Introduction to Integrability in AdS/CFT I

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Perturbative Integrability

N. Beisert et.al. “Review of AdS/CFT Integrability” arXiv:1012.3982-4005

AdS / CFT duality

- Type IIB superstrings on $AdS_5 \times S^5$

dual to

$\mathcal{N} = 4 \quad SU(N_c)$ super-Yang-Mills theory

[Maldacena (1997)]

AdS / CFT duality

- Parameter relations:

$$g_s = \frac{4\pi\lambda}{N_c} \quad \& \quad \frac{R^2}{\alpha'} = \sqrt{\lambda}$$

  't Hooft coupling constant $\lambda = N_c g_{\rm YM}^2$

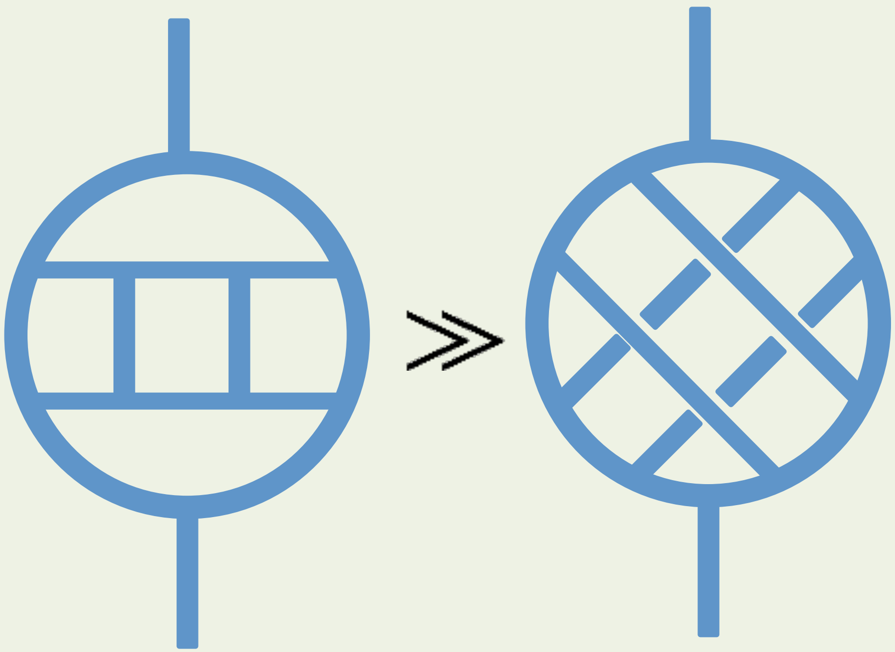
- Free superstring theory corresponds to planar limit of SYM

$$g_s \to 0 \;\equiv\; N_c \to \infty \text{ with fixed } \lambda$$

- Quantitative check is tricky since it is a strong-weak duality
  - SYM perturbation for $\lambda \ll 1$
  - Semiclassical String for $\alpha' \ll 1 \quad \Rightarrow \quad \lambda \gg 1$

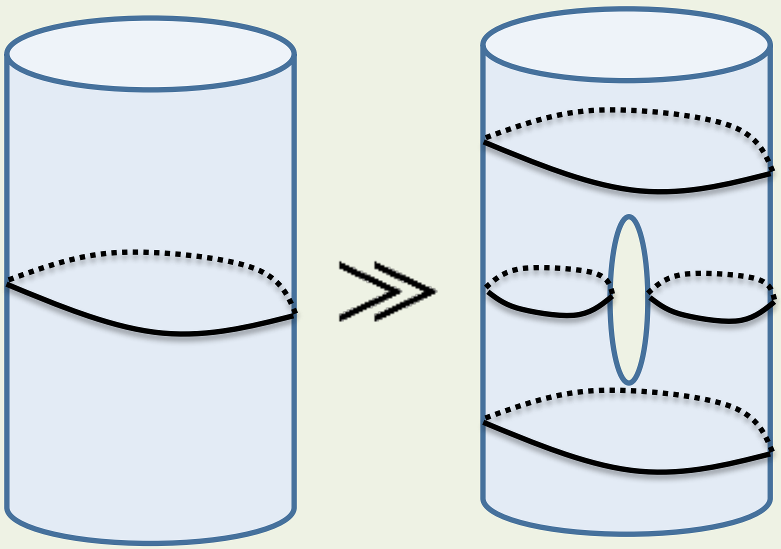
Planar Limit

$$N_c \to \infty$$



$$g_{\text{YM}}^{10} N_c^5 = \lambda^5 \gg g_{\text{YM}}^{10} N_c^3 = \frac{\lambda^5}{N_c^2}$$

