Form Factors in 2D Integrable Models

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based on work with Tristan McLoughlin

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What's a Form Factor?

A Form Factor is a matrix element of an operator in the basis of scattering states

\[ \langle k_1 k_2 \ldots | \mathcal{O}(x) | p_1 p_2 \ldots \rangle \]

and, thus, is a hybrid of

- a correlation function (off-shell) \[ \langle 0 | \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \ldots | 0 \rangle \]

- an S-matrix element (on-shell) \[ \langle k_1 k_2 \ldots | p_1 p_2 \ldots \rangle \]

E.g. QED vertex correction

\[ \langle k | J^\mu(0) | p \rangle = -ie\bar{u}(k) \left[ \gamma^\mu F_1(k - p) + \frac{i\sigma^{\mu\nu}(k - p)_\nu}{2m} F_2(k - p) \right] u(p) \]

\[ J^\mu \rightarrow -ie\bar{\psi}\gamma^\mu \psi \]

Dirac Form Factor

Pauli Form Factor

\[ \approx \text{Fourier transform of charge distribution} \]

„Form of the particle“
Why Form Factors?

... at a workshop for three-point correlation functions?

Indeed, correlation functions can in principle be built from form factors:

\[
\langle 0 | \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) | 0 \rangle_{\text{gauge theory}}
\]

\[
\sum_n \int \prod_i dp_i |p_1...p_n\rangle \langle p_1...p_n|
\]

Although this procedure is much too complicated in general, people do work on **Target Space Form Factors**

at weak coupling


[Boels, Bork, Brandhuber, Engelund, Gehrmann, Gurdogan, Henn, Huber, Kazakov, Kniehl, Loebbert, Moch, Naculich, Nandan, Penante, Roiban, Sieg, Spence, Tarasov, Travaglini, Vartanov, Wilhelm, Yang, ... 2011+]

at strong coupling

[Maldacena, Zhiboedov 2010] [Gao, Yang 2013]
Why Form Factors?

... at a workshop for three-point correlation functions?

In this talk, we are interested in World Sheet Form Factors:

[TK, McLoughlin 2012, 2013]

which may eventually be used to build correlation functions of vertex operators:

[\langle 0 | \mathcal{V}_1(\sigma_1) \mathcal{V}_2(\sigma_2) \mathcal{V}_3(\sigma_3) | 0 \rangle \text{ string theory}]

Operator product expansion coefficients of Heavy-Heavy-Light correlation functions are equal to diagonal world sheet form factors.

[Bajnok, Janik, Wereszczynski 2014]

[Hollo, Jiang, Petrovskii 2015]
Bootstrap Program

- Circumvent perturbation theory

- Write general properties of **S-Matrix** and **Form Factors** using...
  - Symmetry (e.g. Lorentz symmetry, Flavor symmetry, integrability)
  - Unitarity (total probability = 1)
  - Inspiration from Feynman integrals —> Analyticity, Crossing

- Find functions which possess these properties

- Write correlation functions as sums of products of form factors
**S-Matrix in 2D models**

In the Hilbert-space of states, we can choose a basis in terms of **In-states** and **Out-states**.

- **In-states** \( |p_1, p_2\rangle^{(\text{in})} \)
- **Out-states** \( |k_1, k_2\rangle^{(\text{out})} \)

The S-matrix mediates between the two bases

\[ |p_1, p_2, \ldots\rangle^{(\text{in})} = |k_1, k_2, \ldots\rangle^{(\text{out})} S(p_1, p_2, \ldots, k_1, k_2, \ldots) \]

At asymptotic times, the in- and out-states are Fock states

\[ a^{\dagger (\text{in})}(p_1) a^{\dagger (\text{in})}(p_2) |0\rangle^{(\text{in})} \quad \text{or} \quad \quad a^{\dagger (\text{out})}(k_1) a^{\dagger (\text{out})}(k_2) |0\rangle^{(\text{out})} \]

