AdS/CFT Spring School

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Introduction to Integrability in AdS/CFT II: Part 2

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Introduction

Recall:

- 1-loop dilatation operator for single-trace operators of scalars in SU(2) subsector of $\mathcal{N}$=4 SYM is integrable

- can generalize to all single-trace operators in $\mathcal{N}$=4 SYM

$\Rightarrow$

- 1-loop anomalous dimensions are given by a set of Bethe equations

Higher loops?

Plan:

- dilatation operator at higher loops
- all-loop asymptotic S-matrix
- all-loop asymptotic Bethe equations

dilatation operator
at higher loops

Higher loops:

[Beisert, Kristjansen & Staudacher 03, ...]

$$\Gamma = \sum_{n=1}^{\infty} \lambda^n \Gamma^{(n)}$$

n: loop order

n	interaction range
1	2
2	3
⋮	⋮
n	n+1

Interaction range grows with loop order!

Integrability is preserved! $Q_k = \sum_{n=1}^{\infty} \lambda^n Q_k^{(n)}$ $\quad Q_1 = \Gamma$

explicitly known only for low n ~ 3

To get all-loop results, need new approach...

all-loop asymptotic
S-matrix

1 loop SU(2): $$H = \sum_{l=1}^{L} (I - \mathcal{P}_{l,l+1})$$ PBCs

$$H|\psi\rangle = E|\psi\rangle$$

[Bethe 31]

vacuum $|0\rangle = |Z \cdots Z\rangle$ $E = 0$

1 particle ("impurity" or "magnon"):

$$|\psi\rangle = \sum_{x=1}^{L} e^{ipx} |\overset{\overset{1}{\downarrow}}{Z} \cdots \overset{\overset{x}{\downarrow}}{X} \cdots \overset{\overset{L}{\downarrow}}{Z}\rangle \qquad E = 4\sin^2\frac{p}{2} \equiv \epsilon(p)$$

PBCs $\Longrightarrow$ $e^{ipL} = 1$ momentum is quantized

2 particles:

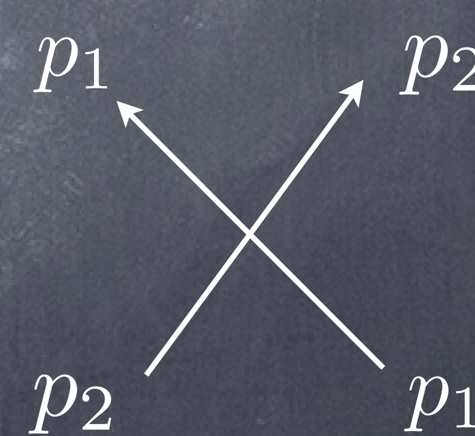
$$|\psi\rangle = \sum_{x_1<x_2} \psi(x_1,x_2)|Z\cdots \overset{x_1}{\overset{\downarrow}{X}} \cdots \overset{x_2}{\overset{\downarrow}{X}} \cdots Z\rangle$$

$$\psi(x_1,x_2) = e^{i(p_1x_1+p_2x_2)} + S(p_2,p_1)e^{i(p_2x_1+p_1x_2)}$$

$$x_1 < x_2$$

$$\boxed{S(p_2\,,p_1) = \frac{u_2-u_1+i}{u_2-u_1-i}}$$

S-matrix



$$u_j = u(p_j)\,, \quad u(p) = \frac{1}{2}\cot(\frac{p}{2})$$

$$E = \epsilon(p_1) + \epsilon(p_2)$$

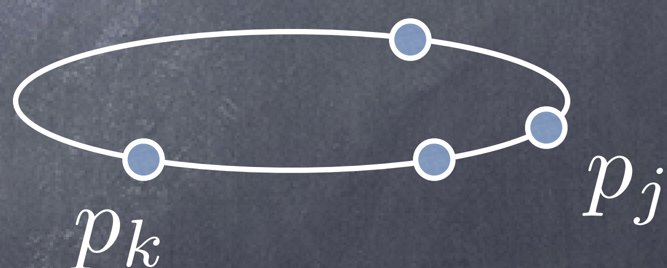
M particles:

ψ constructed using just products of 2-particle S-matrices!

