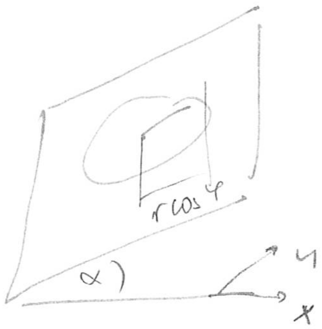
Példák Lagrange-fv-re

Körmozgás ferde lejtőn: x, y, z -vel bonyolult



$$x = r\cos\varphi\cos\alpha$$

$$y = r\sin\varphi$$

$$z = r\cos\varphi\sin\alpha$$

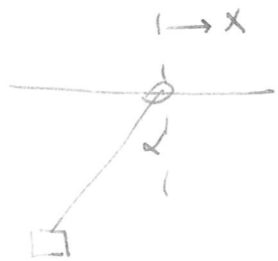
$$L = T - V = \frac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2) - mgz$$

$$= \frac{1}{2}m\left(r^2\cos^2\alpha\sin^2\varphi\,\dot\varphi^2 + r^2\cos^2\varphi\,\dot\varphi^2 + r^2\sin^2\alpha\sin^2\varphi\,\dot\varphi^2\right)$$

$$- mgr\cos\varphi\sin\alpha$$

$$L = \frac{1}{2}mr^2\dot\varphi^2 - mgr\cos\varphi\sin\alpha$$

Nem rögz. tengelyű inga



test helye

$\cancel{x + l\cos\alpha}$

$x + l\sin\alpha$

$-l\cos\alpha$

$\cancel{(\dot x + l\sin\alpha\,\dot\alpha}$

$$T = \frac{1}{2} m_2 \left[(\dot{x} + \ell \cos\varphi \dot{\varphi})^2 + \ell^2 \sin^2\varphi \dot{\varphi}) \right]$$

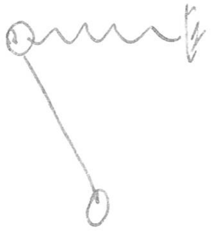
$$+ \frac{1}{2} m_1 \dot{x}^2$$

$$V = -m_2 g \ell \cos\varphi$$

$$L = T - V = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2} m_2 \left(2\dot{x}\ell\cos\varphi\dot{\varphi} + \ell^2\cos^2\varphi\dot{\varphi}^2 + \ell^2\sin^2\varphi\dot{\varphi}^2 \right)$$

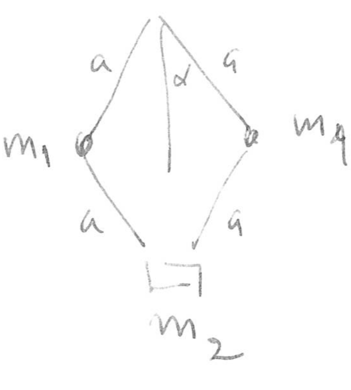
$$+ m_2 g \ell \cos\varphi$$

<u>Rugóval rögz</u>



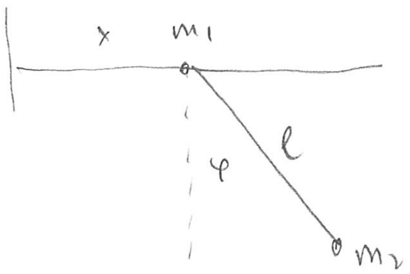
$$-\frac{1}{2} m_1 \omega^2 x^2$$

<u>Centrifugálreg</u>



$$L = m_1 a^2 (\dot{\vartheta}^2 + \omega^2 \sin^2\vartheta) + 2 m_2 a^2 \sin^2\vartheta \dot{\vartheta}^2 + 2 g a (m_1 + m_2)\cos\vartheta$$

Nem rögzített végű inga Landau I, 23.o / 2



$x_1 = x$

$y_1 = 0$

$x_2 = x + l\sin\varphi$

$y_2 = -l\cos\varphi$

$\dot{x}_1 = x \qquad \dot{y}_1 = 0 \qquad \dot{x}_2 = \dot{x} + l\cos\varphi\,\dot{\varphi} \qquad \dot{y}_2 = l\sin\varphi\,\dot{\varphi}$

$$K = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) =$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + 2\dot{x}\dot{\varphi}\,l\cos\varphi + l^2\cos^2\varphi\,\dot{\varphi}^2$$
$$+ l^2\sin^2\varphi\,\dot{\varphi}^2)$$

$$= \frac{m_1+m_2}{2}\dot{x}^2 + \frac{m_2}{2}(l^2\dot{\varphi}^2 + 2l\dot{x}\dot{\varphi}\cos\varphi)$$

