Effective potential for relativistic scattering

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- Quantum Inverse Scattering
- Nucleon potential from lattice QCD
- Sine-Gordon model
- Sine-Gordon effective potential from QIS

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Quantum Inverse Scattering

Inverse scattering problems

Inverse scattering: find potential from scattering data

All physics is inverse scattering:

Newton's law

Rutherford's experiment

Watson & Crick double helix

Direct scattering: given potential (forces) find scattering data

Inverse scattering: given scattering data find forces

1-dimensional quantum mechanics on the half-line

Schrödinger operator: $\ell u = -u'' + qu$

self-adjoint Hamiltonian: ($\hbar$, m etc. scaled out)

$\ell u = -u'' + qu$ and boundary condition $u(0) = 0$

$q(x)$ potential $x \geq 0$

$$q(x) \sim \frac{p(p-1)}{x^2} \qquad x \to 0 \qquad p > 1$$

$u(x) \sim x^p$ regular solution

$u(x) \sim x^{1-p}$ singular solution

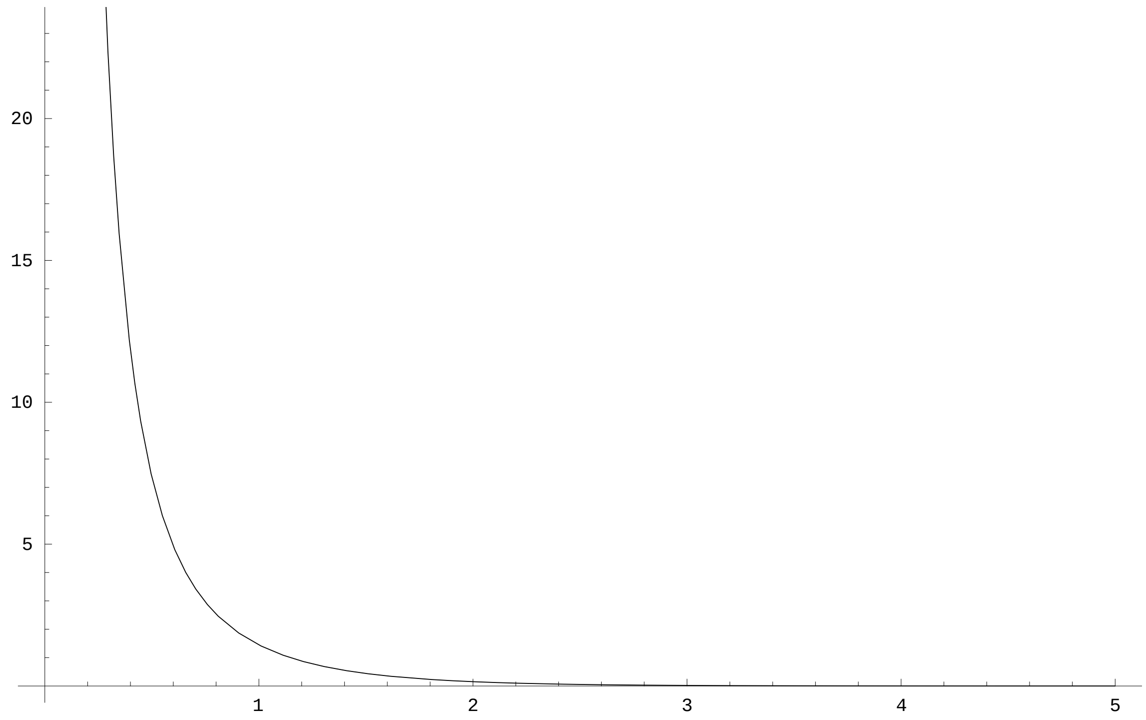


Figure 1: The singular potential $q(x)$

Three solutions of the $\ell u = k^2 u$ Schrödinger equation:

$\phi(x,k) \sim x^p \qquad x \to 0 \qquad$ physical solution

$\tilde{\phi}(x,k) \sim x^{1-p} \qquad x \to 0 \qquad$ singular solution

$f(x,k) \sim \mathrm{e}^{ikx} \qquad x \to \infty \qquad$ Jost solution

$$f^*(x,k) = f(x,-k)$$

Jost function $f(k)$:

$$f(x,k) = \tilde{f}(k)\phi(x,k) + f(k)\tilde{\phi}(x,k)$$

$$\phi(x,k) = \tfrac{2p-1}{2ik}\left\{f(-k)f(x,k) - f(k)f(x,-k)\right\}$$

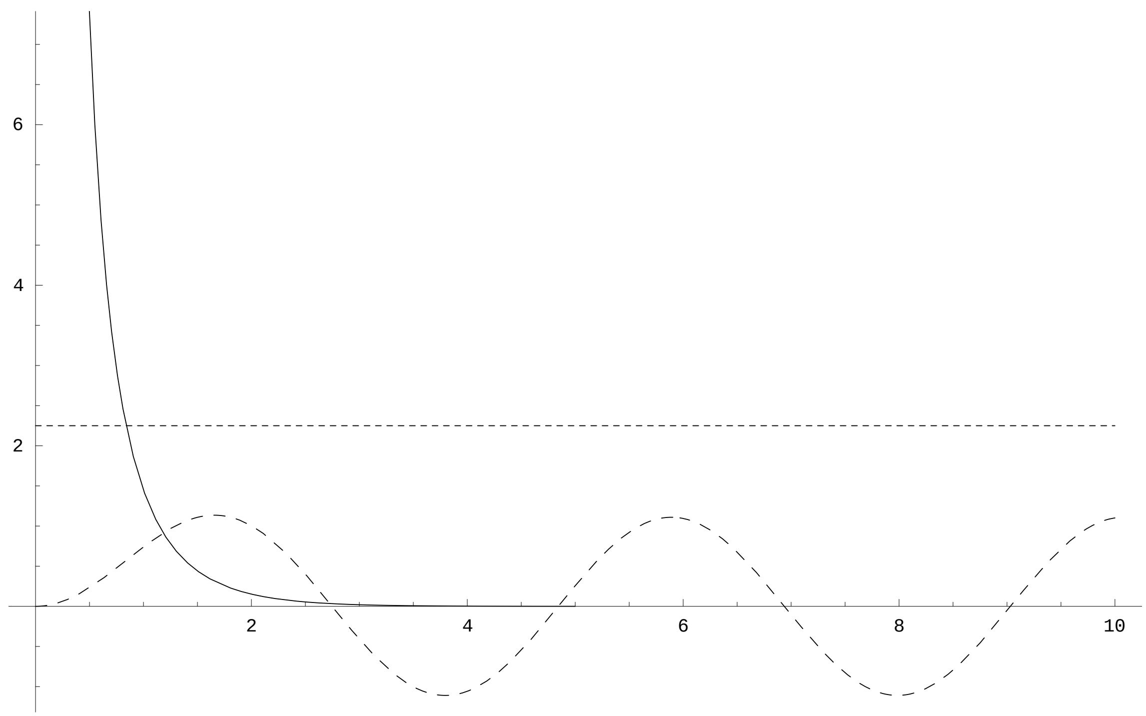


Figure 2: The singular potential $q(x)$ and the regular wave function $\phi(x,k)$ at $k = 1.5$ The constant total energy is also shown.

phase shift $\delta(k)$:

$$f(k) = |f(k)| \mathrm{e}^{-i\delta(k)}$$

S-"matrix":

$$S(k) = \frac{f(-k)}{f(k)} = \mathrm{e}^{2i\delta(k)}$$

asymptotics of the physical solution:

$$\phi(x,k) \sim -\frac{2p-1}{2ik} f(k) \left\{ \mathrm{e}^{-ikx} - S(k) \mathrm{e}^{ikx} \right\}$$

$$\sim \sin\left[kx + \delta(k)\right]$$

(Solvable) example

$$q(x) = \frac{p(p-1)}{\sinh^2(x)}$$

change of variables:

$$u(x) = \mathrm{e}^{ikx} F(z) \qquad\qquad z = \frac{1}{1-\mathrm{e}^{-2x}}$$

hypergeometric equation:

$$z(1-z)F^{''}(z) + [c - (a+b+1)z]F^{'}(z) - abF(z) = 0$$

$$a = p \qquad b = 1-p \qquad c = 1+ik$$

hypergeometric function: ${}_2F_1(a,b,c;z)$

physical solution:

$$\phi(x,k) = \frac{1}{2^p}\left(1-\mathrm{e}^{-2x}\right)^p \mathrm{e}^{ikx}{}_2F_1\left(p, p-ik, 2p; 1-\mathrm{e}^{-2x}\right)$$

Jost solution:

$$f(x,k) = \left(1-\mathrm{e}^{-2x}\right)^p \mathrm{e}^{ikx}{}_2F_1\left(p, p-ik, 1-ik; \mathrm{e}^{-2x}\right)$$

Jost function:

$$f(k) = \frac{1}{2^{p-1}}\frac{\Gamma(1-ik)\Gamma(2p-1)}{\Gamma(p)\Gamma(p-ik)}$$