$$g_s \to 0$$



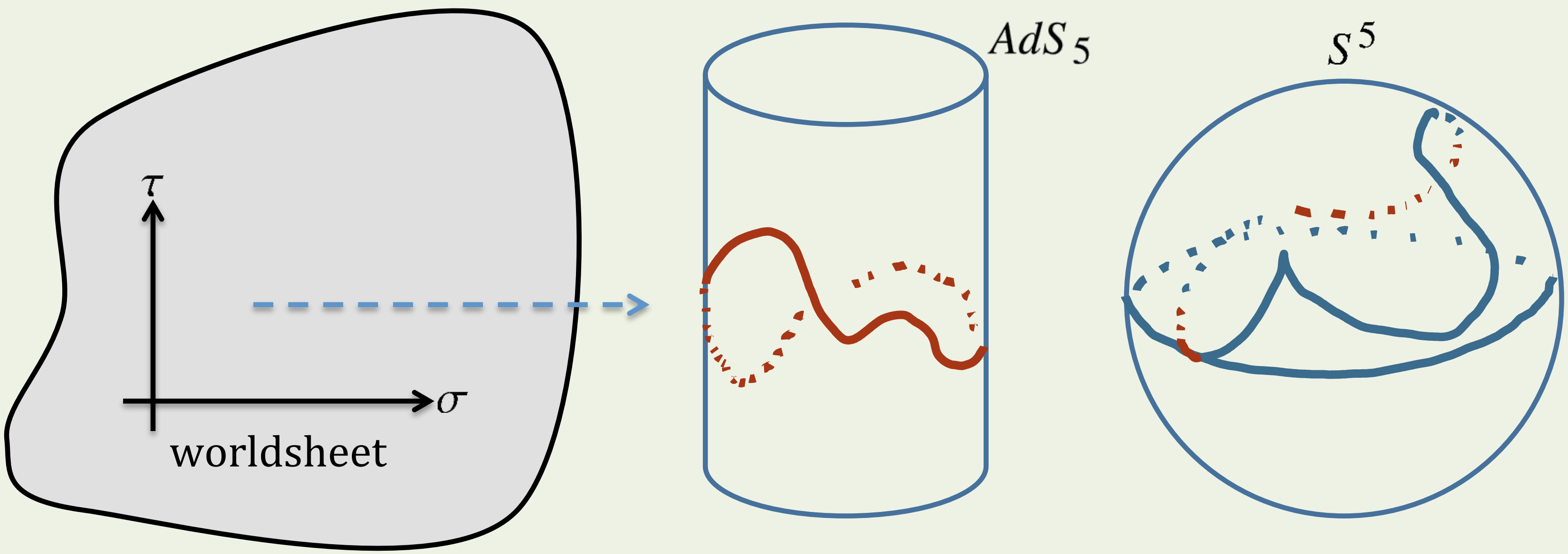
$$\mathcal{O}(g_s^2)$$

SYM operator vs. string configuration

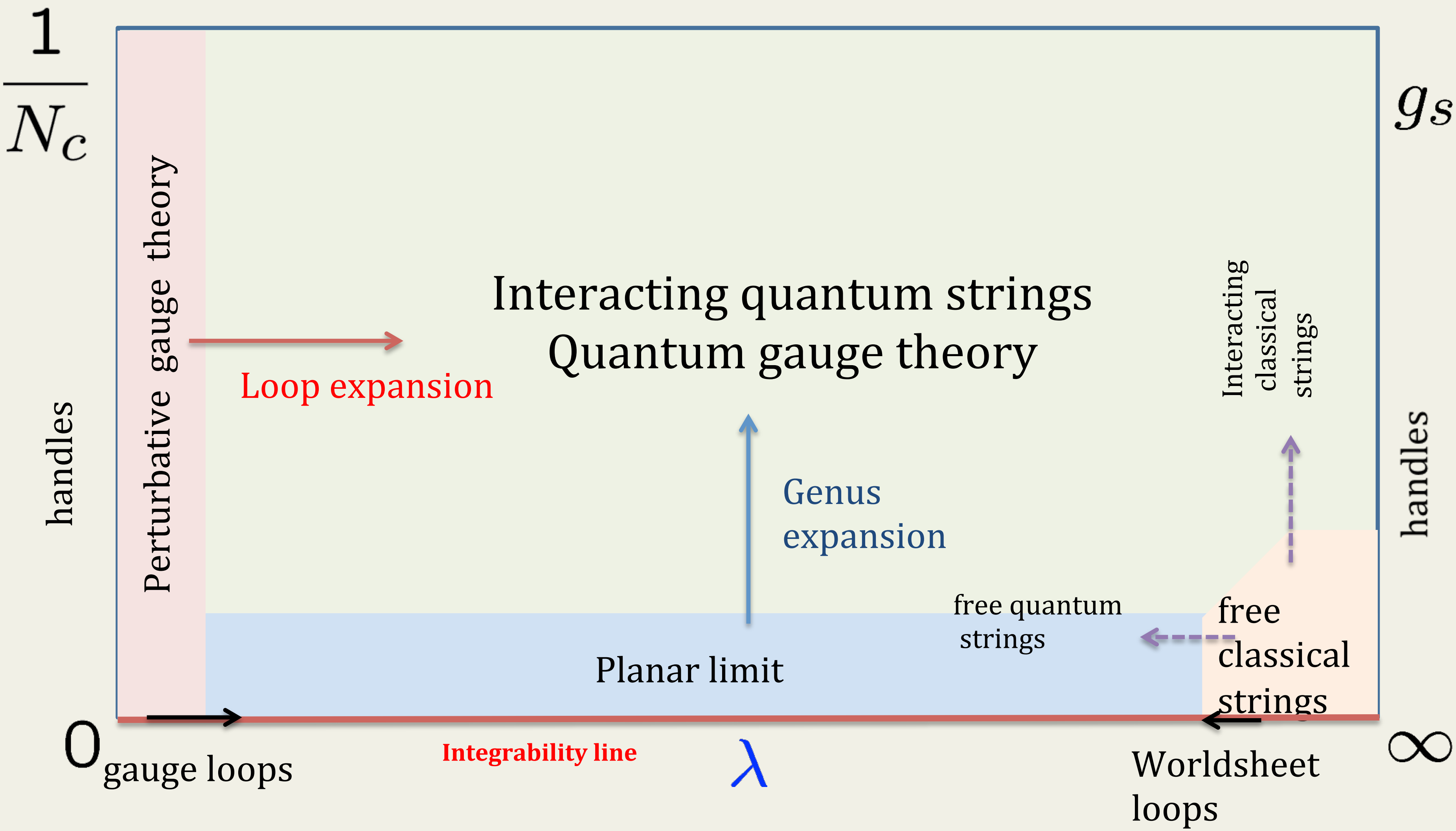
- Composite SYM operator

$$\mathcal{O}(x) = \mathrm{Tr}\left[XYZF_{\mu\nu}\chi^{\alpha}(D_{\mu}Y)\ldots\right]$$

- String configuration in a target space



Parameter space



N=4 Super Yang-Mills theory

- $\mathcal{N}=4\ SU(N_c)$ SYM

$$S = \frac{\mathrm{Tr}}{g_{\mathrm{YM}}^2}\int d^4x\left\{-\frac{1}{4}F_{\mu\nu}^2 + (D_\mu\Phi^a)^2 + \left[\Phi^a,\Phi^b\right]^2 + \bar{\chi}\not{D}\chi - i\bar{\chi}\Gamma_a[\Phi^a,\chi]\right\}$$

- R-symmetry : *N*=4 SUSY so(6) = su(4)
- Scalar fields : $\Phi^a,\ a=1,\ldots,6$ □
- Gauginos : $\chi,\ \bar{\chi}$ fundamental in su(4) □
- All in adjoint rep. in $SU(N_c)$