In general, very non-trivial relation.
In integrable models, the in- and out-states are related in a simple way:

Let \( p_1 > p_2 > \ldots > p_n \), then one can write \[^{[Zamolodchikov 1977]}\]

\[
|p_1, p_2, \ldots, p_n\rangle^{(\text{out})} = A^\dagger(p_n) \ldots A^\dagger(p_1) |\Omega\rangle
\]

\[
|p_1, p_2, \ldots, p_n\rangle^{(\text{in})} = A^\dagger(p_1) \ldots A^\dagger(p_n) |\Omega\rangle
\]

with \[
A^\dagger(p_1) A^\dagger(p_2) = A^\dagger(p_2) A^\dagger(p_1) S(p_1, p_2)
\] “Zamolodchikov-algebra”

This is cooked up such that \[
|p_1, p_2\rangle^{(\text{in})} = |p_1, p_2\rangle^{(\text{out})} S(p_1, p_2)
\]
S-Matrix in 2D integrable models

Even if the interacting oscillators, $A^\dagger$, cannot be constructed explicitly, the Zamolodchikov algebra can be used to “derive” relations for $S(p_1, p_2)$

- **Unitarity**
  \[
  A^\dagger(p_1) A^\dagger(p_2) = \underbrace{A^\dagger(p_2) A^\dagger(p_1)} S(p_1, p_2)
  \]
  \[
  = A^\dagger(p_1) A^\dagger(p_2) S(p_2, p_1)
  \]
  \[
  \Rightarrow S(p_2, p_1) S(p_1, p_2) = \mathbb{1}
  \]

- **Yang-Baxter equation**
  \[
  123 \rightarrow 213 \rightarrow 231 \rightarrow 321
  \]
  \[
  A^\dagger(p_1) A^\dagger(p_2) A^\dagger(p_3) \quad \underbrace{123 \rightarrow 132 \rightarrow 312 \rightarrow 321}
  \]
  \[
  \Rightarrow S(p_1, p_2) S(p_1, p_3) S(p_2, p_3) = S(p_2, p_3) S(p_1, p_3) S(p_1, p_2)
  \]
S-Matrix in 2D integrable models

In the previous discussion, flavor indices were ignored.

In a theory with different particle species, we would have to write

\[ |p_1, p_2, \ldots \rangle^{(\text{in})}_{ij \ldots} = |k_1, k_2, \ldots \rangle^{(\text{out})}_{mn \ldots} S^{mn \ldots}_{ij \ldots} (p_1, p_2, \ldots, k_1, k_2, \ldots) \]

and

\[ A^\dagger_i (p_1) A^\dagger_j (p_2) = A^\dagger_m (p_2) A^\dagger_n (p_1) S^{mn}_{ij} (p_1, p_2) \]

or more specifically, e.g., for O(N)-type indices

\[ A^\dagger_i (p_1) A^\dagger_j (p_2) = A^\dagger_j (p_2) A^\dagger_i (p_1) S_T (p_1, p_2) \quad \text{(transmission)} \]

\[ + A^\dagger_i (p_2) A^\dagger_j (p_1) S_R (p_1, p_2) \quad \text{(reflection)} \]

\[ + \delta_{ij} A^\dagger_k (p_2) A^\dagger_k (p_1) S_A (p_1, p_2) \quad \text{(annihilation)} \]
The Analytic S-Matrix

In a relativistic theory, the two-particle S-matrix is a function of the Mandelstam variables

\[ s = (p_1 + p_2)^2 \geq 4m^2 \quad t = (p_1 - p_3)^2 \leq 0 \quad u = (p_1 - p_4)^2 \leq 0 \]

In 2D, only one of \( \{s, t, u\} \) is independent. Let that be \( s \).