$V = +mgy_2 = -mgl\cos\varphi \qquad L = K - V$

EOM:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi}$$

$$\frac{d}{dt}\left(\frac{m_1+m_2}{2}\dot{x} + m_2 l\dot{\varphi}\cos\varphi\right) = 0$$

$$\frac{m_1+m_2}{2}\ddot{x} = -m_2 l\ddot{\varphi}\cos\varphi$$

$$\frac{d}{dt}\left(m_2 l^2\dot{\varphi} + m_2 l\dot{x}\cos\varphi\right) = -m_2 gl\sin\varphi$$

$$m_2 l^2\ddot{\varphi} + m_2 l\ddot{x}\cos\varphi - m_2 l\dot{x}\dot{\varphi}\sin\varphi = -m_2 gl\sin\varphi$$

$m_1 \to \infty$ határeset:

elsőből $$\ddot{x} = -\underbrace{\frac{2m_2}{m_1+m_2}}_{\to 0} l\ddot{\varphi}\cos\varphi$$

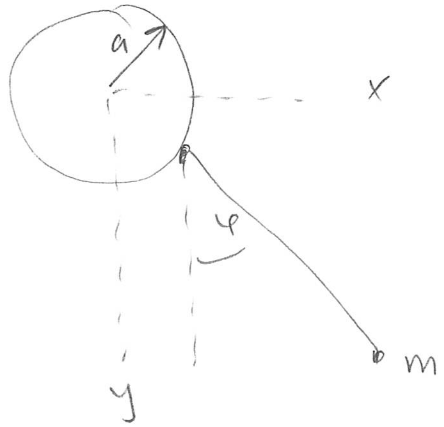
második:

$$m_2 l^2 \ddot{\varphi} = -m_2 g l \sin\varphi$$

ez épp a matematikai inga.

Függőleges körpályán állandó ω körfekv. mozgó felfüggesztési pontú inga

Landau I. 24.o. 3a



$x = a\cdot\cos(\omega t) + l\sin\varphi$

$y = -a\sin(\omega t) + l\cos\varphi$

innen:

$\dot{x} = -a\omega\sin(\omega t) + l\dot{\varphi}\cos\varphi$

$\dot{y} = -a\omega\cos(\omega t) - l\dot{\varphi}\sin\varphi$

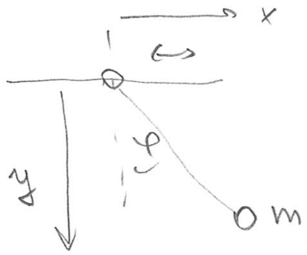
$L = K - V$

$$K = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{m}{2}\Big(a^2\omega^2 + l^2\dot{\varphi}^2 - 2a\omega l\dot{\varphi}\sin(\omega t)\cos\varphi$$
$$+ 2a\omega l\dot{\varphi}\cos(\omega t)\sin\varphi\Big)$$

$$= \frac{m}{2}(a^2\omega^2 + l^2\dot{\varphi}^2 + mla\omega^2\sin(\varphi-\omega t))$$

$V = -mgl\cos\varphi$

Vízszintesen rezgő végpontú inga   Landau 1. 24.0.3b



$$x = a\cos\omega t + l\sin\varphi$$

$$y = l\cos\varphi$$

$$V = K - V \qquad V = -mgl\cos\varphi$$

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = -a\omega\sin\omega t + l\dot{\varphi}\cos\varphi$$

$$\dot{y} = -l\dot{\varphi}\sin\varphi$$

$$K = \frac{m}{2}\left(l^2\dot{\varphi}^2 + a^2\omega^2\sin^2\omega t - 2al\omega\dot{\varphi}\sin\omega t\cos\varphi\right)$$

$$\dot{\varphi}\sin(\omega t)\cos\varphi = \ ?$$

$$\frac{d}{dt}\left(\sin(\omega t)\sin\varphi\right) = \omega\cos(\omega t)\sin\varphi + \dot{\varphi}\sin(\omega t)\cos\varphi$$