		R-charge
A_μ		$\mathbf{1}$
χ_α^A	$\bar{\chi}_{\bar{\alpha}}^{\bar{A}}$	$\mathbf{4}\oplus\bar{\mathbf{4}}$
Φ^a		$\mathbf{6}$

4d conformal field theory

- One-loop β-function

$$\beta \equiv \mu \frac{\partial g_{\text{YM}}}{\partial \mu} = -\frac{g_{\text{YM}}^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{1}{6}\underbrace{\sum_i C_i}_{6N_c} - \frac{1}{3}\underbrace{\sum_j \tilde{C}_j}_{8N_c}\right) = 0$$

- β=0 at all orders of perturbation
  - Three loops in superspace formulation
  - All loops in light-cone gauge

- No scale dependence

$\mathcal{N}$=4 superconformal algebra

- Lorentz generators : $L_{\mu\nu}$
- Translations : P_μ
- Conformal boosts : K_μ
- Dilatation : D
- Supercharges : $Q_{a\alpha}, \quad \overline{Q}^{a}_{\dot\alpha}$
- Superconformal boosts : $S^{a}_{\alpha}, \quad \overline{S}_{\dot\alpha a}$
- R-symmetry : su(4)

su(2,2) ≅ so(2,4)

psu(2,2|4)

32 super charges

$$\left(\begin{array}{c|c|c} L & Q & P \\ \hline S & R & \overline{Q} \\ \hline K & \overline{S} & \overline{L} \end{array}\right)$$

- psu(2,2|4) commutation relations

$$[D, P_\mu] = -iP_\mu, \qquad [D, L_{\mu\nu}] = 0, \qquad [D, K_\mu] = iK_\mu$$

$$[D, Q_{\alpha a}] = -\frac{i}{2}Q_{\alpha a}, \quad [D, \overline{Q}^a_{\dot\alpha}] = -\frac{i}{2}\overline{Q}^a_{\dot\alpha}, \quad [D, S^a_\alpha] = \frac{i}{2}S^a_\alpha, \quad [D, \overline{S}_{\dot\alpha a}] = \frac{i}{2}\overline{S}_{\dot\alpha a}$$

$$[L_{\mu\nu}, P_\lambda] = -i(\eta_{\mu\lambda}P_\nu - \eta_{\lambda\nu}P_\mu), \qquad [L_{\mu\nu}, K_\lambda] = -i(\eta_{\mu\lambda}K_\nu - \eta_{\lambda\nu}K_\mu)$$

$$[P_\mu, K_\nu] = 2i(L_{\mu\nu} - \eta_{\mu\nu}D)$$

$$\{Q_{\alpha a}, \overline{Q}^b_{\dot\alpha}\} = \gamma^\mu_{\alpha\dot\alpha}\delta^b_a P_\mu, \quad \{Q_{\alpha a}, Q_{\alpha b}\} = \{\overline{Q}^a_{\dot\alpha}, \overline{Q}^b_{\dot\alpha}\} = 0$$

$$[P_\mu, Q_{\alpha a}] = [P_\mu, \overline{Q}^a_{\dot\alpha}] = 0, \quad [L^{\mu\nu}, Q_{\alpha a}] = i\gamma^{\mu\nu}_{\alpha\beta}\epsilon^{\beta\gamma}Q_{\gamma a}, \quad [L^{\mu\nu}, \overline{Q}^a_{\dot\alpha}] = i\gamma^{\mu\nu}_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\overline{Q}^a_{\dot\gamma}$$

$$[K^\mu, Q_{\alpha a}] = \gamma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\overline{S}_{\dot\beta a}, \quad [K^\mu, \overline{Q}^a_{\dot\alpha}] = \gamma^\mu_{\alpha\dot\alpha}\epsilon^{\alpha\beta}S^a_\beta$$

$$\{Q_{\alpha a}, \overline{Q}^b_{\dot\alpha}\} = \gamma^\mu_{\alpha\dot\alpha}\delta^b_a P_\mu, \quad \{S^a_\alpha, \overline{S}_{\dot\alpha b}\} = \gamma^\mu_{\alpha\dot\alpha}\delta^a_b K_\mu, \quad \{S^a_\alpha, S^a_\alpha\} = \{\overline{S}_{\dot\alpha a}, \overline{S}_{\dot\alpha b}\} = 0$$

$$[K_\mu, S^a_\alpha] = [K_\mu, \overline{S}_{\dot\alpha a}] = 0$$

$$\{Q_{\alpha a}, S^b_\beta\} = -i\epsilon_{\alpha\beta}\sigma^{IJ\,b}{}_a R_{IJ} + \gamma^{\mu\nu}_{\alpha\beta}\delta^b_a L_{\mu\nu} - \frac{1}{2}\epsilon_{\alpha\beta}\delta^b_a D$$

$$\{\overline{Q}^a_{\dot\alpha}, \overline{S}_{\dot\beta b}\} = i\epsilon_{\dot\alpha\dot\beta}\sigma^{IJ\,a}{}_b R_{IJ} + \gamma^{\mu\nu}_{\dot\alpha\dot\beta}\delta^a_b L_{\mu\nu} - \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\delta^a_b D$$

- Conformal symmetry → No mass spectrum

- Conformal dimension spectrum for a local operator $[D, \mathcal{O}(0)] = -i\,\Delta\,\mathcal{O}(0)$

- Lowered by K : $\mathcal{O}'(0) \equiv \left[K_\mu, \mathcal{O}(0)\right] \to \left[D, \mathcal{O}'(0)\right] = -i\,(\Delta - 1)\,\mathcal{O}'(0)$

$$\left[D, K_\mu\right] = +i\,K_\mu$$

- Primary operator : $\left[K_\mu, \widetilde{\mathcal{O}}(0)\right] = 0$

- Descendent operators : (ex) $\left[P_\mu, \widetilde{\mathcal{O}}\right] = -i\partial_\mu \widetilde{\mathcal{O}}$ $\quad \left[D, P_\mu\right] = -i\,P_\mu$