1. Consider the process
   \[ A + B \longrightarrow A + B \]
   Say, you find \( S_{AB}(s_{AB} \geq 4m^2) \)

2. Now, consider the process
   \[ A + \bar{B} \longrightarrow A + \bar{B} \]
   Say, now you find \( S_{A\bar{B}}(s_{A\bar{B}} \geq 4m^2) \)

In both cases, you will have drawn the same Feynman diagrams and solved the same integrals --- although in different kinematical regions.

As a consequence, we have the crossing relation

\[ S_{A\bar{B}}(s_{A\bar{B}} \geq 4m^2) = S_{AB}(t_{A\bar{B}} \leq 0) \]
The Analytic S-Matrix

There is ONE S-matrix for ALL channels

\[ S_{AB}(s) \]

\[
\begin{align*}
A + \bar{B} & \rightarrow A + \bar{B} \\
\bar{A} + B & \rightarrow \bar{A} + B
\end{align*}
\]
The Rapidity Plane

The relativistic dispersion relation $E^2 = m^2 + p^2$

can be uniformized by introducing the rapidity $\theta$ as

$$E = m \cosh \theta \quad p = m \sinh \theta$$

For two particles, we obtain

$$s = (p_1 + p_2)^2 = 2m^2(1 + \cosh \theta_{12})$$

where

$$\theta_{12} = \theta_1 - \theta_2$$

Now, the crossing relation reads

$$S_{12}(\theta) = S_{2\bar{1}}(i\pi - \theta) = S_{\bar{2}1}(i\pi - \theta) = S_{1\bar{2}}(\theta)$$
S-Matrix Properties

- Unitarity
  \[ S_{12}(\theta)S_{21}(-\theta) = \mathbb{1} \]

- Hermitian analyticity
  \[ S_{12}^\dagger(\theta) = S_{21}(-\theta) \]

- Crossing symmetry
  \[ S_{12}(\theta) = S_{21}(i\pi - \theta) = S_{21}(i\pi - \theta) \]

- Yang-Baxter equation
  \[ S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12} \]

- Bound states
  \[ S_{B3}^B\varphi_{12}^B = \varphi_{12}^B S_{13}S_{23} \]
Exact Relativistic S-Matrices

By solving the previous list of properties under the assumption of certain symmetries, various exact S-matrices were found.

- $O(2) = U(1)$  
  [Zamolodchikov 1977]  
  [Karowski, Thun, Truong, Weisz 1977]

  $\Rightarrow$ Sine-Gordon / Massive Thirring

- $U(n>1)$  
  [Berg, Karowski, Kurak, Weisz 1978]

- $O(n>2)$  
  [Zamolodchikov, Zamolodchikov 1978]

  $\Rightarrow$ $O(n)$ chiral field (minimal sol.) / Gross-Neveu (non-min. sol.)

Use perturbation theory to confirm that the constructed S-matrix is really the S-matrix of the model to which is has been associated.
World-Sheet and Spin-Chain S-Matrix

The world-sheet theory and the spin-chain model are non-relativistic.

The dispersion relation is \( E^2 = 1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2} \)

The S-matrix is a function of two variables \( S(p_1, p_2) \), not just one \( S(\theta_1 - \theta_2) \)

Q: Is there a non-relativistic crossing relation? How does crossing act?

A: Introduce Rapidity Torus

\[
E = \operatorname{dn}(z, -\frac{\lambda}{\pi^2})
\]
\[
\sin \frac{p}{2} = \operatorname{sn}(z, -\frac{\lambda}{\pi^2})
\]

Then, crossing acts as \( z \mapsto z \pm \omega_2 \quad (E \mapsto -E, \quad p \mapsto -p) \)
World-Sheet and Spin-Chain S-Matrix

- **Unitarity**
  \[ S_{12}(z_1, z_2)S_{21}(z_2, z_1) = 1 \]

- **Hermitian analyticity**
  \[ [S_{12}(z_1, z_2)]^\dagger = S_{21}(z_2^*, z_1^*) \]