(integrability)

$$E = \sum_{k=1}^{M} \epsilon(p_k) \qquad \gamma = \frac{\lambda}{8\pi^2} E$$

PBCs $\Longrightarrow$ $$\boxed{e^{ip_k L} = \prod_{\substack{j=1 \\ j\neq k}}^{M} S(p_k, p_j)} \qquad k = 1, \ldots, M$$ "Bethe-Yang equations"



cyclicity $\Longrightarrow$ $$P = \sum_{k=1}^{M} p_k = 0$$

Determine allowed p's

Example: L=4, M=2 $\mathrm{tr}\, XZXZ\,, \quad \mathrm{tr}\, XXZZ$

"SU(2) Konishi" - in same $\mathcal{N}$=4 multiplet

$\therefore$ same anomalous dims

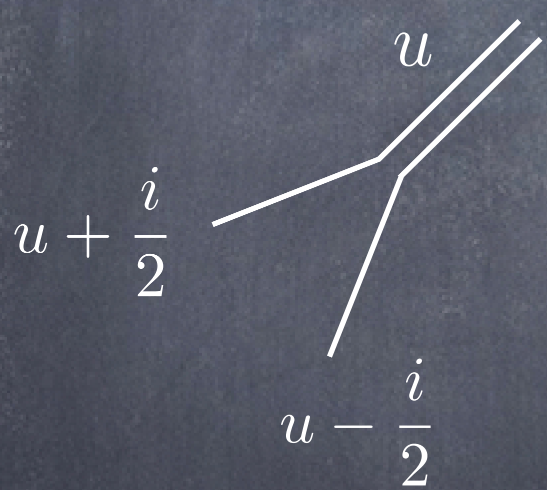
$$p_1 = -p_2 = \frac{2\pi}{3} \qquad \Rightarrow \qquad \gamma = \frac{3\lambda}{4\pi^2} \qquad ✓$$

Remark:

$$S(p_2\,,p_1) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}$$

has pole at

$$u_1 = u - \frac{i}{2}\,, \qquad u_2 = u + \frac{i}{2}$$



2-particle bound state

Similarly, Q-particle bound states (Q=1,2,3, ...)

S-matrices $S^{(Q_1,Q_2)}$

Higher loops:

[Staudacher 04]

long-range interaction $\Longrightarrow$ S-matrix is "asymptotic"

(particles widely separated)

S-matrix gets higher-loop corrections

$\uparrow$

Brute force possible only for low n,
since $\Gamma^{(n)}$ known only for low n ☹

How to get all-loop result?

Audacious idea: "guess" the exact S-matrix! [Beisert 05]

~ sine-Gordon [Zamolodchikov 77]

Guiding principle: symmetry

Global symmetry algebra is psu(2,2|4)

Partially broken by the vacuum $|0\rangle = |Z^L\rangle$:

$$su(2) \to u(1)$$

$$so(6) \to so(4) = su(2) \times su(2)$$

$$psu(2,2|4) \to psu(2|2) \times psu(2|2) \ltimes \underset{\substack{\uparrow \\ \text{common central charge}}}{\mathbb{R}}$$

$$\mathbb{H} \equiv \mathcal{D} - J_{56}$$

$$J_{56}|0\rangle = L|0\rangle \Rightarrow \quad \mathbb{H}|0\rangle = 0 \qquad ✓$$

Elementary excitations:

$$\sum_{x=1}^{L} e^{ipx} \,|\, \overset{\overset{1}{\downarrow}}{Z} \cdots \overset{\overset{x}{\downarrow}}{\chi} \cdots \overset{\overset{L}{\downarrow}}{Z} \rangle \qquad \chi \in \{X, Y, \bar{X}, \bar{Y}, \ldots\}$$

Classified by unbroken symmetry

Fundamental reps $\quad \boxed{\begin{matrix} su(2|2) \\ \square \end{matrix}} \times \begin{matrix} su(2|2) \\ \square \end{matrix}$

Zamolodchikov-Faddeev operators:

$$A^{\dagger}_{k\dot{k}}(p) = \boxed{A^{\dagger}_{k}(p)} \otimes \dot{A}^{\dagger}_{\dot{k}}(p) \qquad k = 1, \ldots, 4, \quad \dot{k} = \dot{1}, \ldots, \dot{4}$$

Let's focus on just 1 copy

$$A_i^\dagger(p_1)\, A_j^\dagger(p_2) = \boxed{S_{i\,j}^{i'\,j'}(p_1, p_2)}\, A_{j'}^\dagger(p_2)\, A_{i'}^\dagger(p_1)$$

S-matrix

16 x 16

Assumptions:

- $A_i^\dagger(p)|0\rangle$ form a rep of $\widetilde{su}(2|2)$

centrally extended

$\widetilde{su}(2|2)$

Generators:

bosonic $\mathbb{L}_a{}^b\,, \quad \mathbb{R}_\alpha{}^\beta \qquad a, b = 1, 2, \quad \alpha, \beta = 3, 4$

SUSY $\mathbb{Q}_\alpha{}^a\,, \quad \mathbb{Q}_a^{\dagger\alpha}$

central charges $\mathbb{H}\,, \quad \mathbb{C}\,, \quad \mathbb{C}^\dagger$

Algebra:

$$\left\{\mathbb{Q}_\alpha{}^a\,,\mathbb{Q}_\beta{}^b\right\} = \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C}\,, \quad \left\{\mathbb{Q}_a^{\dagger\alpha}\,,\mathbb{Q}_b^{\dagger\beta}\right\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathbb{C}^\dagger$$

$$\left\{\mathbb{Q}_\alpha{}^a\,,\mathbb{Q}_b^{\dagger\beta}\right\} = \delta_b^a\mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta\mathbb{L}_b{}^a + \frac{1}{2}\delta_b^a\delta_\alpha^\beta\mathbb{H}$$

$$\vdots$$

Action of symmetry generators on ZF operators:

bosonic:

$$[\mathbb{L}_a{}^b, A_c^\dagger(p)] = (\delta_c^b \delta_a^d - \frac{1}{2}\delta_a^b \delta_c^d) A_d^\dagger(p) , \quad [\mathbb{L}_a{}^b, A_\gamma^\dagger(p)] = 0$$

$$[\mathbb{R}_\alpha{}^\beta, A_\gamma^\dagger(p)] = (\delta_\gamma^\beta \delta_\alpha^\delta - \frac{1}{2}\delta_\alpha^\beta \delta_\gamma^\delta) A_\delta^\dagger(p) , \quad [\mathbb{R}_\alpha{}^\beta, A_c^\dagger(p)] = 0$$

SUSY:

$$\mathbb{Q}_\alpha{}^a \, A_b^\dagger(p) = e^{-ip/2} \left[a(p) \delta_b^a A_\alpha^\dagger(p) + A_b^\dagger(p) \, \mathbb{Q}_\alpha{}^a \right]$$

$$\mathbb{Q}_\alpha{}^a \, A_\beta^\dagger(p) = e^{-ip/2} \left[b(p) \epsilon_{\alpha\beta} \epsilon^{ab} A_b^\dagger(p) - A_\beta^\dagger(p) \, \mathbb{Q}_\alpha{}^a \right]$$

$$\mathbb{Q}_a^{\dagger\alpha} \, A_b^\dagger(p) = e^{ip/2} \left[c(p) \epsilon_{ab} \epsilon^{\alpha\beta} A_\beta^\dagger(p) + A_b^\dagger(p) \, \mathbb{Q}_a^{\dagger\alpha} \right]$$

$$\mathbb{Q}_a^{\dagger\alpha} \, A_\beta^\dagger(p) = e^{ip/2} \left[d(p) \delta_\beta^\alpha A_a^\dagger(p) - A_\beta^\dagger(p) \, \mathbb{Q}_a^{\dagger\alpha} \right]$$

central:

$$\mathbb{C}\, A_i^\dagger(p) = e^{-ip}\left[a(p)b(p)A_i^\dagger(p) + A_i^\dagger(p)\,\mathbb{C}\right]$$

$$\mathbb{C}^\dagger\, A_i^\dagger(p) = e^{ip}\left[c(p)d(p)A_i^\dagger(p) + A_i^\dagger(p)\,\mathbb{C}^\dagger\right]$$

$$\mathbb{H}\, A_i^\dagger(p) = \left[a(p)d(p) + b(p)c(p)\right] A_i^\dagger(p) + A_i^\dagger(p)\,\mathbb{H}$$

Determination of a, b, c, d:

- $A_i^\dagger(p)|0\rangle$ form a rep of algebra $\Longrightarrow \quad ad - bc = 1$
- rep unitary $\Longrightarrow \quad d = a^*\,, c = b^*$
- $\mathbb{C}$ on 2-particle states $\Longrightarrow \quad ab = ig(e^{ip} - 1)$

  g constant

Consistent with

$$a = \sqrt{g}\eta\,, \quad b = \sqrt{g}\frac{i}{\eta}\left(\frac{x^+}{x^-} - 1\right)\,, \quad c = -\sqrt{g}\frac{\eta}{x^+}\,, \quad d = \sqrt{g}\frac{x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right)$$

where

$$\boxed{x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}, \quad \frac{x^+}{x^-} = e^{ip}}, \quad \eta = e^{i\frac{p}{4}}\sqrt{i(x^- - x^+)}$$