S-matrix:

$$S(k) = \frac{\Gamma(1+ik)\Gamma(p-ik)}{\Gamma(1-ik)\Gamma(p+ik)}$$

high energy asymptotics:

$$\delta(k) = \frac{\pi}{2}(1-p) - \frac{d_1}{k} + \ldots \qquad \delta(\infty) = \frac{\pi}{2}(1-p)$$

$p = 2$:

$$f(k) = \frac{1}{1-ik} \qquad S(k) = \frac{1-ik}{1+ik}$$

Inverse scattering in three steps

- step 1: scattering data

$$F(x) = \tfrac{1}{2\pi i x} \int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx}\, S'(k)$$

- step 2: Marchenko equation

$$F(x+y) + A(x,y) + \int_{x}^{\infty} \mathrm{d}s A(x,s) F(s+y) = 0$$

- step 3: potential

$$q(x) = -2 \tfrac{\mathrm{d}}{\mathrm{d}x} A(x,x)$$

$p = 2$ example

- step 1: scattering data

$$F(x) = -2\,\mathrm{e}^{-x}$$

- step 2: Marchenko equation

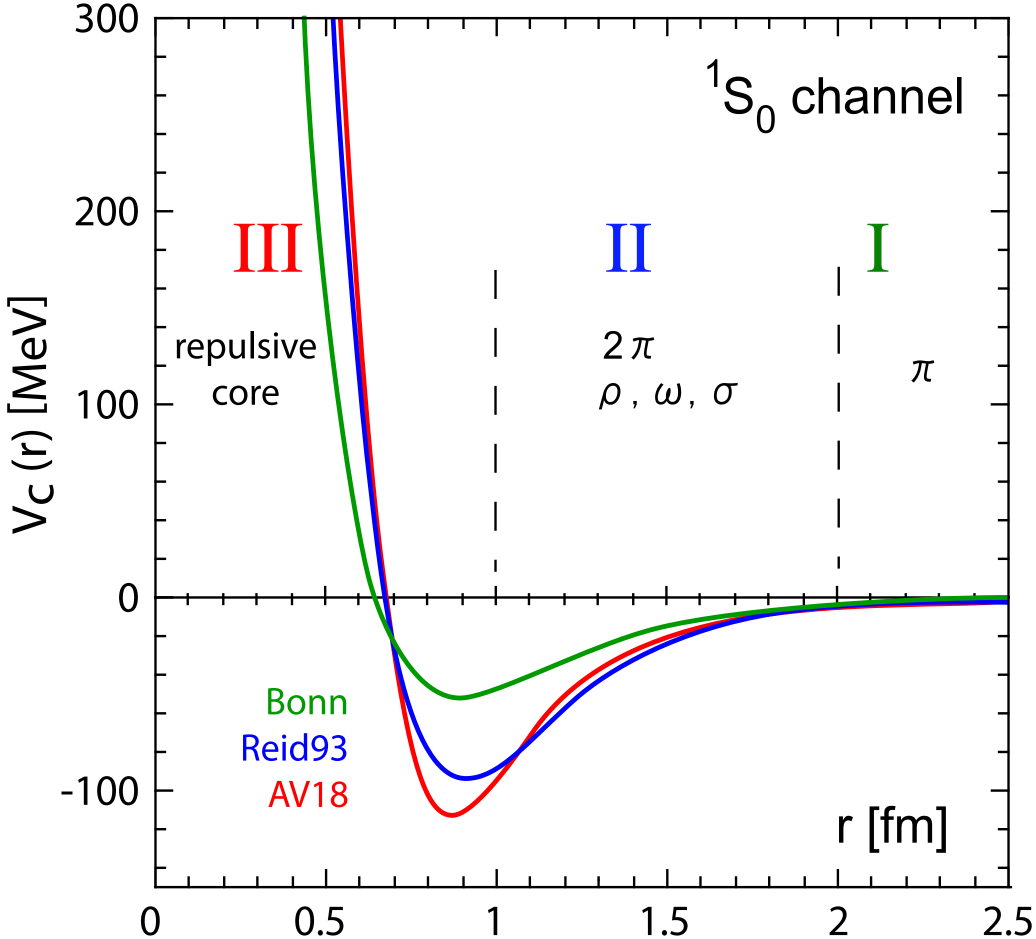
$$A(x,y) = \tfrac{\mathrm{e}^{-y}}{\sinh(x)}$$

- step 3: potential

$$A(x,x) = \coth(x) - 1 \qquad\qquad q(x) = -2\tfrac{\mathrm{d}}{\mathrm{d}x}A(x,x) = \tfrac{2}{\sinh^2(x)}$$

Nucleon potential from first principles

Modern nucleon-nucleon potential



I Long range part

one pion exchange potential (OPEP)

II Medium range part

$\sigma,\ \rho,\ \omega$ exchange

2π exchange

III Short range part

repulsive core (RC)

R. Jastrow(1951)

quark ?