- **Crossing symmetry**
  \[ S_{12}(z_1 + \omega_2, z_2) = S_{12}(z_1, z_2 - \omega_2) = [S_{12}(z_1, z_2)]^{-1} \]  
  \[ \text{[Janik, 2006]} \]

- **Yang-Baxter**
  \[ S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12} \]
For momenta $p_1 > p_2 > \ldots > p_n$ define the Form Factor as

$$F(\theta_{ij}) = \langle \Omega | \mathcal{O}(0) | p_1, \ldots, p_n \rangle^{(in)}$$

and consider it as a function of all pairs $\theta_{ij} = \theta_i - \theta_j > 0$ with $i < j$ where $p = m \sinh \theta$

Then define the Form Factor $F(\theta_{ij})$ by analytic extension for complex $\theta_{ij}$
Analytic properties:

- CPT invariance \((+i\varepsilon \leftrightarrow -i\varepsilon)\)

\[
\langle \Omega | \mathcal{O} | p_1, \ldots, p_n \rangle^{(\text{out})} = F(-\theta_{ij}^*) = F(-\theta_{ij})
\]

- Crossing symmetry

\[
^{(\text{out})} \langle p_1, \ldots, p_m | \mathcal{O} | p_{m+1}, \ldots, p_n \rangle^{(\text{in})} = F(\theta_{ij}, i\pi - \theta_{kr}, \theta_{st})
\]

\[
\psi_i, j, k \quad \psi_r, s, t
\]
Watson’s Equations

Let’s focus on the two-particle case and compute...

\[ F(\theta_{12}) = \langle 0 | \mathcal{O} | p_1, p_2 \rangle^{(\text{in})} = \sum_n \langle 0 | \mathcal{O} | n \rangle^{(\text{out})} \langle n | p_1, p_2 \rangle^{(\text{in})} = F(-\theta_{12}) S(\theta_{12}) \]

\[ F(i\pi - \theta_{12}) = \langle p_1 | \mathcal{O} | p_2 \rangle^{(\text{in})} = \sum_{n,m} (\langle p_1 | n \rangle^{(i)(i)} \langle n | \mathcal{O} | m \rangle^{(o)(o)} \langle m | p_2 \rangle^{(i)} = F(i\pi + \theta_{12}) \]
Solving Watson’s Equations

We are looking for a solution of \( F(\theta) = F(-\theta)S(\theta) = F(\theta + 2\pi i)S(\theta) \)

assuming that \( F(\theta) \)

- is meromorphic in the physical strip
- has poles only along the imaginary axis
- falls off sufficiently rapidly for \( |\text{Re}\ \theta| \)

Factorize \( F(\theta) = NK(\theta)F_{\text{min}}(\theta) \) with \( F_{\text{min}}(\theta) \) analytic in physical strip.

1. By Cauchy’s theorem for a contour around the strip \( 0 \leq \text{Im}\ \theta \leq 2\pi \)

\[
\ln F_{\text{min}}(\theta) = \int_{C} \frac{dz}{4\pi i} \coth \frac{z - \theta}{2} \ln F_{\text{min}}(z) = \int_{-\infty}^{\infty} \frac{dz}{4\pi i} \coth \frac{z - \theta}{2} \ln \frac{F_{\text{min}}(z)}{F_{\text{min}}(z + 2\pi i)} S(z)
\]

\( \Rightarrow \) Integral formula for the Form Factor in terms of the S-matrix!

2. see Tristan’s talk...
Auxiliary Form Factor Function

Definition: The auxiliary form factor function is defined as

\[ f(\theta_1, \ldots, \theta_n) = F(\theta_{ij}) = \langle \Omega | \mathcal{O}(0) | p_1, \ldots, p_n \rangle^{\text{in}} \quad \text{for} \quad \theta_1 > \ldots > \theta_n \]

and by analytic continuation for all other orderings.