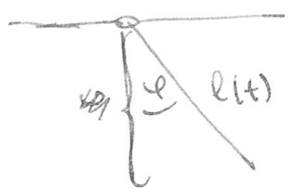
$$-2al\omega\dot{\varphi}\sin\omega t\cos\varphi = -\frac{d}{dt}\left(2al\omega\sin\omega t\sin\varphi\right)$$

$$+ 2al\omega^2\cos(\omega t)\sin\varphi$$

az első teljes derivált, L-ből elhagyható

$$L' = \frac{m}{2}\left(l^2\dot{\varphi}^2 + 2al\omega^2\cos(\omega t)\sin\varphi\right) + mgl\cos\varphi$$

Időben változó hosszúságú inga



$$x = l(t)\sin\varphi$$

$$y = l(t)\cos\varphi$$

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V$$

$$V = -mgy$$

$$\dot{x} = \dot{l}\sin\varphi + l\dot{\varphi}\cos\varphi$$

$$\dot{y} = \dot{l}\cos\varphi - l\dot{\varphi}\sin\varphi$$

$$\dot{x}^2 = \dot{l}^2\sin^2\varphi + l^2\dot{\varphi}^2\cos^2\varphi + 2l\dot{l}\dot{\varphi}\sin\varphi\cos\varphi$$

$$\dot{y}^2 = \dot{l}^2\cos^2\varphi + l^2\dot{\varphi}^2\sin^2\varphi - 2l\dot{l}\dot{\varphi}\sin\varphi\cos\varphi$$

$$L = \frac{m}{2}(\dot{l}^2 + l^2\dot{\varphi}^2) + mg\,l(t)\cos\varphi$$

mozgásegyenletek:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi}$$

$$\frac{\partial L}{\partial \dot{\varphi}} = ml^2\dot{\varphi} \qquad \frac{\partial L}{\partial \varphi} = -mgl\sin\varphi$$

$$ml^2\ddot{\varphi} = -mgl\sin\varphi$$

$$+2ml\dot{l}\dot{\varphi}$$

Potenciálos erők 1D-ban

$$m\ddot{x} = F(x) \qquad (\text{és nem } F(x,\dot{x},\dots,t))$$

ekkor legyen $V(x)$ olyan, hogy $F(x) = -V'(x)$

$$m\ddot{x} = -V'(x) \qquad \Big/ \; \dot{x}(t)$$

$$\underbrace{m\dot{x}\ddot{x}} = \underbrace{-V'(x)\dot{x}}$$

összetett fv. deriváltja: $\frac{d}{dt}V(x(t)) = V'(x(t))\,\dot{x}(t)$

$$\frac{d}{dt}\dot{x}^2 = 2\dot{x}\ddot{x}$$

$$\frac{1}{2}\frac{d}{dt}m\dot{x}^2 = -\frac{d}{dt}V(x(t))$$

azaz $$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + V(x)\right) = 0$$

$$\frac{1}{2}m\dot{x}^2 + V(x) = E = \text{áll.}$$

"a mozgásegyenlet első integrálja"

MJ: $\cancel{H = \frac{\partial L}{\partial \dot{y}}\dot{y}}$

$$H = \frac{\partial L}{\partial \dot{x}}\dot{x} - L \qquad \text{Beltrami-f. } (-1)\text{-szerese}$$

$\rightarrow$ időben állandó $H = E$

hogy lehet ezt használni?

$$\frac{1}{2} m \dot{x}^2 = E - V(x)$$

$$|\dot{x}| = \sqrt{\frac{2}{m}(E-V(x))}$$

adott helyen <u>meghatározható a sebesség nagysága</u>

$$\dot{x} = \pm\sqrt{\frac{2}{m}(E-V(x))}$$

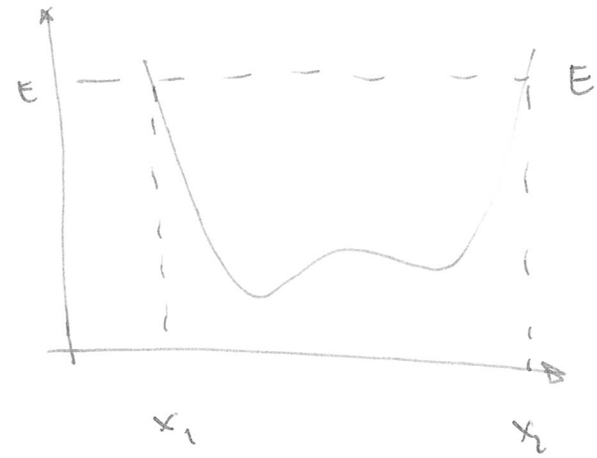
ha a mozgás egy szakasza során végig ki tudjuk találni az előjelet:

$$\dot{x} = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}(E-V(x))}$$

$$\pm \frac{dx}{\sqrt{\frac{2}{m}(E-V(x))}} = dt$$

$$\pm \int_{x_0}^{x(t)} \frac{dx}{\sqrt{\frac{2}{m}(E-V(x))}} = t - t_0$$

Periódusidő számolása



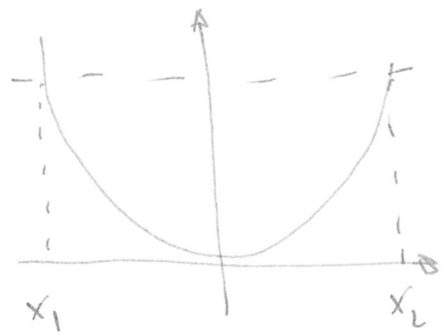
x_1, x_2: megfordulási pontok

$K = \frac{1}{2} m \dot{x}^2 = 0$

$\boxed{V(x_{1,2}) = E}$

$$T = 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2}{m}\left(E - V(x)\right)}}$$

alkalmazás harmonikus oszcillátorra:



$V(x) = \frac{1}{2} k x^2$

végpontok: $E - V(x_{1,2}) = 0$

$E - \frac{1}{2} k x_{1,2}^2 = 0$

megoldásai: $x_{1,2} = \mp\sqrt{\frac{2E}{k}}$

periódusidő-számolás

$$T = 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{\left(E - V(x)\right)\frac{2}{m}}} = 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2}{m}\left(E - \underbrace{V(x)}_{\frac{1}{2}kx^2}\right)}}$$

$x = \sqrt{\frac{2E}{k}}\,\xi \qquad dx = \sqrt{\frac{2E}{k}}\,d\xi$

$$T = 2\sqrt{\frac{2E}{k}} \int_{-1}^{1} \frac{d\xi}{\sqrt{\underbrace{\frac{2}{m}\left(E - \frac{1}{2}k^2\frac{E}{k}\xi^2\right)}_{\frac{2E}{m}(1-\xi^2)}}} = 2\sqrt{\frac{m}{k}} \int_{-1}^{1} \frac{d\xi}{\sqrt{1-\xi^2}}$$

$$= 2\sqrt{\frac{m}{k}}\Big[\arcsin \xi\Big]_{-1}^{1} = 2\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{\omega}$$

$$\omega^2 = \frac{k}{m}$$

Visszakaptuk az ismert formulát, de:

A MOZGÁSEGYENLET

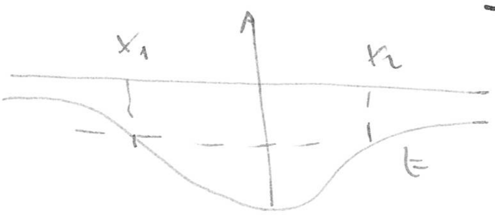
MEGOLDÁSA

NÉLKÜL!

Ez fontos: más potenciálok esetén sokszor nagyon nehéz, vagy lehetetlen a mozgásegyenletet zárt alakban (képlettel) megoldani.

$1/\mathrm{ch}^2$ potenciál

$$V(x) = -\frac{V_0}{\mathrm{ch}^2 \alpha x}$$



$$-V < E < 0$$

$$x_{1,2} = \mp x_0$$

$$x_0 = \frac{1}{\alpha} \operatorname{arch} \sqrt{-\frac{V_0}{E}}$$

$$T = \sqrt{2m} \int_{-x_0}^{x_0} \frac{dx}{\sqrt{E - V(x)}} = 2\sqrt{2m} \int_0^{x_0} \frac{dx}{\sqrt{E + \frac{V_0}{\mathrm{ch}^2 \alpha x}}} =$$

$$= \sqrt{\frac{8M}{V_0}} \int_0^{x_0} \frac{dx}{\sqrt{\frac{E}{V_0} + \frac{1}{\mathrm{ch}^2 \alpha x}}} =$$

$y = \alpha x$

$$= \sqrt{\frac{8M}{V_0}} \int_0^{x_0} \frac{dx}{\sqrt{\frac{1}{\mathrm{ch}^2 \alpha x} - \frac{1}{\mathrm{ch}^2 \alpha x_0}}} = \sqrt{\frac{8M}{V_0}} \frac{1}{\alpha} \int_0^{y_0} \frac{dy}{\sqrt{\frac{1}{\mathrm{ch}^2 y} - \frac{1}{\mathrm{ch}^2 y_0}}}$$