Bonn: Machleidt, Phys.Rev. C63('01)024001

Reid93: Stoks et al., Phys. Rev. C49('94)2950.

AV18: Wiringa et al., Phys.Rev. C51('95) 38.

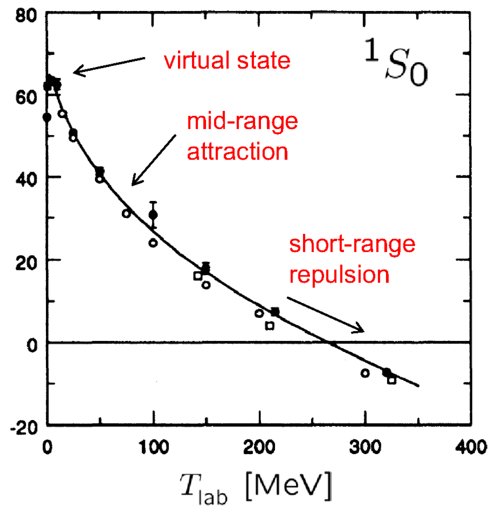
Potentials in QCD ?

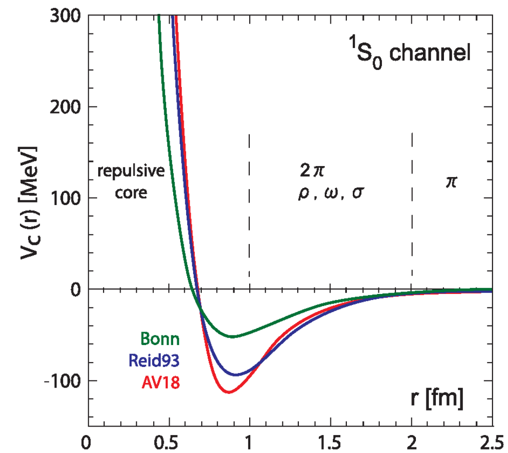
What are "potentials" in quantum field theories such as QCD ?

"Potentials" themselves can NOT be directly measured. analogy: running coupling in QCD

scheme dependent, Unitary transformation

experimental data of scattering phase shifts → potentials, but not unique





useful to "understand" physics

analogy: asymptotic freedom

"Potentials" are useful tools to extract observables such as scattering phase shift. → One may adopt a convenient definition of potentials as long as they reproduce correct physics of QCD.

Our strategy in lattice QCD

Full details: Aoki, Hatsuda & Ishii, PTP123(2010)89.

Consider "elastic scattering"

$$NN \to NN \qquad \sout{NN \to NN + \text{ others}} \quad \sout{(NN \to NN + \pi, NN + \bar{N}N, \cdots)}$$

energy $\quad W_k = 2\sqrt{\mathbf{k}^2 + m_N^2} < W_{\text{th}} = 2m_N + m_\pi \quad$ Elastic threshold

Quantum Field Theoretical consideration

- S-matrix below inelastic threshold. Unitarity gives $\quad S = e^{2i\delta}$

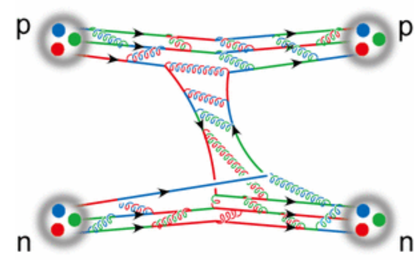
Step 1

define (Equal-time) Nambu-Bethe-Salpeter (NBS) Wave function

Spin model: Balog et al., 1999/2001

$$\varphi_{\mathbf{k}}(\mathbf{r}) = \langle 0|N(\mathbf{x}+\mathbf{r},0)N(\mathbf{x},0)|NN, W_k\rangle$$

