Rugós rendszerek L-fves tárgyalása

$$L = K - V$$

ált. koordináták: $\Delta l, \varphi$

Δl : rugó nyúlása (nyugalmi helyzethez képest)

φ : függőlegessel bezárt szög

$$x = (l + \Delta l)\cos\varphi$$

$$y = (l + \Delta l)\sin\varphi$$

$$\dot{x} = \Delta\dot{l}\cos\varphi - (l+\Delta l)\dot{\varphi}\sin\varphi$$

$$\dot{y} = \Delta\dot{l}\sin\varphi + (l+\Delta l)\dot{\varphi}\cos\varphi$$

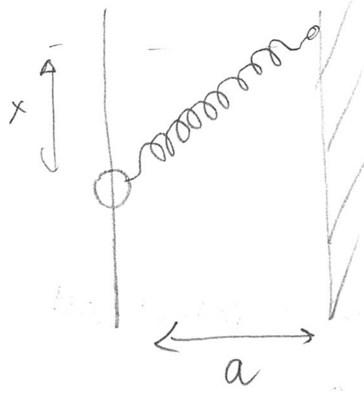
$$\dot{x}^2 = (\Delta\dot{l})^2\cos^2\varphi + (l+\Delta l)^2\dot{\varphi}^2\sin^2\varphi - 2(l+\Delta l)\Delta\dot{l}\dot{\varphi}\cos\varphi\sin\varphi$$

$$\dot{y}^2 = (\Delta\dot{l})^2\sin^2\varphi + (l+\Delta l)^2\dot{\varphi}^2\cos^2\varphi + 2(l+\Delta l)\Delta\dot{l}\dot{\varphi}\sin\varphi\cos\varphi$$

$$K = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{m}{2}\left((\Delta\dot{l})^2 + (l+\Delta l)^2\dot{\varphi}^2\right)$$

$$V = \frac{1}{2}k\Delta l^2 - mgx = -mg(l+\Delta l)\cos\varphi + \frac{1}{2}k\Delta l^2$$

$$L = K - V = \frac{m}{2}\left[(\Delta\dot{l})^2 + (l+\Delta l)^2\dot{\varphi}^2\right] + mg(l+\Delta l)\cos\varphi - \frac{1}{2}k\Delta l^2$$



általános koordináta: x

$$L = K - V = \frac{1}{2} m \dot{x}^2 - V$$

$$V = -mgx + \frac{k}{2}\left(\sqrt{x^2+a^2} - \ell\right)^2$$

$$L = K - V = \frac{m}{2}\dot{x}^2 + mgx - \frac{k}{2}\left(\sqrt{x^2+a^2} - \ell\right)^2$$

egyensúlyi helyzet: $V'(x_0) = 0$

$$\cancel{V'(x_0) = mg - \frac{k}{2}\left(\frac{x}{\sqrt{x^2+a^2}}\right.}$$

$$V'(x_0) = mg - k\left(\sqrt{x^2+a^2} - \ell\right)\frac{x}{\sqrt{x^2+a^2}}$$

$$= mg - k\left(1 - \frac{\ell}{\sqrt{x^2+a^2}}\right)x \stackrel{!}{=} 0$$

$$mg = k\left(1 - \frac{\ell}{\sqrt{x^2+a^2}}\right)x$$

$$mg - kx = -\frac{kx\ell}{\sqrt{x^2+a^2}} \qquad |\,()^2$$

$$(mg - kx)^2 = \frac{k^2x^2\ell^2}{x^2+a^2}$$

$$(mg - kx)^2(x^2+a^2) = k^2x^2\ell^2 \rightarrow x_0 = \dots$$

<u>Kis rezgések frekvenciája</u> $\quad x = x_0 + \delta x$

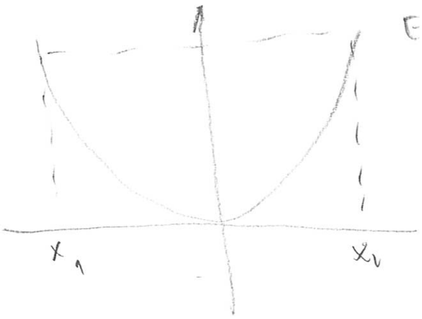
$$L \approx \frac{1}{2} mg (\delta\dot{x})^2 - V''(x_0)\,\delta x^2 - \underbrace{V(x_0)}_{\text{elhanyagolható}}$$