$z = \mathrm{ch}\, y$

$y = \alpha x$

legyen $z = \mathrm{ch}\, y$ $\qquad z_0 = \mathrm{ch}\, y_0 = \sqrt{-\frac{V_0}{E}}$

$$dz = \mathrm{sh}\, y\, dy \qquad dy = \frac{dz}{\mathrm{sh}\, y} = \frac{dz}{\sqrt{z^2 - 1}}$$

ui. $\mathrm{ch}^2 y - \mathrm{sh}^2 y = 1$

$$T = \sqrt{\frac{8M}{V_0}}\ \frac{1}{\alpha} \int_1^{z_0} \frac{dz}{\sqrt{z^2-1}}\ \frac{1}{\sqrt{\frac{1}{z^2} - \frac{1}{z_0^2}}} =$$

$$= \sqrt{\frac{8M}{V_0}}\ \frac{1}{\alpha} \int_1^{z_0} \frac{dz}{\sqrt{z^2-1}}\ \frac{z_0}{\sqrt{\frac{z_0^2}{z^2} - 1}} =$$

$$= \sqrt{\frac{8M}{V_0}}\ \frac{1}{\alpha} \int_1^{z_0} \frac{dz}{\sqrt{z^2-1}}\ \frac{z_0 z}{\sqrt{z_0^2 - z^2}}$$

$u = z^2$

$$u = z^2 \qquad du = 2z\,dz$$

$w = u-1$

$$= \sqrt{\frac{2M}{V_0}}\ \frac{1}{\alpha} \int_1^{z_0^2} \frac{z_0\, du}{\sqrt{u-1}\ \sqrt{z_0^2-u}}$$

$$w = u-1 \qquad dw = du$$

$$= \sqrt{\frac{2M}{V_0}}\ \frac{z_0}{\alpha} \int_0^{z_0^2-1} \frac{dw}{\sqrt{w}\ \sqrt{\underbrace{z_0^2-1}_{w_0^2}-w}} = \sqrt{\frac{-2M}{E}}\ \frac{1}{\alpha} \int_0^{w_0^2} \frac{dw}{\sqrt{w}\ \sqrt{w_0^2-w}}$$

$s = \sqrt{w}$

$$s = \sqrt{w} \qquad ds = \frac{1}{2\sqrt{w}}\, dw = \frac{1}{2s}\, dw \qquad dw = 2s\,ds$$

$$T = \sqrt{-\frac{2M}{E}}\ \frac{1}{\alpha} \int_0^{w} \frac{2\cancel{s}\, ds}{\cancel{s}\sqrt{w_0^2 - s^2}}$$ alapintegrál! ld Bronstejn

$$T = \sqrt{-\frac{2M}{E}}\ \frac{1}{\alpha} \left[\arcsin \frac{s}{w_0}\right]_0^{w_0} = \sqrt{-\frac{2M}{E}}\ \frac{1}{\alpha}\, \pi = \frac{2\pi}{\alpha} \sqrt{\frac{M}{2|E|}}$$

másik megoldás: változótranszformációval

harmonikus oszcillátorrá alakítjuk:

$$\frac{1}{2} m \dot{x}^2 - \frac{V_0}{ch^2 \alpha x} = E$$

$$y = sh\, \alpha x \qquad dy = ch(\alpha x)\, \alpha\, dx$$

$$\frac{dx}{dt} = \frac{dy}{dt} \frac{1}{\alpha\, ch(\alpha x)}$$

$$\frac{m}{2} (\dot{y})^2 \frac{1}{\alpha^2 ch^2 \alpha x} - \frac{V_0}{ch^2 \alpha x} = E$$

$$\frac{m}{2\alpha^2} \dot{y}^2 - V_0 = E\, ch^2(\alpha x) = E + E\, sh^2 \alpha x = E + E y^2$$

$$\frac{m}{2\alpha^2} \dot{y}^2 + |E|\, y^2 = V_0 - |E|$$

$$\frac{m}{2} \dot{y}^2 + \underbrace{|E| \alpha^2}_{k = \frac{1}{2} m \omega^2} y^2 = \alpha^2 (V_0 - |E|)$$

$$\omega^2 = \frac{m}{2|E|\alpha^2}$$