QCD eigen-state

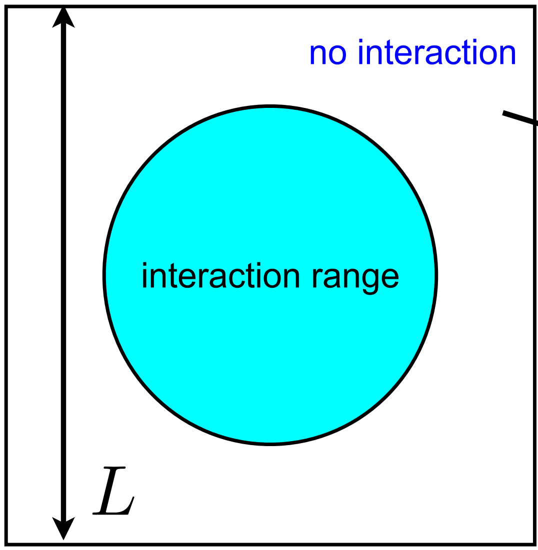


$N(x) = \varepsilon_{abc} q^a(x) q^b(x) q^c(x)$: local operator

"scheme"

Asymptotic behavior of NBS wave function

Lin et al., 2001; CP-PACS, 2004/2005



$$r = |\mathbf{r}| \to \infty$$

$$\varphi^l_{\mathbf{k}} \to A_l \frac{\sin(kr - l\pi/2 + \delta_l(k))}{kr}$$

partial wave

scattering phase shift (phase of the S-matrix by unitarity) in QCD !

NBS wave function ⟺ scattering wave function in quantum mechanics

cf. Luescher's finite volume method: allowed k at L ⇒ $\delta_l(k_n)$

Step 2

• Define a **potential** through application of a Schrödinger operator:

$$V_{\mathbf{k}}(\mathbf{r}) = \frac{\left[E_{\mathbf{k}} + \frac{1}{\mu}\nabla^2\right]\varphi_{\mathbf{k}}(\mathbf{r})}{\varphi_{\mathbf{k}}(\mathbf{r})}$$

where $E_{\mathrm{k}} = \frac{\mathrm{k}^2}{\mu}$ is the kinetic energy. Note the energy dependence of potential!

Step 3

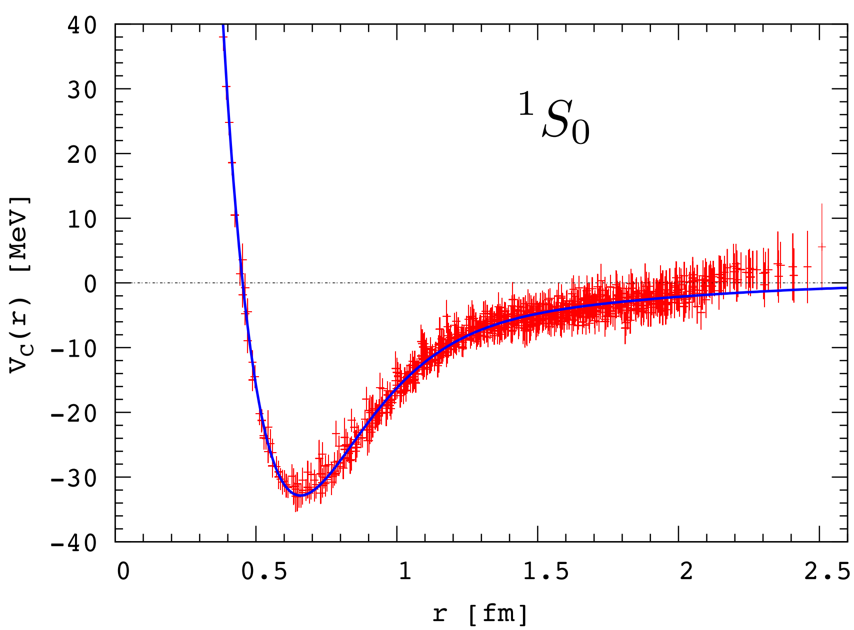
• Solve the Schrödinger equation with this (zero energy) potential in infinite volume to find phase shifts and possible bound states below the inelastic threshold.

Qualitative features of NN potential reproduced!

