# A ZH-feladatok megoldása

## 1. feladat

$$\ddot{x} + \omega_0^2 x = f \qquad f(t) = \begin{cases} f_0 \sin^2(\omega_0 t) & 0 < t < \frac{\pi}{\omega_0} \\ 0 & \text{egyébként} \end{cases}$$

$$G(t,t') = \frac{\Theta(t-t')}{\omega_0} \sin(\omega_0 t)$$

a partikuláris megoldás

$$x(t) = \int_{-\infty}^{\infty} f(t')\, G(t-t')\, dt' = \int_{-\infty}^{\infty} \frac{\Theta(t-t')}{\omega_0} \sin(\omega_0(t-t')) \cdot$$

$$\cdot\, \Theta(t')\, \Theta\left(\frac{\pi}{\omega_0} - t'\right) f_0 \sin^3(\omega_0 t')\, dt'$$

- az integrál alsó határa 0 lehet, mert $t' < 0$-ra $f(t'<0) = 0$
- a felső határ: $t' > t$-re a Green-fv. eltűnik
  $t' > \frac{\pi}{\omega_0}$-ra $f\left(t' > \frac{\pi}{\omega_0}\right) = 0$

tehát a felső határ $\min\left(t, \frac{\pi}{\omega_0}\right)$

legyen először $t < \frac{\pi}{\omega_0}$

$$x(t) = \int_0^t \frac{f_0}{\omega_0} \sin[\omega_0(t-t')] \sin^2(\omega_0 t')\, dt'$$

$$= \frac{f_0}{\omega_0} \int_0^t \left(\sin(\omega_0 t)\cos(\omega_0 t') - \cos(\omega_0 t)\sin(\omega_0 t')\right) \sin^2(\omega_0 t')\, dt'$$

$$= \frac{f_0}{\omega_0} \Big[ \sin(\omega_0 t) \int_0^t \cos(\omega_0 t') \sin^2(\omega_0 t')\, dt' -$$

$$- \cos(\omega_0 t) \int_0^t \sin^3(\omega_0 t')\, dt' \Big]$$

$$= \frac{f_0}{w_0}\left\{ \sin(w_0 t)\left[\frac{\sin^3(w_0 t)}{3 w_0}\right]_0^t - \frac{\cos(w_0 t)}{w_0}\left[-\cos(w_0 t) + \frac{1}{3}\cos^3 w_0 t\right]_0^{t}\right\}$$

$$= \frac{4 f_0}{3 w_0^2} \sin^4\left(\frac{w_0 t}{2}\right)$$

$$\frac{1}{w_0}\int \sin^3\alpha \, d\alpha = \frac{1}{w_0}\int (1-\cos^2\alpha)\underbrace{\sin\alpha \, d\alpha}_{-d(\cos\alpha)} = \frac{1}{w_0}\left(-\cos\alpha + \frac{1}{3}\cos^3\alpha\right)$$

$\alpha = w_0 t$

$d\alpha = w_0 dt$

hely.

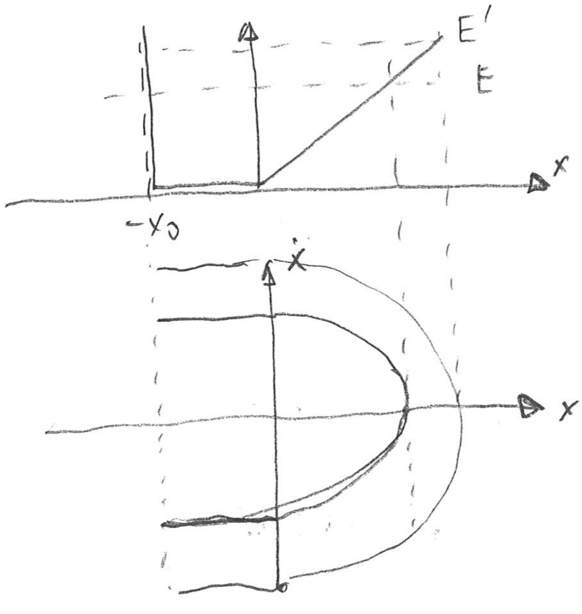
$t > t'$-re a fent integrált $\frac{\pi}{w_0}$-ig kell tekinteni

$$x(t) = -\frac{4}{3}\frac{f_0}{w_0^2}\cos(w_0 t)$$

[ $\sin(w_0 t')$, $\sin(2w_0 t')$, ... $t' = \frac{\pi}{w_0}$-ban eltűnnek ]