$$\omega = \sqrt{\frac{V''(x_0)}{m}}$$

Mozgás hatványfv-potenciálban

$$V(x) = b|x|^\beta \qquad \text{megáll: } E = b x_0^\beta \qquad (\beta > 0)$$

$$T = 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}} = 2\int_{-x_0}^{x_0} \frac{dx}{\sqrt{\frac{2}{m}(b x_0^\beta - b|x|^\beta)}} =$$



$$= 4\sqrt{\frac{m}{2}} \int_0^{x_0} \frac{dx}{\sqrt{b x_0^\beta - \underbrace{b|x|^\beta}_{b x^\beta}}}$$

legyen $x = x_0 \xi$

$$x_0 = (E/b)^{1/\beta} \qquad dx = x_0 \, d\xi$$

$$T = 4\sqrt{\frac{m}{2}} \, \frac{x_0}{\sqrt{b}} \, x_0^{-\beta/2} \underbrace{\int_0^1 \frac{d\xi}{\sqrt{1-\xi^\beta}}}_{\text{numerikusan kiszámolható integrál}}$$

$$x_0^{-\beta/2} = \left(\frac{E}{b}\right)^{-1/2}$$

$$T = \sqrt{\frac{8m}{b}} \left(\frac{E}{b}\right)^{1/\beta - \frac{1}{2}} \int_0^1 \frac{d\xi}{\sqrt{1-\xi^\beta}}$$

$$\int_0^1 (1-x^\mu)^{-1/\nu} \, dx = \frac{1}{\mu} B\left(\frac{1}{\mu}, 1-\frac{1}{\nu}\right)$$

$$\mu = \beta, \quad \nu = 2 \qquad \frac{1}{2} B\left(\frac{1}{2}, 1-\frac{1}{\beta}\right) =$$

Gradshteyn & Ryzhik

$$= \frac{\Gamma(1/2)\,\Gamma(1-1/\beta)}{\Gamma(3/2 - 1/\beta)}$$

Table of Integrals, Series and products

Fordított probléma : $U(x) = ?$

$$T(E) = a\, E^{\alpha}$$

(előzőből: $U(x) = b\,|x|^{\beta}$ $\quad$ $\alpha = \frac{1}{\beta} - \frac{1}{2}$)

Előadáson szerepelt:

$\alpha = \frac{1}{\beta} - \frac{1}{2}$

$$x_2(U) - x_1(U) = \frac{1}{\pi\sqrt{2m}} \int_0^U \frac{T(E)\,dE}{\sqrt{U-E}}$$

ide ~~behelyettesítünk~~

$$\int_0^U \frac{a\cdot E^{\alpha}}{\sqrt{U - aE^{\alpha}}}\,dE = \sqrt{a}\int_0^U \frac{E^{\alpha}\,dE}{\sqrt{U/a - E^{\alpha}}}$$

$E = U e$

$dE = U\,de$

$$= \sqrt{a}\,U \int_0^1 \frac{U^{\alpha} e^{\alpha}\,de}{\sqrt{U/a - U^{\alpha}e^{\alpha}}}$$

$$\int_0^U \frac{T(E)\,dE}{\sqrt{U-E}} = \int_0^U \frac{a\cdot E^{\alpha}\,dE}{\sqrt{U-E}}$$