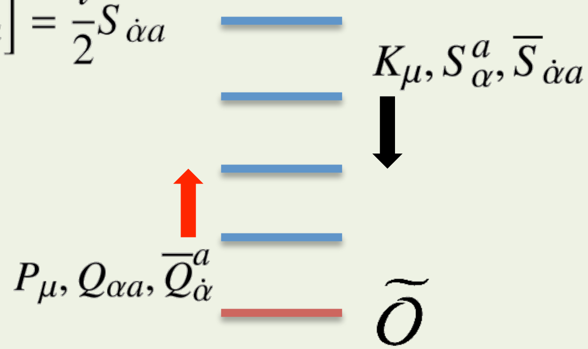
$$\left[D, \partial_\mu \tilde{\mathcal{O}}\right] = -i\,(\Delta + 1)\,\partial_\mu \tilde{\mathcal{O}}$$

- Superconformal raising and lowering ops.

$$[D, Q_{\alpha a}] = -\frac{i}{2} Q_{\alpha a}, \quad \left[D, \overline{Q}^a_{\dot{\alpha}}\right] = -\frac{i}{2}\overline{Q}^a_{\dot{\alpha}}, \quad [D, S^{\ a}_{\alpha}] = \frac{i}{2} S^{\ a}_{\alpha}, \quad \left[D, \overline{S}_{\dot{\alpha} a}\right] = \frac{i}{2}\overline{S}_{\dot{\alpha} a}$$

- Superconformal primary :

$$\left[S^{\ a}_{\alpha}, \widetilde{\mathcal{O}}(0)\right] = \left[\overline{S}_{\dot{\alpha} a}, \widetilde{\mathcal{O}}(0)\right] = 0$$



- Cartan subalgebra $$[D,R]=\left[L_{\mu\nu},R\right]=\left[D,L_{\mu\nu}\right]=0$$

- Irreducible rep. are given by eigenvalues of these operators

$$(\overbrace{\quad\Delta\quad}^{D},\overbrace{S_1,S_2}^{L_{\mu\nu}}\,|\,\overbrace{J_1,J_2,J_3}^{R})$$

- Scalar fields

$$Z\equiv\Phi_1+i\Phi_2,\quad Y\equiv\Phi_3+i\Phi_4,\quad X\equiv\Phi_5+i\Phi_6$$
$$\overline{Z}\equiv\Phi_1-i\Phi_2,\quad \overline{Y}\equiv\Phi_3-i\Phi_4,\quad \overline{X}\equiv\Phi_5-i\Phi_6$$
$$(1,0,0|\pm1,0,0),\ (1,0,0|0,\pm1,0),\ (1,0,0|0,0,\pm1)$$

- Gauginos and gauge fields

$$\underset{\chi^A_\alpha}{\left(\frac{3}{2},\pm\frac{1}{2},0|\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}\right)},\ \underset{F_+}{(2,m,0|0,0,0)},\ \underset{\mathcal{D}}{\left(1,\pm\frac{1}{2},\pm\frac{1}{2}|0,0,0\right)}$$

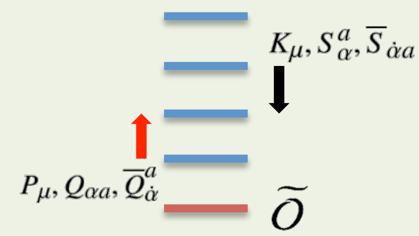
- General gauge invariant composite operators

$$\tilde{O}(x)=\mathrm{Tr}\left[O_1(x)O_2(x)\ldots O_L(x)\right]$$

- ½ -BPS operator $\mathrm{Tr}\left[Z^L\right]\quad\rightarrow\quad(L,0,0|L,0,0)$

Chiral primary or BPS operator

- Impose further condition $\left[Q_{a\alpha},\widetilde{O}(0)\right]=0,\quad$ for some α, a
  - Jacobi identity $\left[\left\{Q_{a\alpha},S^b_\beta\right\},\widetilde{O}(0)\right]=\left[-i\varepsilon_{\alpha\beta}\left(\sigma^{IJ}\right)^b_a R_{IJ}-\varepsilon_{\alpha\beta}\delta^b_a D+\sigma^{\mu\nu}_{\alpha\beta}\delta^b_a L_{\mu\nu},\widetilde{O}(0)\right]=0$
  - For the Lorentz scalar operator : $\left[L_{\mu\nu},\widetilde{O}(0)\right]=0$



$$\left(\sigma^{IJ}\right)^b_a\left[R_{IJ},\widetilde{O}(0)\right]=\Delta\,\delta^b_a\,\widetilde{O}(0)\qquad \sigma^{12}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

  - Satisfied if R-charge = conformal dimension $\Delta = J_1$

$$\mathrm{Tr}\left[Z^L\right]\quad\rightarrow\quad(L,0,0|L,0,0)$$

  - This commutes with half SUSY charges (a=1,2) "half-BPS"
  - conformal dimension is protected and gets no quantum corrections

Anomalous Dimension

- Conformal dimensions of composite operators :

$$\langle \mathcal{O}_n(x)\mathcal{O}_m(0)\rangle = \frac{\delta_{mn}}{|x|^{2\Delta_n}}$$

- Anomalous dimension is defined by $\Delta = \Delta_0 + \gamma$

- Operator mixing by RG dilatation

RG method

- Under dilatation $x \to x/\Lambda$

$$\mathcal{O}(x) = \Lambda^{\Delta}\mathcal{O}(x/\Lambda)$$

- One-loop corrections

- However, operators are mixed by RG

$$\mathcal{O}_a = \mathcal{Z}_a^b(\Lambda)\mathcal{O}_b$$

- Dilatation matrix

$$\Gamma = \frac{d\mathcal{Z}}{d\ln\Lambda}\cdot\mathcal{Z}^{-1}$$

Mapping to integrable spin chain

- Finding the eigenvalues of the dilatation matrix is a difficult problem but fortunately …
- Mapping the matrix to a Hamiltonian of integrable spin chain has been discovered [(ex) so(6), su(2) spin chains]
- (ex) su(2) sector $\left\{\mathrm{Tr}\left[Z^L\right], \mathrm{Tr}\left[Z^{L-1}X\right], \mathrm{Tr}\left[Z^{L-n-1}XZ^{n-1}X\right], \ldots, \mathrm{Tr}\left[X^L\right]\right\}$

- One-loop dilatation → Heisenberg spin chain model
  - Map: $|\Uparrow\rangle \equiv |Z\rangle, \quad |\Downarrow\rangle \equiv |X\rangle$
  - Vacuum state: Ferromagnetic vac. ←→ BPS $|\Uparrow \ldots \Uparrow\rangle \equiv \mathrm{Tr}\left[Z^L\right]$
  - Excited states:

$$|\Uparrow \Downarrow \Uparrow \Downarrow \Uparrow\Uparrow \ldots\rangle + \ldots \equiv \mathrm{Tr}\,[ZXZXZZ + \ldots] + \ldots$$