Theorem: From LSZ reduction formalism and maximal analyticity, it follows:

- Permutation \( f_{\ldots ij\ldots}(\ldots, \theta_i, \theta_j, \ldots) = f_{\ldots ji\ldots}(\ldots, \theta_j, \theta_i, \ldots) S_{ij}(\theta_i - \theta_j) \)
- Periodicity \( f_{12\ldots n}(\theta_1, \theta_2, \ldots, \theta_n) = f_{2\ldots n1}(\theta_2, \ldots, \theta_n, \theta_1 - 2\pi i) \)
- One-particle pole

\[
\text{Res}_{\theta_{12}=i\pi} f_{12\ldots n}(\theta_1, \theta_2, \ldots, \theta_n) = 2iC_{12} f_{3\ldots n}(\theta_3, \ldots, \theta_n) \left(1 - S_{2n} \cdots S_{23}\right)
\]
- Bound states

\[
\text{Res}_{\theta_{12}=\theta_B} f_{12\ldots n}(\theta_1, \theta_2, \ldots, \theta_n) = f_{B\ldots n}(\theta_B, \theta_3, \ldots, \theta_n) \sqrt{2i} \text{Res}_{\theta_{12}=\theta_B} S_{12}
\]
Worldsheet Form Factors

- Permutation

\[
f_{...ij...}(\cdots, z_i, z_j, \cdots) = f_{...ji...}(\cdots, z_j, z_i, \cdots)S_{ij}(z_i, z_j)
\]

- Periodicity

\[
f_{12...n}(z_1, z_2, \cdots, z_n) = f_{2...n1}(z_2, \cdots, z_n, z_1 - 2\omega_2)
\]

- One-particle pole

\[
\text{Res}_{p_{12}=0} f_{1...n}(z_1, z_2, \cdots, z_n) = 2iC_{12}f_{3...n}(z_3, \cdots, z_n)\left(1 - S_{2n} \cdots S_{23}\right)
\]

- Bound states

\[
\text{Res}_{z_{12}=z_B} f_{12...n}(z_1, z_2, \cdots, z_n) = f_{B...n}(z_B, z_3, \cdots, z_n)\sqrt{2i\text{Res}_{z_{12}=z_B} S_{12}}
\]
Perturbative Checks

- **Strong coupling:** tree-level in *near-plane-wave model*, and two-loop in *near-flat-space model*  
  - Vacuum \( |0\rangle \)
  - Particle excitation \( |p\rangle = \tilde{Y}(p)|0\rangle \)
  - Anti-particle excitation \( |\bar{p}\rangle = Y(p)|0\rangle \)

  We chose to restrict ourselves to external states with these types of excitations

- **Weak coupling:** leading-\( \lambda \) in SU(2)-sector = *Heisenberg XXX*  
  - Vacuum \( |0\rangle = |\uparrow \uparrow \uparrow \uparrow \cdots \uparrow \rangle \)
  - Particle excitation \( |\psi(p)\rangle = \sum_x e^{ipx} S_{-x}|0\rangle = \sum_x e^{ipx} |\uparrow \cdots \uparrow \downarrow \uparrow \cdots \uparrow \rangle \)

- **Relation via Landau-Lifshitz model**  
  \[ S_+ \doteq \frac{1 - YY/2}{(1 + YY/2)^2} \times \begin{cases} \tilde{Y} \\ Y \end{cases} \]

  

  [Berenstein, Maldacena, Nastase 2002]

  [Maldacena, Swanson 2006]

  [Minahan, Zarembo 2002]

  [Kruczenski 2003]
Perturbative Computations

- $f(p_1, p_2) = \langle \Omega | Y^2 | p_1, p_2 \rangle$

- $f(\bar{p}_1, p_2, p_3) = \langle \Omega | Y | \bar{p}_1, p_2, p_3 \rangle$

- $f(\bar{p}_1, p_2, p_3, p_4) = \langle \Omega | Y^2 | \bar{p}_1, p_2, p_3, p_4 \rangle$
Bubble Integral