legyen $E = U\cdot e$

$dE = U\,de$

$$= \int_0^1 \frac{a U^{\alpha} U e^{\alpha}\,de}{\sqrt{U - Ue}} =$$

$$= a\cdot U^{\alpha+1/2} \int_0^1 \frac{e^{\alpha}\,de}{\sqrt{1-e}}$$

és kaptunk egy összefüggést két numerikus integrál között is ! ✓

$x(U) \propto U^{\alpha+1/2}$

$x^{\frac{1}{\alpha+1/2}} \propto U$

$\beta = \frac{1}{\alpha + 1/2}$

$\frac{1}{\beta} = \alpha + \frac{1}{2}$

A $V(x) = -V_2 x^2 + V_4 x^4$ potenciál

Egyensúlyok: $V'(x) = 0$

$$V'(x) = -2V_2 x + 4V_4 x^3$$

$$= x(4V_4 x^2 - 2V_2)$$

$V_2 > 0$

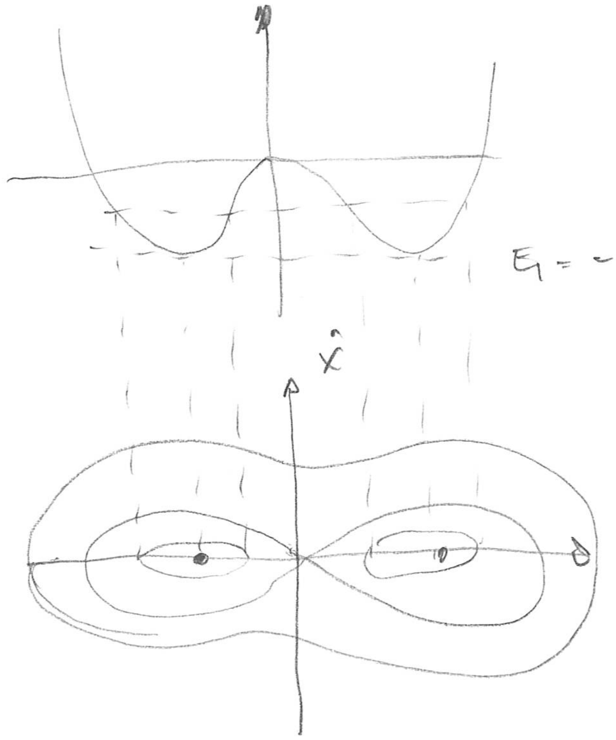
$$x_0 = 0$$

$$x_{1,2} = \pm \sqrt{V_2 / 2V_4}$$

egyensúlyok stabilitása: fázistérkép rajzolásával

$$V(x_0) = 0$$

$$V(x_{1,2}) = -V_2 \cdot \frac{V_2}{2V_4} + V_4 \frac{V_2^2}{4V_4^2} = \frac{V_2^2}{2V_4} \underbrace{\left(-1 + \frac{1}{2}\right)}_{-1/2} = -\frac{V_2^2}{4V_4}$$



$\frac{1}{2}\dot{x}^2 + V(x) = E$ lerajzolása

$$\dot{x} = \pm\sqrt{\frac{2}{m}(E - V)}$$

$$V(x) = -V_0 + V_4 (x^2 - x_0^2)^2$$

$x_{1,2}$ stabil

x_0 instab

rezg. frekv: $V(x) \approx -V_0 + \frac{1}{2}\overbrace{V''(x_1)}^{-V_2}(x - x_1)^2 + \ldots$

$$\omega^2 = \frac{2|V_2|}{m}$$

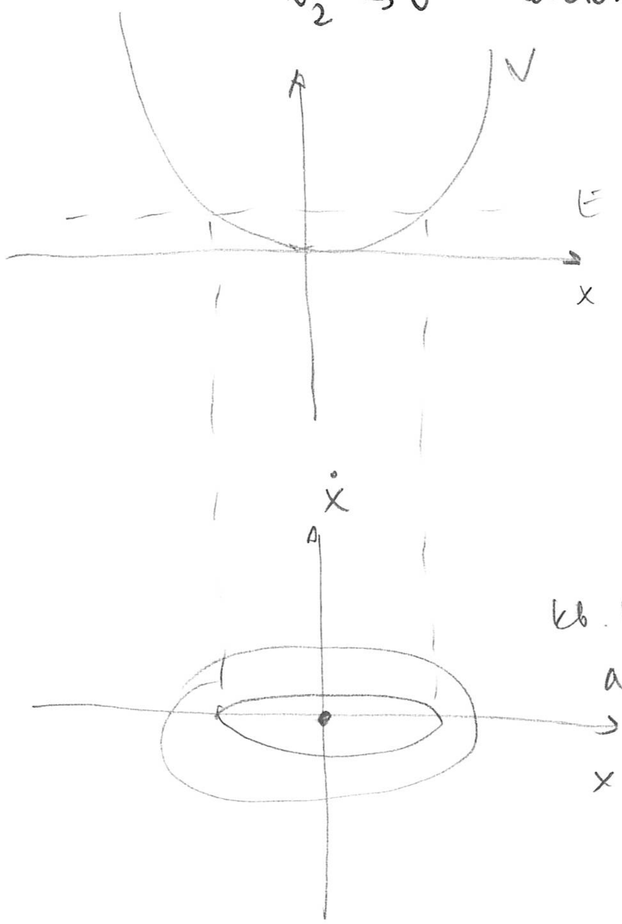
$V_0 \to 0$ $\omega \to 0$ $T = \frac{2\pi}{\omega} \to \infty$