  - Dilatation $$\Gamma = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}[1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}]$$

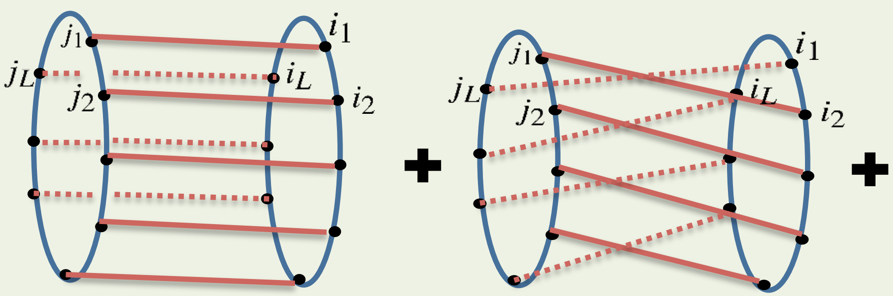
(ex) SO(6) sector

- Scalar fields $\{Z,\ Y,\ X,\ \bar{Z},\ \bar{Y},\ \bar{X}\}$
- Composite operators

$$\left\{\mathrm{Tr}\left[XYZ\bar{X}YZX\bar{Z}\cdots\right],\ldots\right\} = \mathrm{Tr}\left[\Phi_{i_1}\cdots\Phi_{i_L}\right] \equiv \mathcal{O}_{i_1\cdots i_L}(x)$$

- Two-point function $\left\langle \bar{\mathcal{O}}^{j_1\cdots j_L}(x)\,\mathcal{O}_{i_1\cdots i_L}(y)\right\rangle$

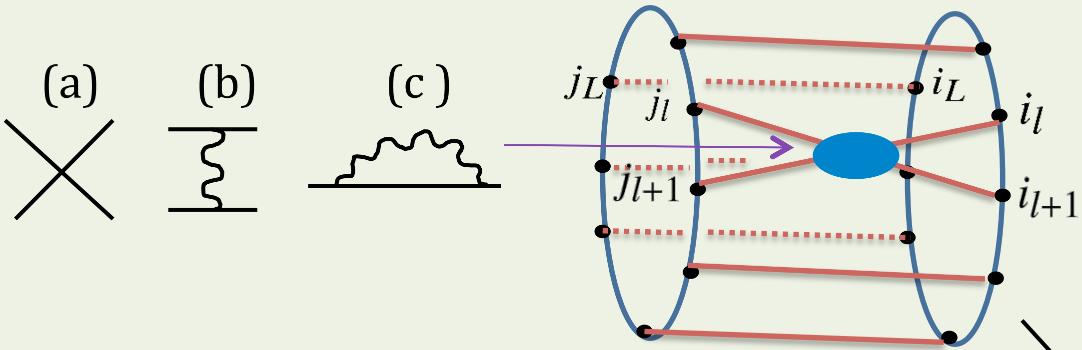
- Tree level

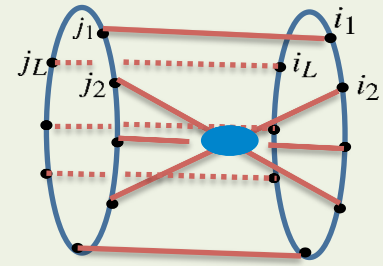


$$\left(\frac{\lambda}{8\pi^2}\right)^L \frac{1}{|x-y|^{2L}}\left[\delta_{i_1}^{j_1}\cdots\delta_{i_L}^{j_L} + \delta_{i_2}^{j_1}\cdots\delta_{i_1}^{j_L} + \ldots\right]$$

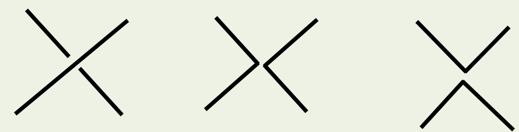
cyclic permutations

- One-loop level : nearest neighbor only





- Wave-function renormalization



$$Z^{(a)} = 1 - \frac{\lambda}{16\pi^2}\ln\Lambda\cdot\left(2\delta_{i_l}^{j_{l+1}}\delta_{i_{l+1}}^{j_l} - \delta_{i_l}^{j_l}\delta_{i_{l+1}}^{j_{l+1}} - \delta_{i_l,i_{l+1}}\delta^{j_l j_{l+1}}\right)$$

$$Z^{(b)} = 1 - \frac{\lambda}{16\pi^2}\ln\Lambda\cdot\delta_{i_l}^{j_l}\delta_{i_{l+1}}^{j_{l+1}}$$

$$Z^{(c)} = 1 + \frac{\lambda}{8\pi^2}\ln\Lambda\cdot\delta_{i_l}^{j_l}\delta_{i_{l+1}}^{j_{l+1}}$$

$$Z = 1 + \frac{\lambda}{16\pi^2}\ln\Lambda\cdot\left(\delta_{i_l,i_{l+1}}\delta^{j_l j_{l+1}} - 2\delta_{i_l}^{j_{l+1}}\delta_{i_{l+1}}^{j_l} + 2\delta_{i_l}^{j_l}\delta_{i_{l+1}}^{j_{l+1}}\right)$$

- Dilatation matrix

$$\Gamma = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(1 - \mathbf{P}_{l,l+1} + \frac{1}{2}\mathbf{K}_{l,l+1}\right)$$

Minahan, Zarembo (2003)

Bethe ansatz equation for su(2) sector

- Eigenvectors

$$|p_1, p_2, \cdots\rangle = \sum_{n_1, n_2, \cdots = 1}^{L} A(p_1, p_2, \cdots) e^{i(n_1 p_1 + n_2 p_2 + \cdots)} |\cdots \Uparrow \underset{n_1}{\Downarrow} \Uparrow \cdots \Uparrow \underset{n_2}{\Downarrow} \Uparrow \cdots\rangle + \cdots$$

$$|\Uparrow \Downarrow \Uparrow \Downarrow \Uparrow\Uparrow \ldots\rangle + \ldots \equiv \mathrm{Tr}\,[ZXZXZZ + \ldots] + \ldots$$

