} \left( \xi_k \frac{\partial \eta_l}{\partial x_k} - \eta_k \frac{\partial \xi_l}{\partial x_k} \right) \frac{\partial f}{\partial x_l}$$

így $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$

baloldalt: a másodrendű tagok kölcsönösen kiejtik egymást. De akkor ugyanígy $g, h$ második deriváltjai is → az egész eltűnik.

HF: Lie-algebra

Példa: egy koordináta függvénye:

$f = f(q_j)$ $\qquad \{f,g\} = ?$

$$\{f,g\} = \sum_{i=1}^{N} \Big( \underbrace{\frac{\partial f}{\partial q_i}}_{\substack{f'(q)\,\delta_{ij} \\ \underbrace{\phantom{x}}_{\frac{\partial q_j}{\partial q_i}}}} \frac{\partial g}{\partial p_i} - \underbrace{\frac{\partial f}{\partial p_i}}_{0} \frac{\partial g}{\partial q_i} \Big) = f'(q_j)\{q_j, g\}$$

$$\frac{\partial q_j}{\partial p_i} = 0$$

Még egy érdekesség:

$$\boxed{\begin{array}{l} \{q_i, q_j\} = 0 \\ \{p_i, p_j\} = 0 \end{array}}$$

mi. $\dfrac{\partial q_i}{\partial q_j} = \delta_{ij}$ $\quad \dfrac{\partial q_i}{\partial p_j} = 0$

$\dfrac{\partial p_i}{\partial p_j} = \delta_{ij}$ $\quad \dfrac{\partial p_i}{\partial q_j} = 0$

$$\boxed{\{q_i, p_j\}} = \sum_{l=1}^{N} \Big( \underbrace{\frac{\partial q_i}{\partial q_l} \frac{\partial p_j}{\partial p_l}}_{\delta_{il}\delta_{jl}} - \underbrace{\frac{\partial q_i}{\partial p_l} \frac{\partial p_j}{\partial q_l}}_{0} \Big) = \boxed{\delta_{ij}}$$

$$\dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}$$

## 5. Poisson-tétele



ismétlés: $f = f(q,p,t)$

$$\dot{f} = \frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}$$

időtől expliciten nem függő $\left(\frac{\partial f}{\partial t} = 0\right)$ mozgásállandó $\left(\frac{df}{dt} = 0\right)$

PZ-je H-val 0

$$\frac{df}{dt} = 0 \quad \& \quad \frac{\partial f}{\partial t} = 0 \quad \Rightarrow \quad \{f, H\} = 0$$

Tétel: ha $f, g$ 2 időtől expliciten nem függő mozgásállandó,

akkor $\{f, g\}$ is az

Biz: Jacobi-azonosság

$$\{\{f,g\},H\} = -\{\underbrace{\{g,H\}}_{0},f\} - \{\underbrace{\{H,f\}}_{0},g\}$$

általános eset:

Tétel: $\dot{f} = \dot{g} = 0 \Rightarrow \frac{d}{dt}\{f,g\} = 0$

Biz: deriváltak szorzásszabálya

$$\frac{d}{dt}\{f,g\} = \frac{\partial}{\partial t}\{f,g\} + \underbrace{\{\{f,g\},H\}}_{-\{\underbrace{\{g,H\}}_{-\frac{\partial g}{\partial t}},f\} - \{\underbrace{\{H,f\}}_{+\frac{\partial f}{\partial t}},g\}}$$

$$= \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\} + \left\{\frac{\partial g}{\partial t}, f\right\} - \left\{\frac{\partial f}{\partial t}, g\right\} = 0$$

kiesik

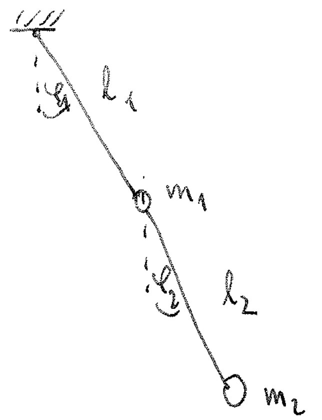
## 6. Megmaradó mennyiségek

$$\frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial H}{\partial x} = 0$$