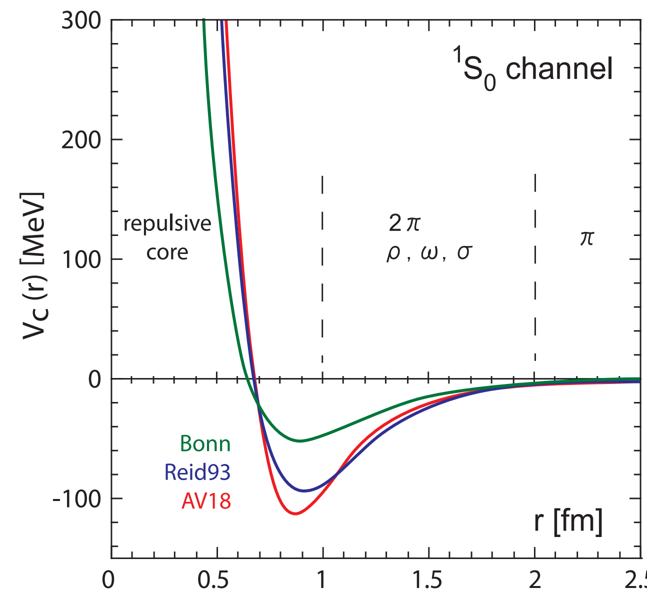
NN potential

2+1 flavor QCD, spin-singlet potential (PLB712(2012)437)

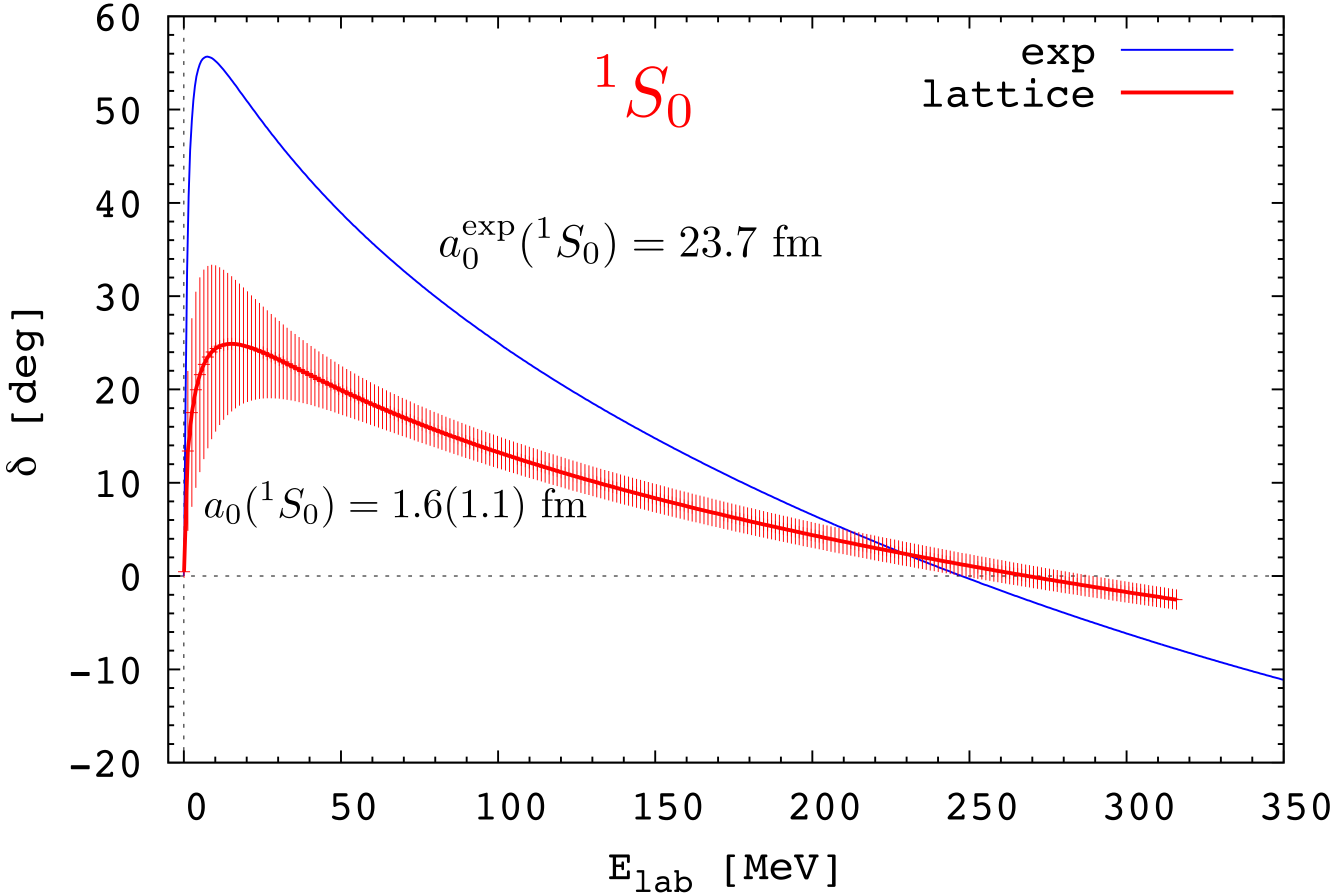
a=0.09fm, L=2.9fm $m_\pi \simeq 700\ \mathrm{MeV}$



phenomenological potential



NN potential ⇨ phase shift



It has a reasonable shape. The strength is weaker due to the heavier quark mass.