- BAE

$$e^{ip_j L} = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{\cot\frac{p_j}{2} - \cot\frac{p_k}{2} + 2i}{\cot\frac{p_j}{2} - \cot\frac{p_k}{2} - 2i}, \qquad j = 1, \ldots, M$$

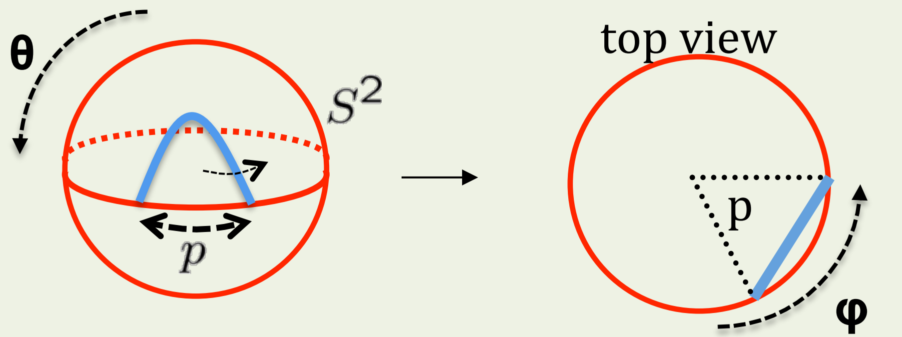
$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}, \qquad \frac{u + i/2}{u - i/2} \equiv e^{ip}$$

- Anomalous dimensions are given by the eigenvalues

$$\gamma = \frac{\lambda}{2\pi^2} \sum_{j=1}^{M} \sin^2 \frac{p_j}{2} = \frac{\lambda}{2\pi^2} \sum_{j=1}^{M} \frac{1}{u_j^2 + \frac{1}{4}}$$

Giant magnon

- Classical string configuration in R x S² **Hofman, Maldacena (2006)**

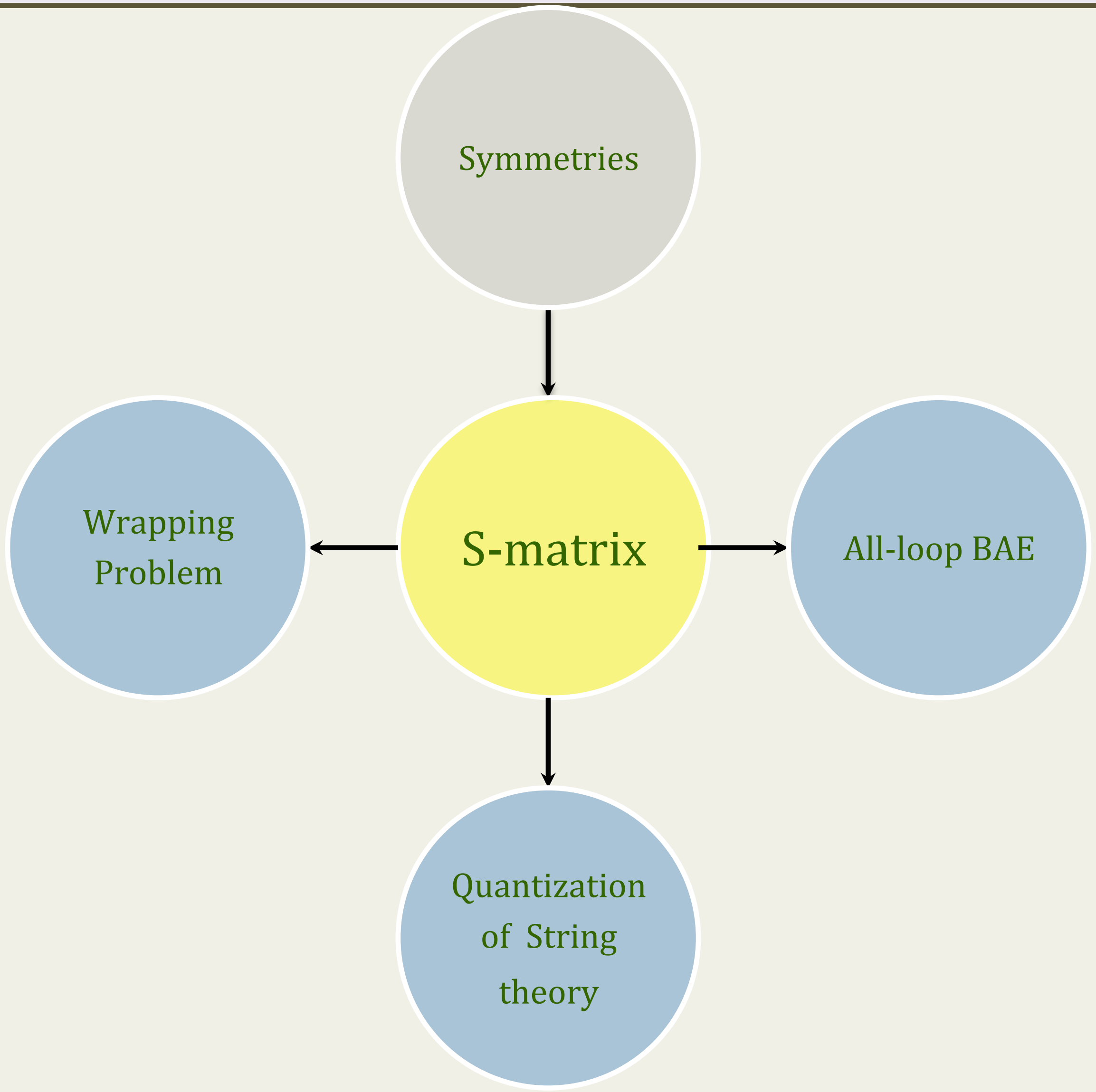


$$\cos\theta = \frac{\sin\frac{p}{2}}{\cosh\xi}, \qquad \tan\varphi = \tan\frac{p}{2}\tanh\xi, \qquad \xi \equiv \frac{\sigma - \cos\frac{p}{2}\tau}{\sin\frac{p}{2}}$$