$V_2 < 0$: egy valós gyök $x = 0$

ott a potenciál $V(x) \approx |V_2| x^2 + \dots$

$$\omega = \sqrt{\frac{2|V_2|}{m}}$$

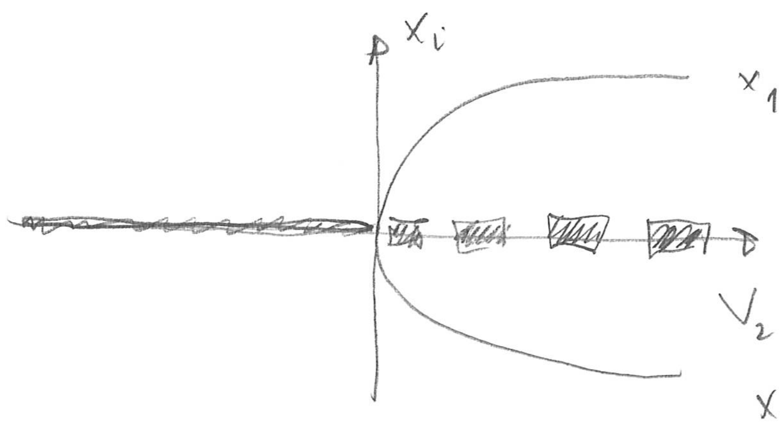
$V_2 \to 0$ esetén itt is $\omega \to 0$ $\quad T = \frac{2\pi}{\omega} \to \infty$



kb. mint a harmonikus oszcillátornál

Egyensúlyok: ——— stabil

- - - - - instabil



$$x_1 = \sqrt{\frac{V_2}{2V_4}}$$

$$x_2 = -\sqrt{\frac{V_2}{2V_4}}$$

Fourier-transzformáció

adott $g(t)$ függvény összerakása $e^{i\omega t}$-kből

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega)\, e^{i\omega t}\, d\omega$$

akkor teljesül, ha

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, dt$$

a Fourier-tf. néhány tulajdonsága:

$$\dot{g}(t) = \frac{d}{dt} \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega)\, e^{i\omega t}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega)\, i\omega\, e^{i\omega t}\, d\omega$$

$$\boxed{\tilde{\dot{g}}(\omega) = i\omega\, \tilde{g}(\omega)}$$

konvolúció:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-t')\, g(t')\, dt'$$

mi ennek a Fourier-transzformáltja?

$$\widetilde{(f * g)}(\omega) = \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \int_{-\infty}^{\infty} dt'\, f(t-t')\, g(t')\, dt'$$

$$= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \; e^{-i\omega t} \; f(t-t') \; g(t') dt'$$

integrálhelyettesítés:

$\tau = t - t'$

$d\tau = dt$

$$= \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} dt' \; e^{-i\omega(t'+\tau)} \; f(\tau) \; g(t') dt'$$

$$= \int_{-\infty}^{\infty} e^{-i\omega\tau} \; f(\tau) d\tau \; \int_{-\infty} e^{-i\omega t'} \; g(t') dt'$$

$$= \tilde{f}(\omega) \; \tilde{g}(\omega)$$

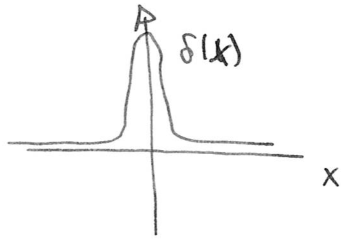
azaz

$$\widetilde{f * g} = \tilde{f} \cdot \tilde{g}$$

konvolúció F.T.-je = F.T.-ak szorzata

A Dirac-delta:

$\delta(t)$ elképzelése:



tulajdonságai:

- $t \neq 0$ -ra $\delta(t) = 0$
- $t = 0$ $\delta(0) = \infty$
- $$\int_{-\infty}^{\infty} dt\, \delta(t) = 1$$

sőt:

$$\boxed{\int_{-\infty}^{\infty} f(t)\, \delta(t)\, dt = f(0)}$$

tetszőleges $f(t)$ folyt. függvényre

Mi a δ-fv. Fourier-trf-ja

$$\boxed{\tilde{\delta}(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t}\, dt = 1}$$

de akkor hogy értsük az inverzet?