## 3. Periódusidő energiafüggése

$$V(x) = \begin{cases} \infty & x < -x_0 \\ 0 & -x_0 < x < 0 \\ ax & x > 0 \end{cases}$$




fordulópontok: $V(x_i) = E$

$$x_1 = -x_0$$

$$x_2 = \frac{E}{a}$$

a periódusidő innen az ismert formulával számolható

$$T = \sqrt{2m}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E-V(x)}} = \sqrt{2m}\int_{x_1}^{0}\frac{dx}{\sqrt{E}} + \sqrt{2m}\int_{0}^{x_2}\frac{dx}{\sqrt{E-ax}}$$

$$= \underbrace{\sqrt{\frac{2m}{E}}}_{1/v} x_0 + \sqrt{2m}\int_0^{E/a}\frac{dx}{\sqrt{E-ax}} =$$

(itt áll. a seb.)

$$x = \frac{E}{a}\xi$$

$$dx = \frac{E}{a}d\xi$$

$$= \sqrt{\frac{2m}{E}}x_0 + \sqrt{2m}\,\frac{E}{a}\int_0^1\frac{d\xi}{\sqrt{E}\sqrt{1-\xi}}$$

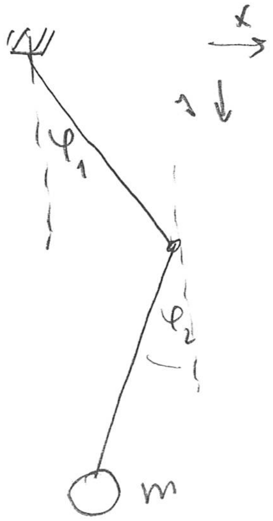
$$\int\frac{1}{\sqrt{1-\xi}}d\xi = -2\sqrt{1-\xi}$$

$$\left[-2\sqrt{1-\xi}\right]_0^1 = 2$$

tehát

$$T = \frac{x_0}{\sqrt{\frac{E}{2m}}} + 2\frac{\sqrt{2mE}}{a} = \frac{x_0}{\sqrt{\frac{E}{2m}}} + \frac{\sqrt{8mE}}{a}$$

2. Síkinga körpályán mozgó felf. ponttal




általános koord.: a függőlegessel bezárt szögek

$$x = l_1 \sin\varphi_1 + l_2 \sin\varphi_2$$

$$y = l_1 \cos\varphi_1 + l_2 \cos\varphi_2$$

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + mgy$$

$$\dot{x}^2 = \left(l_1 \dot{\varphi}_1 \cos\varphi_1 + l_2 \dot{\varphi}_2 \cos\varphi_2\right)^2$$

$$\dot{y}^2 = \left(-l_1 \dot{\varphi}_1 \sin\varphi_1 - l_2 \dot{\varphi}_2 \sin\varphi_2\right)^2$$

ezzel

$$L = \frac{m}{2}\left(l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \overbrace{\left(\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2\right)}^{\cos(\varphi_1 - \varphi_2)}\right)$$

$$- mg\left(l_1 \cos\varphi_1 + l_2 \cos\varphi_2\right)$$

a $\varphi_1, \varphi_2$ koordináták egyike sem ciklikus $\quad \frac{\partial L}{\partial \varphi_i} \neq 0$

ugyanakkor $\frac{\partial L}{\partial t} = 0$, az energia megmarad

$$\frac{\partial L}{\partial \dot{\varphi}_1} = m l_1 \dot{\varphi}_1 + m l_1 l_2 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \qquad \frac{\partial L}{\partial \varphi_1} = -m l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) + m g l_1 \sin\varphi_1$$

Mozgásegyenlet:

$$m l_1 \ddot{\varphi}_1 + m l_1 l_2 \ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) - m l_1 l_2 \dot{\varphi}_2 \sin(\varphi_1 - \varphi_2)(\dot{\varphi}_1 - \dot{\varphi}_2)$$

$$+ m l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) - m g l_1 \sin\varphi_1 = 0$$

hasonlóan:

$$\frac{\partial L}{\partial \dot\varphi_2} = m l_2^2 \dot\varphi_2 + m l_1 l_2 \dot\varphi_1 \cos(\varphi_1-\varphi_2)$$

$$\frac{\partial L}{\partial \varphi_2} = + m l_1 l_2 \dot\varphi_1 \dot\varphi_2 \sin(\varphi_1-\varphi_2) + m g l_2 \sin\varphi_2$$

Mozgásegyenlet

$$m l_2^2 \ddot\varphi_2 + m l_1 l_2 \ddot\varphi_1 \cos(\varphi_1-\varphi_2) + m l_1 l_2 \dot\varphi_1 \cancel{\dot\varphi_2 \sin}(\varphi_1-\varphi_2)$$

$$- m l_1 l_2 \dot\varphi_1^2 \sin(\varphi_1-\varphi_2)$$

$$- m l_1 l_2 \dot\varphi_1 \dot\varphi_2 \cancel{\sin}(\varphi_1-\varphi_2) - m g l_2 \sin\varphi_2$$