  - Energy of the string $E = \frac{\sqrt{\lambda}}{\pi}\sin\frac{p}{2}$
  - Pohlmeyer reduction: GM is mapped to the SG "Soliton"
  - Dual to magnons in the SYM spin chain $\cdots \Uparrow\Uparrow \Downarrow \Uparrow\Uparrow \cdots$

$$\Delta = 1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2} \xleftarrow{\lambda \ll 1} \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} \xrightarrow{\lambda \gg 1} E = \frac{\sqrt{\lambda}}{\pi}\sin\frac{p}{2}$$

S-matrix program



Perturbative spin-chain S-matrix

Excitations

- Vacuum

$$\mathrm{Tr}\,[ZZZ\cdots ZZZ]$$

- Excitations

$$\mathrm{Tr}\,[Z\cdots\chi_1 Z\cdots\chi_2 Z\cdots\chi_3 Z\cdots Z\chi_n Z\cdots]$$

- Examples
  - su(2): $\chi = \{X\}$
  - su(3): $\chi = \{X,\ Y\}$
  - so(6): $\chi = \{X,\ Y,\ \overline{X},\ \overline{Y}\}$
- spin-chain in the weak coupling limit

su(2) spin-chain

- XXX Hamiltonian

$$H = \sum_{l=1}^{L} \left[1 - \mathbf{P}_{l,l+1}\right]$$

- 2-magnon states

$$|\psi(p_1, p_2)\rangle = A_{XX}(12)|X(p_1)X(p_2)\rangle + A_{XX}(21)|X(p_2)X(p_1)\rangle,$$

$$|X(p_i)X(p_j)\rangle = \sum_{n_1<n_2} e^{i(p_i n_1 + p_j n_2)} \mathrm{Tr}[\overset{1}{\underset{\downarrow}{}}Z \cdots \overset{n_1}{\underset{\downarrow}{}}X \cdots \overset{n_2}{\underset{\downarrow}{}}X \cdots \overset{L}{\underset{\downarrow}{}}Z]$$

  - satisfy $H|\psi\rangle = E(p_1, p_2)|\psi\rangle = \left(4\sin^2\frac{p_1}{2} + 4\sin^2\frac{p_2}{2}\right)|\psi\rangle$

  - if $A_{XX}(21) = S(p_2, p_1) A_{XX}(12)$ with $S(p_2, p_1) = \dfrac{u_2 - u_1 + i}{u_2 - u_1 - i}$

One-loop X-X scattering amplitude

su(3) spin-chain

- Hamiltonian
$$H=\sum_{l=1}^{L}\left[1-\mathbf{P}_{l,l+1}\right]$$
- 2-magnon states
$$|\psi\rangle = A_{XY}(12)|X(p_1)Y(p_2)\rangle + A_{XY}(21)|X(p_2)Y(p_1)\rangle + A_{YX}(12)|Y(p_1)X(p_2)\rangle + A_{YX}(21)|Y(p_2)X(p_1)\rangle$$
$$|\phi_1(p_i)\phi_2(p_j)\rangle = \sum_{n_1<n_2} e^{i(p_i n_1 + p_j n_2)}\,\mathrm{Tr}[\overset{1}{\overset{\downarrow}{Z}}\cdots\overset{n_1}{\overset{\downarrow}{\phi_1}}\cdots\overset{n_2}{\overset{\downarrow}{\phi_2}}\cdots\overset{L}{\overset{\downarrow}{Z}}]$$
  - satisfy $H|\psi\rangle = E(p_1,p_2)|\psi\rangle = \left(4\sin^2\frac{p_1}{2}+4\sin^2\frac{p_2}{2}\right)|\psi\rangle$
  - if
$$\begin{pmatrix} A_{XY}(21) \\ A_{YX}(21)\end{pmatrix} = \boxed{\begin{pmatrix} R(p_2,p_1) & T(p_2,p_1) \\ T(p_2,p_1) & R(p_2,p_1)\end{pmatrix}}\begin{pmatrix} A_{XY}(12) \\ A_{YX}(12)\end{pmatrix}$$

One-loop X-Y scattering amplitude

$$T(p_2,p_1)=\frac{u_2-u_1}{u_2-u_1-i},\qquad R(p_2,p_1)=\frac{i}{u_2-u_1-i}$$

- S-matrix
$$\mathbf{S}=\begin{pmatrix} S & & & \\ & T & R & \\ & R & T & \\ & & & S\end{pmatrix} \propto \begin{pmatrix} u+i & & & \\ & u & i & \\ & i & u & \\ & & & u+i\end{pmatrix}$$

su(2) R-matrix

so(6) spin-chain

- Hamiltonian $H = \sum_{l=1}^{L}\left(1 - \mathbf{P}_{l,l+1} + \frac{1}{2}\mathbf{K}_{l,l+1}\right)$ $\quad \mathbf{K}\,\Phi_i \otimes \Phi_j = \delta_{ij}\left(\sum_{k=1}^{6}\Phi_k \otimes \Phi_k\right)$
- Same as su(3) except the cases like $|X(p_2)\bar{X}(p_1)\rangle$

$$|\psi\rangle = \sum_{\phi=X,Y}\left[A_{\phi\bar\phi}(12)|\phi(p_1)\bar\phi(p_2)\rangle + A_{\phi\bar\phi}(21)|\phi(p_2)\bar\phi(p_1)\rangle + A_{\bar\phi\phi}(12)|\bar\phi(p_1)\phi(p_2)\rangle + A_{\bar\phi\phi}(21)|\bar\phi(p_1)\phi(p_1)\rangle\right] + A_{\bar Z}|\bar Z(p_1+p_2)\rangle$$

$$|\phi(p_i)\bar\phi(p_j)\rangle = \sum_{n_1<n_2} e^{i(p_i n_1 + p_j n_2)}\,\mathrm{Tr}[\overset{1}{\overset{\downarrow}{Z}}\cdots\overset{n_1}{\overset{\downarrow}{\phi}}\cdots\overset{n_2}{\overset{\downarrow}{\bar\phi}}\cdots\overset{L}{\overset{\downarrow}{Z}}] \qquad |\bar Z(p)\rangle = \sum_n e^{ipn}\,\mathrm{Tr}[\overset{1}{\overset{\downarrow}{Z}}\cdots\overset{n}{\overset{\downarrow}{\bar Z}}\cdots\overset{L}{\overset{\downarrow}{Z}}]$$