Need calculations at physical quark mass.

(courtesy of Sinya Aoki)

Problems with the NBS approach

- Dependence of potential on choice of nucleon operator
- Energy dependence of the NBS potential

Possible solutions:

define nonlocal, but energy-independent potential

define the zero-momentum potential

$$U_o(\mathbf{r}) = \lim_{\mathbf{k}\to 0} V_{\mathbf{k}}^{\mathrm{NBS}}(\mathbf{r})$$

correctly reproduces scattering lengths, but effective range different

Sine-Gordon model

Sine-Gordon model

Very well known: RFT (quantum/classical)

RM (quantum/classical) [alias RS model]

Integrable (solvable) $\Longrightarrow$ (almost) everything calculable

spectrum (solitons, anti-solitons, breathers)

S-matrix

Form-factors, correlators, free energy, ...

Lagrangian ($\hbar = c = 1$ RQFT conventions)

$$\mathcal{L} = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}\phi'^2 + \tfrac{\mu^2}{\beta^2}\cos(\beta\phi)$$

SG coupling β: (equivalence to Thirring model)

$$0 < \beta < \sqrt{8\pi} \qquad\qquad \beta = 2\sqrt{\pi} \quad \text{FF point}$$

equations of motion: (Kl)Sine-Gordon equation

$$\Box\varphi + \mu^2 \sin\varphi = 0 \qquad\qquad (\varphi = \beta\phi)$$

parameters:

$$p = \tfrac{4\pi}{\beta^2} \qquad\qquad \nu = \tfrac{1}{2p-1}$$

soliton mass:

$$m = \frac{2p-1}{\pi}\,\mu$$

bound states (breathers):

$$m_k = 2m\sin\left(\frac{\pi}{2}\nu k\right) \qquad\qquad k = 1, 2, \cdots < 2p-1$$

Ruijsenaars-Schneider RQM description: zero-momentum potential

$$q_o(x) = \frac{4}{\sinh^2(\pi\nu x)}$$

Sine-Gordon S-matrix

Rapidity parametrization: $\theta = \theta_1 - \theta_2$

$$p_i = mc \sinh(\theta_i) \qquad E_i = mc^2 \cosh(\theta_i) \qquad E_i = \sqrt{(mc^2)^2 + (p_i c)^2}$$

soliton-soliton S-matrix (no bound states):

$$\Sigma(\theta) = \exp\left\{ i \int_0^\infty \frac{\mathrm{d}\omega}{\omega} \sin\left(\tfrac{2}{\pi}\theta\omega\right) \frac{\sinh((\nu-1)\omega)}{\cosh(\omega)\sinh(\nu\omega)} \right\}$$

phase shift:

$$\Sigma(\theta) = \mathrm{e}^{2i\delta(\theta)} \qquad \delta(\infty) = \tfrac{\pi}{2}(1-p)$$

for integer p explicit:

$$\Sigma(\theta) = \prod_{m=1}^{p-1} \frac{s_m - i\sinh(\theta)}{s_m + i\sinh(\theta)} \qquad s_m = \sin(m\nu\pi)$$

2-particle scattering

$$\underset{x_1}{\bullet} \longrightarrow p_1 \qquad\qquad p_2 \longleftarrow \underset{x_2}{\bullet}$$

initially: $p_1 > p_2$ $\qquad$ $x_2 > x_1$ all times

asymptotic wave function $(x_2 - x_1) \to \infty$:

$$\Phi(x_1, x_2) \approx \mathrm{e}^{i(k_1 x_1 + k_2 x_2)} + S(p_1, p_2)\mathrm{e}^{i(k_2 x_1 + k_1 x_2)}$$

$k_i = \frac{p_i}{\hbar}$ $\qquad$ $i = 1, 2$ $\qquad$ wave numbers

Relativistic:

$$S_{\mathrm{R}}(p_1, p_2) = -\Sigma(\theta_1 - \theta_2) \qquad \theta_i = \operatorname{arcsinh}\left(\frac{p_i}{mc}\right)$$