$$p_x = \frac{\partial L}{\partial \dot{x}}$$

$$\{p_x, H\} = -\frac{\partial H}{\partial x} = 0 \qquad p_x = \text{áll.}$$

## 7. A kettősinga kis rezgései




$$K_1 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2)$$
$$2 \qquad\quad 2 \quad 2 \qquad\quad 2$$

$$x_1 = l_1 \sin\varphi_1$$
$$y_1 = l_1 \cos\varphi_1$$
$$x_1 = l_1 \sin\varphi_1 + l_2 \sin\varphi_2$$
$$y_2 = l_1 \cos\varphi_1 + l_2 \cos\varphi_2$$

deriválás, behelyettesítés

$$K_1 = \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2 \qquad V_1 = -m_1 g l_1 \cos\varphi_1$$
$$K_2 = \frac{1}{2} m_2 \left[ l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2 l_1 l_2 \cos(\varphi_1 - \varphi_2) \dot{\varphi}_1 \dot{\varphi}_2 \right)$$
$$V_2 = - m_2 g l_1 \cos\varphi_1 - m_2 g l_2 \cos\varphi_2$$

$$L = K_1 + K_2 - V_1 - V_2$$

kis rezgések a $\varphi_1 = \varphi_2 = 0$ egyensúly körül

$$K = K_1 + K_2 = \frac{1}{2} m_1 l_1 \dot{\varphi}_1^2 + \frac{1}{2} m_2 \left[ l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2 l_1 l_2 \cos(\varphi_1 - \varphi_2) \dot{\varphi}_1 \dot{\varphi}_2 \right)$$

ezt sorbafejtjük, $\varphi_1, \varphi_2, \dot{\varphi}_1, \dot{\varphi}_2$ kis mennyiségek

— lineáris tag nincs (K eleve kvadratikus)

— kvadratikus tagok: $\dot{\varphi}^2$ mindben van, a többinek a nulladrendje kell

$$\cos(\varphi_1 - \varphi_2) = 1 \quad \text{ha } \varphi_1 = \varphi_2 = 0$$

$$K = \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 \left[ l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \right]$$

ezt $\frac{1}{2} (\dot{\varphi}_1, \dot{\varphi}_2) \underline{\underline{M}} \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix}$ alakba írjuk: $\underline{\underline{M}} = \begin{pmatrix} m_1 l_1^2 + m_2 l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix}$

a potenciált is sorbafejtjük

$$V = V_1 + V_2 = -m_1 g l_1 \cos\varphi_1 - m_2 g l_1 \cos\varphi_1 - m_2 g l_2 \cos\varphi_2$$

$$\cos\varphi \approx 1 - \frac{\varphi^2}{2} + \dots$$

$$V = V_1 + V_2 \approx -m_1 g l_1 - m_2 g l_1 - m_2 g l_2$$
$$+ \frac{1}{2}(m_1 + m_2) g l_1 \frac{\varphi_1^2}{2} + \frac{1}{2} m_2 g l_2 \frac{\varphi_2^2}{2} + \dots$$

itt nincs keresztag; a kvadratikus részt

$\frac{1}{2}(\varphi_1, \varphi_2) \underline{\underline{V}} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ alakba írhatjuk

$$\underline{\underline{V}} = \begin{pmatrix} (m_1+m_2) g l_1 & \\ & m_2 g l_2 \end{pmatrix}$$

mozgásegyenlet: az $L = K - V$ variációjával

$$\underline{\underline{M}} \begin{pmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{pmatrix} = -\underline{\underline{V}} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

egy oldalra rendezve

$$-\underline{\underline{V}}^{-1} M \begin{pmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

tehát $\underline{\underline{V}}^{-1} \underline{\underline{M}}$ sajátértékei a sajátfrekvenciák reciprokénégyzetei

$$\omega_1^2 = \frac{2g}{l_1 + l_2 - \frac{1}{\sqrt{m_1+m_2}}\sqrt{(l_1-l_2)^2 m_1 + (l_1+l_2)^2 m_2}} \xrightarrow[l_1 \to 0]{} \infty$$

$$\omega_2^2 = \frac{2g}{l_1 + l_2 + \frac{1}{\sqrt{m_1+m_2}}\sqrt{(l_1-l_2)^2 m_1 + (l_1+l_2)^2 m_2}} \xrightarrow[l_2 \to 0]{} \frac{g}{l_2}$$