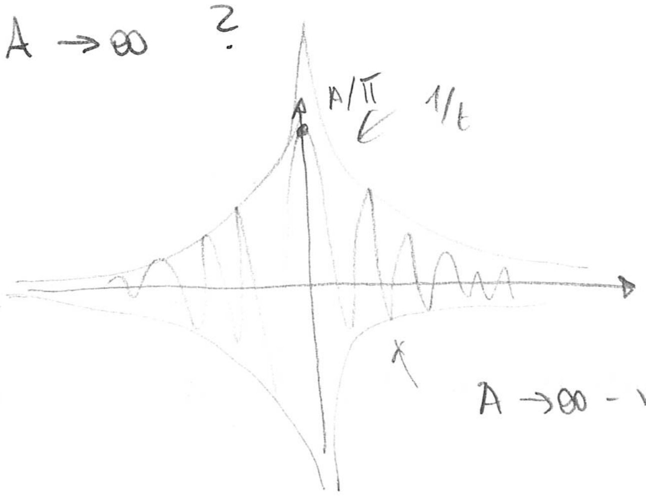
$$\text{"} \quad \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t}\, d\omega \quad \text{"}$$

regularizáció:

$$\delta_A(t) = \frac{1}{2\pi}\int_{-A}^{A} e^{i\omega t}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{i\omega t}}{it}\right]_{-A}^{A} = \frac{e^{iAt} - e^{-iAt}}{2\pi\, it}$$

$$= \frac{\sin(At)}{\pi t}$$

a milyen értelemben tart a Dirac-deltához, ha $A \to \infty$?



$$\lim_{t\to 0} \frac{\sin(At)}{t} = \lim_{t\to 0} \frac{A\cos(At)}{1} = A$$

$A \to \infty$ - re egyre gyorsabban oszcillál

$$\int_{-\infty}^{\infty} f(t) \frac{\sin(At)}{\pi t}\, dt = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)-f(0)}{t} \sin(At)\, dt$$

$$+ \underbrace{\frac{\varphi(0)}{\pi}\int_{-\infty}^{\infty} \frac{\sin(At)}{t}\, dt}_{\varphi(0)}$$

$$\int_{-\infty}^{\infty} \frac{f(t)-f(0)}{t} \sin(At)\, dt = 0$$

"kioszcillálja" magát

(Riemann-lemma)

$\tau = At$

$d\tau = A dt$

hasonlóképpen:

~~fla~~ $$\delta_\varepsilon(t) = \frac{1}{\sqrt{2\pi}\,\varepsilon} e^{-\frac{t^2}{2\varepsilon^2}}$$

$$\delta_\varepsilon(t) \xrightarrow[\varepsilon \to 0]{} \delta(t)$$

milyen értelemben?

$$\int_{-\infty}^{\infty} f(t)\,\delta_\varepsilon(t)\,dt \to f(0)$$

A Dirac-δ néhány tulajdonsága

def. volt: $$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = 0$$

$$\int_{-\infty}^{\infty} f(t+t')\,\delta(t')\,dt' = f(t)$$

$$\parallel$$

$$\int_{-\infty}^{\infty} f(t-t')\,\delta(t')\,dt'$$

$$\boxed{f * \delta = f}$$

Megjegyzés: $\int_{-\infty}^{\infty} e^{-x^2} dx = ?$

$$\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy$$

$$= \int_{-\infty}^{\infty} e^{-r^2} r\, dr\, d\varphi = 2\pi \int_{-\infty}^{\infty} e^{-r^2} r\, dr$$

$$e^{-r^2} r = -\frac{1}{2} \frac{d}{dr} e^{-r^2}$$

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) = -\cancel{2}\pi \left[e^{-r^2} \right]_0^{\infty} = \pi$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Gauss-görbe Fourier-transzformáltja:

$$g(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-i\omega t} e^{-t^2/2\sigma^2} dt = ?$$

alakítsuk teljes négyzetté!