For 1-particle states:

$$\mathbb{H} = ig\left(x^- - \frac{1}{x^-} - x^+ + \frac{1}{x^+}\right) = \sqrt{1 + 16g^2\sin^2\frac{p}{2}}$$

Compare with 1 loop to determine g:

$$\begin{aligned}\Delta - J &= (\Delta_0 + \gamma) - J \\ &= 1 + \gamma \\ &= 1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2} + \cdots\end{aligned}$$

$$\boxed{g = \frac{\sqrt{\lambda}}{4\pi}}$$

Consistent with

$$a=\sqrt{g}\eta\,,\quad b=\sqrt{g}\frac{i}{\eta}\left(\frac{x^+}{x^-}-1\right)\,,\quad c=-\sqrt{g}\frac{\eta}{x^+}\,,\quad d=\sqrt{g}\frac{x^+}{i\eta}\left(1-\frac{x^-}{x^+}\right)$$

where

$$\boxed{x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}\,, \qquad \frac{x^+}{x^-} = e^{ip}}\,, \qquad \eta = e^{i\frac{p}{4}}\sqrt{i(x^- - x^+)}$$

For 1-particle states:

$$\mathbb{H} = ig\left(x^- - \frac{1}{x^-} - x^+ + \frac{1}{x^+}\right) = \boxed{\sqrt{1+16g^2\sin^2\frac{p}{2}}}$$

Compare with 1 loop to determine g:

exact!

$$\begin{aligned}\Delta - J &= (\Delta_0+\gamma) - J\\ &= 1+\gamma\\ &= 1+\frac{\lambda}{2\pi^2}\sin^2\frac{p}{2}+\cdots\end{aligned}$$

$$\boxed{g=\frac{\sqrt{\lambda}}{4\pi}}$$

- $\widetilde{su}(2|2)$ generators $\mathcal{J}$ commute with 2-particle scattering

i.e, consider $\mathcal{J}\, A_i^\dagger(p_1)\, A_j^\dagger(p_2)|0\rangle$

Can first exchange $A^\dagger$'s, then move $\mathcal{J}$ to right $\mathcal{J}|0\rangle = 0$

Or first move $\mathcal{J}$ to right, then exchange $A^\dagger$'s

Consistency $\Rightarrow$ linear equations for S-matrix elements bosonic generators $\Rightarrow$

$$S_{a\,a}^{a\,a} = A\,, \quad S_{\alpha\,\alpha}^{\alpha\,\alpha} = D\,,$$

$$S_{a\,b}^{a\,b} = \frac{1}{2}(A - B)\,, \quad S_{a\,b}^{b\,a} = \frac{1}{2}(A + B)\,,$$

$$S_{\alpha\,\beta}^{\alpha\,\beta} = \frac{1}{2}(D - E)\,, \quad S_{\alpha\,\beta}^{\beta\,\alpha} = \frac{1}{2}(D + E)\,,$$

$$S_{a\,b}^{\alpha\,\beta} = -\frac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}\,C\,, \quad S_{\alpha\,\beta}^{a\,b} = -\frac{1}{2}\epsilon^{ab}\epsilon_{\alpha\beta}\,F\,,$$

$$S_{a\,\alpha}^{a\,\alpha} = G\,, \quad S_{a\,\alpha}^{\alpha\,a} = H\,, \quad S_{\alpha\,a}^{a\,\alpha} = K\,, \quad S_{\alpha\,a}^{\alpha\,a} = L\,,$$

$a\,, b \in \{1\,, 2\}, \quad a \neq b$ $\qquad\qquad$ $\alpha\,, \beta \in \{3\,, 4\}, \quad \alpha \neq \beta$