The bubble integral with two in-flowing momenta evaluates to

\[ B(p_1, p_2) = \frac{i}{2\pi} \frac{p_1 p_2}{p_1^2 - p_2^2} \ln \left| \frac{p_2}{p_1} \right| - \frac{p_1 p_2}{4(p_1 + p_2)|p_1 - p_2|} \left( \frac{p_1}{|p_1|} + \frac{p_2}{|p_2|} \right) \]

The form factor is defined with a certain ordering, say \( p_1 > p_2 > 0 \), s.t.

\[ B_>(p_1, p_2) = \frac{i}{2\pi} \frac{p_1 p_2}{p_1^2 - p_2^2} \left( \ln \frac{p_2}{p_1} + i\pi \right) \]

and by analytic continuation we use the same expression for any \( p_1 \) and \( p_2 \).

Permutation and periodicity are then a consequence of

\[ B_>(p_2, p_1) - B_>(p_1, p_2) = B_>(p_1 e^{2\pi i}, p_2) - B_>(p_1, p_2) = \frac{p_1 p_2}{p_1^2 - p_2^2} \]
Spin-Chain Form Factors

Form factors $\langle \psi(p_1, \ldots) | O_x | \psi(p_1, \ldots) \rangle$, typically $O_x = S_x^+ \cdot O_x = S_x^+ S_{x+1}^+$. 

Bethe states:

$$|\psi(p_1, p_2, \ldots)\rangle = \sum_{1 \leq x_1 < x_2 < \ldots \leq L} \chi(p_1, p_2, \ldots)_{x_1, x_2, \ldots} \left| \begin{array}{cccccccc} \uparrow & \ldots & \uparrow & \uparrow \\ \downarrow & \uparrow & \uparrow & \uparrow \end{array} \vphantom{\begin{array}{cccccccc} \uparrow & \ldots & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow \end{array}} \right\rangle_{x_1} \left| \begin{array}{cccccccc} \uparrow & \ldots & \uparrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \uparrow \end{array} \vphantom{\begin{array}{cccccccc} \uparrow & \ldots & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow \end{array}} \right\rangle_{x_2} \cdots \left| \begin{array}{cccccccc} \uparrow & \ldots & \uparrow & \uparrow \\ \uparrow & \uparrow & \downarrow & \uparrow \end{array} \vphantom{\begin{array}{cccccccc} \uparrow & \ldots & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow \end{array}} \right\rangle_{x_L}$$

where, e.g. for two magnons, the wave-function is

$$\chi(p_1, p_2)_{x_1, x_2} = e^{i(p_1 x_1 + p_2 x_2)} + S(p_2, p_1) e^{i(p_2 x_1 + p_1 x_2)}$$

- Permutation $|\psi(p_2, p_1)\rangle = |\psi(p_1, p_2)\rangle S(p_1, p_2)$

- Bound states $\text{Res}_{p_{1,2} = \hat{p}_{1,2}} |\psi(p_1, p_2)\rangle = |\psi_B(\hat{p}_1, \hat{p}_2)\rangle \text{ Res}_{p_{1,2} = \hat{p}_{1,2}} S(p_1, p_2)$

- Periodicity

- One-particle pole $\}$ Not observable at weak coupling, rapidity torus degenerates for $\lambda \to 0$
Conclusions and Outlook

- Formulated a set of Consistency Conditions for Form Factors in the worldsheet theory of light-cone gauge-fixed strings in AdS$_5\times$S$^5$: two-dimensional, massive, integrable, non-relativistic QFT.

- Perturbative computation of various worldsheet form factors at strong and weak coupling, SU(2) sector. This provides a first check of the axioms.

- Showed match of thermodynamic limit of spin-chain form factors and small momentum limit of worldsheet form factors (via Landau-Lifshitz).

- Hope of possibility to find all-loop or non-perturbative results by solving form factor equations directly (analogous to S-matrix).

- Make contact with holographic correlation functions by considering form factors of vertex operators.