$$\frac{t^2}{2\sigma^2} + i\omega t + \dots = \frac{1}{2\sigma^2} (t + i\sigma^2 \omega)^2$$

$$\frac{1}{2\sigma^2}(t+i\sigma^2\omega)^2 = \frac{t^2}{2\sigma^2} + i\omega t - \frac{\sigma^2\omega^2}{2}$$

$$e^{-\frac{t^2}{2\sigma^2} - i\omega t} = e^{-\frac{1}{2\sigma^2}(t+i\sigma^2\omega)^2} \, e^{-\frac{\sigma^2\omega^2}{2}}$$

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}\underbrace{(t+i\sigma^2\omega)^2}_{\tau^2}} e^{-\frac{\sigma^2\omega^2}{2}} dt$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{\sigma^2\omega^2}{2}} \int_{-\infty + i\sigma^2\omega}^{\infty + i\sigma\omega} e^{-\frac{\tau^2}{2\sigma^2}} d\tau$$

$$= e^{-\frac{\sigma^2\omega^2}{2}}$$

A Dirac-δ még néhány tulajdonsága

$$\int_{-\infty}^{\infty} f(t)\,\delta(at)\,dt = \int_{-\infty}^{\infty} f\left(\frac{\tau}{a}\right)\delta(\tau)\frac{d\tau}{a} = \frac{1}{a} f(0)$$

$$\tau = at \qquad d\tau = a\,dt$$

$$\delta(at) = \frac{1}{a}\delta(t)$$

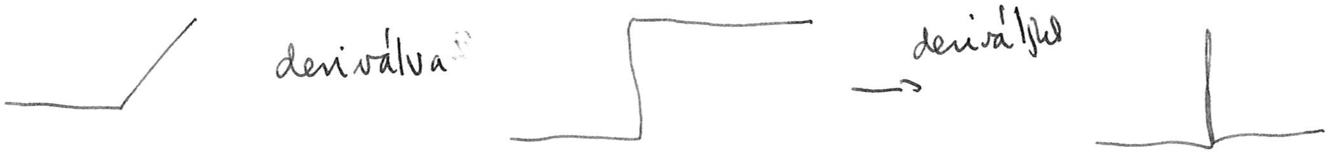
hasonlóan

$$\delta(g(t)) = \sum_{t_i : g(t_i)=0} \frac{\delta(t-t_i)}{|g'(t_i)|}$$

és $\delta'(t) = ?$

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = -\int_{-\infty}^{\infty} f'(t)\,\delta(t)\,dt = -f'(0)$$

másrészt:



$f(t)$ $\Theta(t) = H(t)$ Heaviside $\delta(t)$

innen: $i\omega \tilde{f}(\omega) = \tilde{\Theta}(\omega)$ $i\omega \tilde{\Theta}(\omega) = \tilde{\delta}(\omega) = 1$

Mi lehet ennek Fourier-transzformáltja?

$$i\omega \tilde{\Theta}(\omega) = 1$$

$$\Rightarrow \tilde{\Theta}(\omega) = \frac{1}{i\omega} + \underbrace{(\ldots \text{valami} \ldots)}_{\substack{\text{at } i\omega\text{-val szorozva} \\ 0\text{-t kapunk} \\ \delta(\omega),\ \delta'(\omega), \ldots \text{ ilyenek}}}$$

$$\tilde{\Theta}(\omega) = -\frac{i}{\omega} + \pi\delta(\omega)$$

stb.

Periódusidő perturbációszámítás x^3 potenciálra

$$\mathcal{U}(x) = \frac{k}{2}x^2 + \varepsilon v(x) \qquad \text{(itt majd } v(x) = x^3\text{)}$$

$$E = \mathcal{U}(A_1) = \mathcal{U}(-A_2)$$

$$T = T_1 + T_2 \qquad T_1 = \sqrt{2m}\int_0^{A_1}\frac{dx}{\sqrt{E-\mathcal{U}(x)}}$$