SUSY generators $\Rightarrow$

$$A = S_0 \frac{x_2^- - x_1^+}{x_2^+ - x_1^-} \frac{\eta_1 \eta_2}{\tilde{\eta}_1 \tilde{\eta}_2} \,,$$

$$B = -S_0 \left[\frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2 \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)} \right] \frac{\eta_1 \eta_2}{\tilde{\eta}_1 \tilde{\eta}_2} \,,$$

$$C = S_0 \frac{2i x_1^- x_2^- (x_1^+ - x_2^+) \eta_1 \eta_2}{x_1^+ x_2^+ (x_1^- - x_2^+)(1 - x_1^- x_2^-)} \,, \qquad D = -S_0 \,,$$

$$E = S_0 \left[1 - 2 \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)} \right] \,,$$

$$F = S_0 \frac{2i (x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-) \tilde{\eta}_1 \tilde{\eta}_2} \,,$$

$$G = S_0 \frac{(x_2^- - x_1^-)}{(x_2^+ - x_1^-)} \frac{\eta_1}{\tilde{\eta}_1} \,, \qquad H = S_0 \frac{(x_2^+ - x_2^-)}{(x_1^- - x_2^+)} \frac{\eta_1}{\tilde{\eta}_2} \,,$$

$$K = S_0 \frac{(x_1^+ - x_1^-)}{(x_1^- - x_2^+)} \frac{\eta_2}{\tilde{\eta}_1} \,, \qquad L = S_0 \frac{(x_1^+ - x_2^+)}{(x_1^- - x_2^+)} \frac{\eta_2}{\tilde{\eta}_2}$$

$$\eta_1 = \eta(p_1) e^{i p_2/2} \,, \quad \eta_2 = \eta(p_2) \,, \quad \tilde{\eta}_1 = \eta(p_1) \,, \quad \tilde{\eta}_2 = \eta(p_2) e^{i p_1/2}$$

- Satisfies YBE
- Does not have "difference" property; can be mapped to R-matrix for Hubbard model [Shastry 86]
- Has Yangian symmetry [Beisert 07]
- su(2|2) symmetry determines S-matrix only up to overall scalar factor S_0

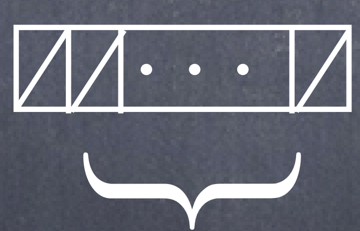
Full S-matrix: 2 copies

$$\mathbb{S}(p_1, p_2) = S(p_1, p_2) \otimes S(p_1, p_2)$$

Scalar factor determined from crossing symmetry

[Janik 06, BES 06, BHL 06, DHM 07, Volin 09]

Q-particle bound states: [Dorey 06]



Q boxes

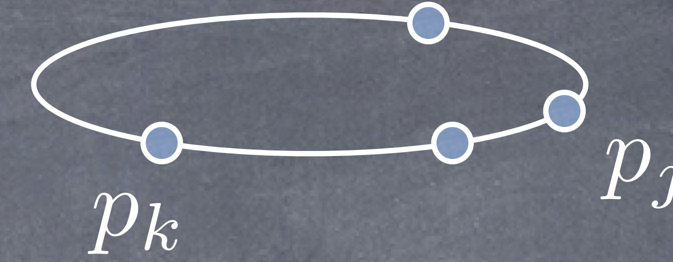
4Q-dimensional reps of su(2|2)

Can determine $S^{(Q_1,Q_2)}$ using also Yangian symmetry

[Beisert 07, Arutyunov, de Leeuw & Torrielli 09]

all-loop asymptotic
Bethe equations

M particles in ring of length L



$\mathbb{S}$ [Beisert 05, 06; Martins & Melo 07]

PBCs $\Longrightarrow$ all-loop asymptotic Bethe eqs [Beisert & Staudacher 05]

Example: SU(2) Konishi $\mathrm{tr}\,XZXZ\,, \quad \mathrm{tr}\,XXZZ$

L=4, M=2 up to L-1=3 loops L > # loops "long"

$$p_1 = -p_2 = \frac{2\pi}{3} - \sqrt{3}g^2 + \frac{9\sqrt{3}}{2}g^4 + \dots \qquad g \equiv \frac{\sqrt{\lambda}}{4\pi}$$

$$\Longrightarrow \quad \gamma = \underbrace{12g^2}_{\frac{3\lambda}{4\pi^2}\ \checkmark} - 48g^4 + 336g^6 + \dots$$

1 loop 2 loops 3 loops

Matches with perturbative gauge theory!

Works for any long operator!

Epilogue

- We don't know the all-loop dilatation operator for single-trace operators in $\mathcal{N}$=4 SYM

- Nevertheless, we know that the (all-loop) anomalous dimensions of "long" operators are given by a set of BEs!

- Key: all-loop S-matrix

- Based on su(2|2) symmetry

- Anomalous dimensions of "short" operators: lectures of Arutyunov & Volin