## 4. Kényszermozgás

$y = f(x)$ alakú pályán, grav. erőtérben


a) Lagrange-fv

$$L = \frac{m}{2}(\dot x^2 + \dot y^2) + m g y = \frac{m}{2}(1+f'^2)\dot x^2 + m g f(x)$$

$$y = f(x) \qquad \dot y = f'(x)\dot x$$

mozgásegyenlet:

$$\frac{\partial L}{\partial \dot x} = m(1+f'^2)\dot x \qquad \frac{\partial L}{\partial x} = m \dot x^2 f' f'' + m g f'$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot x} = m(1+f'^2)\ddot x + 2m f' f'' \dot x^2$$

$$\underline{m(1+f'^2)\ddot x + m f' f'' \dot x^2 - m g f'(x) = 0}$$

## b.) Dissipációs függvény

Sebességgel arányos súrlódás dissipációs fv-e:

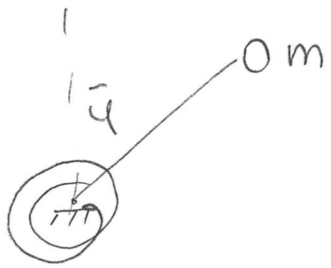
$$R=\frac{1}{2}\gamma(\dot{x}^2+\dot{y}^2) = \frac{1}{2}\gamma(1+f'^2)\dot{x}^2$$

$$\frac{\partial R}{\partial \dot{x}} = \gamma(1+f'^2)\dot{x}$$

ezzel a mozgásegyenlet $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = -\frac{\partial R}{\partial \dot{x}}$, azaz

$$m(1+f'^2)\ddot{x} + m f' f'' \dot{x}^2 - mg f'(x) = -\gamma(1+f'^2)\dot{x}$$

## 5. Metronóm




forgatónyomaték $M=-k\varphi$

tárolt energia: $V_r=\frac{1}{2}k\varphi^2$

potenciál: $V=V_r+V_g$

sebesség: $l\dot{\varphi}$ $\quad V_g=mgy=mgl\cos\varphi$

(természetesen $k>0$ visszatérítő erő)

a.) $$L=\frac{1}{2}ml^2\dot{\varphi}^2 - \frac{1}{2}k\varphi^2 - mgl\cos\varphi$$

Euler-Lagrange-egyenlet: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0$

$$ml^2\ddot{\varphi} + k\varphi - mgl\sin\varphi = 0$$

b.) $\varphi=0$ egyensúly stabilitása: $V=\frac{1}{2}k\varphi^2+mgl\cos\varphi$

$$V'(\varphi)=k\varphi-mgl\sin\varphi \qquad V''(\varphi)=k-mgl\cos\varphi$$

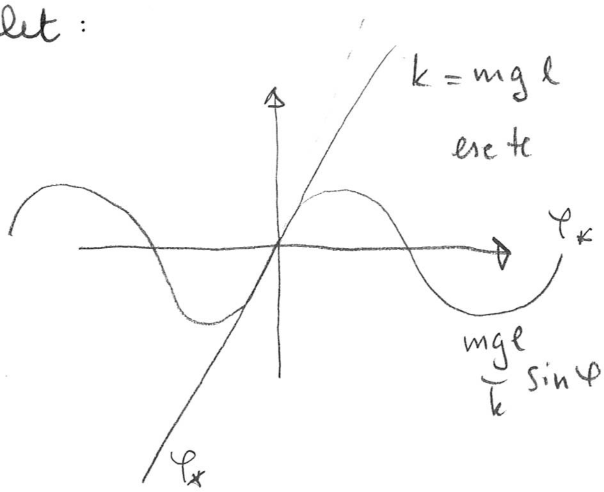
$V'(0)=0$ teljesül, ez tényleg egyensúly $\quad V''(0)=k-mgl$

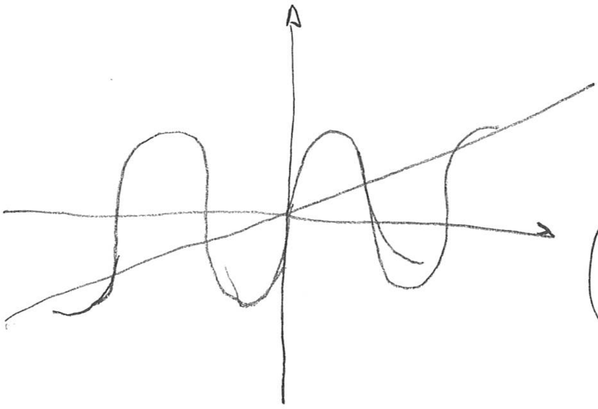
ha $V''(0)=k-mgl>0$ akkor stabil

egyéb egyensúlypkat meghatározó egyenlet:

$$V'(\varphi_*) = 0$$

$$\varphi_* = \frac{mgl}{k} \sin\varphi$$



mgl > k eset
(pont amikor
$\varphi_* = 0$ instabil)

c.) $k_c = mgl$ $\quad k < k_c \quad (k-k_c) \ll k_c$ eset

a potenciált sorbafejtjük

$$V(\varphi) = \frac{1}{2} k\varphi^2 + \underbrace{mgl}_{k_c} \cos\varphi$$

$$\approx \frac{1}{2} k\varphi^2 + k_c\left(1 - \frac{\varphi^2}{2} + \frac{\varphi^4}{24}\right)$$

$$= k_c - \frac{1}{2}(k_c - k)\varphi^2 + \frac{k_c \varphi^4}{24}$$

$$= \frac{k_c}{24}(\varphi^2 - \varphi_*^2)^2 + \text{konst.}$$

$$\underbrace{-2 \cdot \frac{k_c}{24}}_{k_c/12} \cancel{\varphi^2} \varphi_*^2 \overset{!}{=} -\frac{1}{2}(k_c - k)\cancel{\varphi^2}$$

$$\varphi_*^2 = 6\,\frac{k_c - k}{k_c} \qquad \varphi_* = \pm\sqrt{6\,\frac{k_c - k}{k_c}}$$

stabilitás:

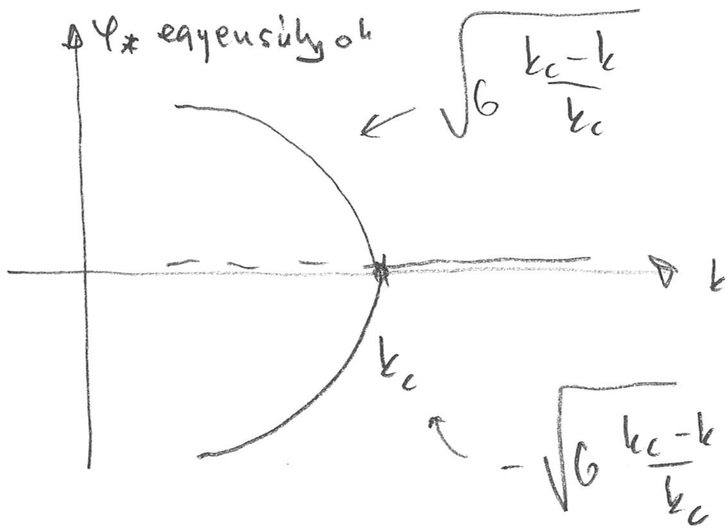
$$V(\varphi) = \frac{k_c}{24}(\varphi^2 - \varphi_*^2)^2 + \text{konst.}$$

$$V'(\varphi) = \frac{k_c}{6}(\varphi^2 - \varphi_*^2)\varphi \qquad V'(\varphi_*) = 0$$

$$V''(\varphi) = \frac{k_c}{6}(3\varphi^2 - \varphi_*^2) \qquad V''(\varphi_*) = \frac{k_c}{3}\varphi_*^2 > 0$$

$\rightarrow$ ezek az egyensúlyok stabilak

bifurkációs diagram



# 6. 2D harmonikus oscillátor

a.) L-f, mozgásegyenletek

$$L = K - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} k (x^2 + y^2)$$

mozgásegyenletek:

$$m\ddot{x} = -kx$$

$$m\ddot{y} = -ky$$

(két független egyenlet!) sem x, sem y nem ciklikus

áttérés polárkoordinátákra:

$$x = r\cos\vartheta$$

$$y = r\sin\vartheta$$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{1}{2} k r^2$$

$$\frac{\partial L}{\partial \varphi} = 0 \quad \varphi \text{ ciklikus koord,} \quad N = p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} = \text{áll.}$$

(ez épp az impulzusmomentum)

$$m\ddot{r} - m r \dot{\varphi}^2 - kr = 0$$

b.) Sebességgel arányos közegellenállás

$$R = \frac{\gamma}{2} (\dot{x}^2 + \dot{y}^2) = \frac{\gamma}{2} (\dot{r}^2 + r^2\dot{\varphi}^2)$$

$$m\ddot{x} = -kx - \gamma\dot{x}$$

$$m\ddot{y} = -ky - \gamma\dot{y}$$

$$m r^2 \ddot{\varphi} + 2 m r \dot{r} \dot{\varphi} = -\gamma r^2 \dot{\varphi}$$

$$m\ddot{r} - m r \dot{\varphi}^2 - kr = -\gamma \dot{r}$$