$$T_2 = \sqrt{2m}\int_0^{A_2}\frac{dx}{\sqrt{E-\mathcal{U}(-x)}}$$

$$E = \frac{k}{2}A_1^2 + \varepsilon v(A_1)$$

$$E = \frac{k}{2}A_2^2 + \varepsilon v(A_2)$$

$$T_1(E) = \sqrt{2m}\int_0^{A_1}\frac{dx}{\sqrt{E-V(x)}} = \sqrt{2m}\int_0^{A_1}\frac{dx}{\sqrt{\frac{k}{2}(A_1^2-x^2)+\varepsilon(v(A_1)-v(x))}}$$

helyettesítünk: $\quad x = A_1\sin u$

$$dx = A_1\cos u\, du$$

$$A_1^2 - x^2 = A_1^2(1-\sin^2 u) = \cos^2 u$$

$$T_1(E) = \sqrt{2m}\int_0^{\pi/2}\frac{A_1\cos u\, du}{\sqrt{\frac{kA_1^2}{2}\cos^2 u + \varepsilon(v(A_1)-v(A_1\sin u))}}$$

$$= \underbrace{2\sqrt{\frac{m}{k}}}_{\frac{2}{\omega}}\int_0^{\pi/2}\frac{du}{\sqrt{1+\frac{2\varepsilon}{kA_1^2}\frac{v(A_1)-v(A_1\sin u)}{\cos^2 u}}}$$

sorbafejtjük $(1+y)^{-1/2} = 1 - \frac{y}{2} + \dots \; O(y^2)$

$$T_1(E) \approx \frac{2}{\omega} \int_0^{\pi/2} \left[1 - \frac{\varepsilon}{kA_1^2} \frac{v(A_1) - v(A_1 \sin u)}{\cos^2 u} \right] du \quad + O(\varepsilon^2)$$

az első részét $\int$ kiintegráljuk

$$T_1(E) = \frac{\pi}{\omega} - \frac{2\varepsilon}{\omega k A_1^2} \int_0^{\pi/2} \frac{v(A_1) - v(A_1 \sin u)}{\cos^2 u} du$$

és $A_1 = A_0 + O(\varepsilon)$

$$A_0 = \sqrt{\frac{2E}{k}}$$

$$T_1(E) \approx \frac{\pi}{\omega} + \underbrace{\frac{2\varepsilon}{\omega k A_0^2}}_{\frac{\varepsilon}{\omega E}} \int_0^{\pi/2} \frac{v(+A_0 \sin u) - v(+A_0)}{\cos^2 u} du$$

$$T_2(E) \approx \frac{\pi}{\omega} + \frac{\varepsilon}{\omega E} \int_0^{\pi/2} \frac{v(-A_0 \sin u) - v(-A_0)}{\cos^2 u} du$$

konkrétan $v(x) = x^3$

$$T_1 = \frac{\pi}{\omega} + \frac{2\varepsilon}{\omega k A_0^2} \int_0^{\pi/2} \frac{A_0^3 (\sin^3 u - 1)}{\cos^2 u} du$$

$$T_2 = \frac{\pi}{\omega} + \frac{2\varepsilon}{\omega k A_0^2} \int_0^{\pi/2} \frac{A_0^3 (1 - \sin^3 u)}{\cos^2 u} du \qquad \Bigg| \; \int \frac{1}{\cos^2 u} du = \operatorname{tg} u$$

$$-\int \frac{\sin^3 u}{\cos^2 u} du = -\int \sin u \operatorname{tg}^2 u \, du = -\int \underbrace{\sin u \, du}_{-dt} \left(\frac{1}{\cos^2 u} - 1 \right) =$$

$$\cos u = t$$

$$-\sin u \, du = dt$$

$$= \int \left(\frac{1}{t^2} - 1 \right) dt = -\frac{1}{t} - t \qquad \text{és } t = \cos u \qquad = -\frac{1}{\cos u} - \cos u$$

$$\cancel{\left[\frac{1}{t^2} \right.}$$

$$\int \frac{1-\sin^3 u}{\cos^2 u}\, du = \operatorname{tg} u - \cos u - \frac{1}{\cos u}$$

$$\Big[\ldots \Big]_0^{\pi/2} = 8$$

$$T_{1,2} = \frac{\pi}{\omega} \left(1 \mp \frac{4 \varepsilon A_0^2}{\pi k} \right) + \mathcal{O}(\varepsilon^2)$$

$$T = T_1 + T_2 = \frac{2\pi}{\omega} + \mathcal{O}(\varepsilon^2)$$

nincs ε-ban lineáris korrekció