  - satisfy $H|\psi\rangle = E(p_1,p_2)|\psi\rangle = \left(4\sin^2\frac{p_1}{2} + 4\sin^2\frac{p_2}{2}\right)|\psi\rangle$
  - if

$$\begin{pmatrix} A_{X\bar X}(21) \\ A_{\bar X X}(21) \\ A_{Y\bar Y}(21) \\ A_{\bar Y Y}(21)\end{pmatrix} = \boxed{\begin{pmatrix} \mathcal{R}(p_2,p_1) & \mathcal{T}(p_2,p_1) & \mathcal{S}(p_2,p_1) & \mathcal{S}(p_2,p_1) \\ \mathcal{T}(p_2,p_1) & \mathcal{R}(p_2,p_1) & \mathcal{S}(p_2,p_1) & \mathcal{S}(p_2,p_1) \\ \mathcal{S}(p_2,p_1) & \mathcal{S}(p_2,p_1) & \mathcal{R}(p_2,p_1) & \mathcal{T}(p_2,p_1) \\ \mathcal{S}(p_2,p_1) & \mathcal{S}(p_2,p_1) & \mathcal{T}(p_2,p_1) & \mathcal{R}(p_2,p_1)\end{pmatrix}} \begin{pmatrix} A_{X\bar X}(12) \\ A_{\bar X X}(12) \\ A_{Y\bar Y}(12) \\ A_{\bar Y Y}(12)\end{pmatrix}$$

$$\mathcal{T}(p_2,p_1) = \frac{(u_2-u_1)^2}{(u_2-u_1-i)(u_2-u_1+i)},\quad \mathcal{R}(p_2,p_1) = \frac{-1}{(u_2-u_1-i)(u_2-u_1+i)},\quad \mathcal{S}(p_2,p_1) = \frac{i(u_2-u_1)}{(u_2-u_1-i)(u_2-u_1+i)}$$

One-loop so(6) scattering amplitude

- so(6) S-matrix can be factorized into a tensor product !
- Define $X = \color{blue}{1}\color{red}{\dot{2}}, \quad \bar{X} = \color{blue}{2}\color{red}{\dot{1}}, \quad Y = \color{blue}{2}\color{red}{\dot{2}}, \quad \bar{Y} = \color{blue}{1}\color{red}{\dot{1}}$

$$\mathbf{S}^{XX}_{XX} = \mathbf{S}^{(1\dot{2})(1\dot{2})}_{(1\dot{2})(1\dot{2})} = S_0 \mathbf{S}^{11}_{11}\mathbf{S}^{\dot{2}\dot{2}}_{\dot{2}\dot{2}}, \quad \mathbf{S}^{YX}_{XY} = \mathbf{S}^{(2\dot{2})(1\dot{2})}_{(1\dot{2})(2\dot{2})} = S_0 \mathbf{S}^{21}_{12}\mathbf{S}^{\dot{2}\dot{2}}_{\dot{2}\dot{2}}, \quad \mathbf{S}^{XY}_{XY} = \mathbf{S}^{(1\dot{2})(2\dot{2})}_{(1\dot{2})(1\dot{2})} = S_0 \mathbf{S}^{12}_{12}\mathbf{S}^{\dot{2}\dot{2}}_{\dot{2}\dot{2}},$$

$$\mathbf{S}^{X\bar{X}}_{X\bar{X}} = \mathbf{S}^{(1\dot{2})(2\dot{1})}_{(1\dot{2})(2\dot{1})} = S_0 \mathbf{S}^{12}_{12}\mathbf{S}^{\dot{2}\dot{1}}_{\dot{2}\dot{1}}, \quad \mathbf{S}^{\bar{X}X}_{X\bar{X}} = \mathbf{S}^{(2\dot{1})(1\dot{2})}_{(1\dot{2})(2\dot{1})} = S_0 \mathbf{S}^{21}_{12}\mathbf{S}^{\dot{1}\dot{2}}_{\dot{2}\dot{1}}, \quad \mathbf{S}^{\bar{Y}Y}_{X\bar{X}} = \mathbf{S}^{(2\dot{2})(1\dot{1})}_{(1\dot{2})(2\dot{1})} = S_0 \mathbf{S}^{21}_{12}\mathbf{S}^{\dot{2}\dot{1}}_{\dot{2}\dot{1}}$$

$$\frac{u+i}{u-i} = \frac{1}{(u-i)(u+i)}\cdot(u+i)\cdot(u+i), \quad \frac{u}{u+i} = \frac{1}{(u-i)(u+i)}\cdot u\cdot(u+i), \quad \frac{i}{u+i} = \frac{1}{(u-i)(u+i)}\cdot i\cdot(u+i)$$

$$\frac{i^2}{(u-i)(u+i)} = \frac{1}{(u-i)(u+i)}\cdot i\cdot i, \quad \frac{u^2}{(u-i)(u+i)} = \frac{1}{(u-i)(u+i)}\cdot u\cdot u, \quad \frac{iu}{(u-i)(u+i)} = \frac{1}{(u-i)(u+i)}\cdot u\cdot i$$

$$\mathbf{S}_{so(6)} = S_0 \cdot \color{blue}{\mathbf{S}} \otimes \color{red}{\mathbf{S}}, \qquad S_0 = (u^2+1)^{-1}$$

$$\color{blue}{\mathbf{S}} = \color{red}{\mathbf{S}} = \begin{pmatrix} u+i & & & \\ & u & i & \\ & i & u & \\ & & & u+i \end{pmatrix}$$

$$SO(6) \to SO(4) \simeq \color{blue}{SU(2)} \times \color{red}{SU(2)}$$

Worldsheet S-matrix

- So far we considered S-matrix from gauge theory spin chains
- String perturbative computation (large λ) of S-matrix is also possible
  - Fluctuation around BMN in light-cone gauge
  - Effective Lagrangian contains
    - Quadratic terms in terms of oscillator algebra [BMN limit]
    - Quartic interaction terms

$$\mathcal{L}_{\text{int.}} \sim \frac{1}{\sqrt{\lambda}}\left[x^2(p_y^2 + y'^2) - y^2(p_x^2 + x'^2) + 2x^2x'^2 - 2y^2y'^2\right]$$

- Can compute scattering amplitudes on the worldsheet

Klose,McLoughlin,Roiban,Zarembo (2007)