NR Schrödinger equation:

$$\hat{\mathcal{H}}\Phi = E\Phi \qquad \hat{\mathcal{H}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + U(x_2 - x_1)$$

separating COM and relative motion:

$$\Phi(x_1, x_2) = \mathrm{e}^{iK(x_1 + x_2)}\,\Psi(x_2 - x_1)$$

effective 1-particle Schrödinger equation:

$$-\frac{\hbar^2}{m}\Psi''(x) + U(x)\Psi(x) = \frac{\hbar^2}{m}\kappa^2\Psi(x)$$

total energy:

$$E = \frac{\hbar^2}{2m}(k_1^2 + k_2^2) = \frac{\hbar^2}{m}(K^2 + \kappa^2)$$

$$k_1 = K + \kappa \qquad\qquad k_2 = K - \kappa$$

asymptotic wave function:

$$\Psi(x) \approx -\mathcal{A}(\kappa)\mathrm{e}^{i\kappa x} + \mathrm{e}^{-i\kappa x} \qquad x \to \infty$$

NR S-matrix:

$$S_{\mathrm{NR}}(p_1, p_2) = -\mathcal{A}\left(\frac{p_1 - p_2}{2\hbar}\right) = -S\left(\frac{p_1 - p_2}{mc}\right)$$

rescaling between physics $\longleftrightarrow$ maths conventions:

$$\mathcal{A}(\kappa) = S(2\kappa L) \qquad\qquad L = \frac{\hbar}{mc} \quad \text{Compton length}$$

Identification?

$$S_{\mathrm{NR}}(p_1, p_2) \sim S_{\mathrm{R}}(p_1, p_2)$$

$$S\left(\frac{p_1 - p_2}{mc}\right) \sim \Sigma\left(\operatorname{arcsinh}\left(\frac{p_1}{mc}\right) - \operatorname{arcsinh}\left(\frac{p_2}{mc}\right)\right)$$

2 special cases of interest:

I (fixed target) $\qquad p_2 = 0 \qquad p_1 = kmc$

$$S_I(k) = \Sigma\left(\operatorname{arcsinh}(k)\right)$$

II (centre of mass) $\qquad p_1 + p_2 = 0 \qquad p_1 - p_2 = kmc$

$$S_{II}(k) = \Sigma\left(2\operatorname{arcsinh}(k/2)\right)$$

Effective Sine-Gordon potential

effective Sine-Gordon potentials

Sine-Gordon NR S-matrix, I determination:

$$S_I(k) = \prod_{m=1}^{p-1} \frac{s_m - ik}{s_m + ik} \qquad s_m = \sin(m\nu\pi)$$

Sine-Gordon NR S-matrix, II determination:

$$S_{II}(k) = \prod_{m=1}^{p-1} \frac{s_m - ik\sqrt{1+k^2/4}}{s_m + ik\sqrt{1+k^2/4}} \qquad s_m = \sin(m\nu\pi)$$

simplest case: I; $p = 2$

$$S_I(k) = \frac{s_1 - ik}{s_1 + ik} = \frac{1 - ik/s_1}{1 + ik/s_1} \qquad s_1 = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

rescaling of basic $p = 2$ case:

$$q_I(x) = \frac{3}{2}\frac{1}{\sinh^2\left(\frac{\sqrt{3}}{2}x\right)}$$

zero-momentum potential:

$$q_o(x) = \frac{4}{\sinh^2\left(\frac{\pi}{3}x\right)}$$

Inverse scattering case I, general p

step1: Fourier transformation

$$F(x) = -\sum_{m=1}^{p-1} R_m \, \mathrm{e}^{-s_m x}$$

residues:

$$R_m = 2s_m \prod_{n \neq m} \frac{s_n + s_m}{s_n - s_m}$$

step2: Marchenko equation

Ansatz:

$$A(x, y) = \sum_{m=1}^{p-1} R_m \, b_m(x) \, \mathrm{e}^{-s_m (x+y)}$$

Marchenko equation reduced to system of algebraic equations:

$$b_m = 1 + \sum_{n=1}^{p-1} \frac{z_n b_n}{s_m + s_n} \qquad z_m(x) = R_m \, \mathrm{e}^{-2 s_m x}$$

step3: calculation of potential

$$A(x,x) = \sum_{m=1}^{p-1} b_m(x) z_m(x)$$

$p = 3$ solution

$$A(x,x) = -s_1 - s_2 + \frac{(s_1^2 - s_2^2)\sinh(s_1 x)\sinh(s_2 x)}{\mathcal{D}(x)}$$

determinant:

$$\mathcal{D}(x) = s_2 \cosh(s_2 x)\sinh(s_1 x) - s_1 \cosh(s_1 x)\sinh(s_2 x)$$

short distance:

$$A(x,x) \approx \frac{3}{x}$$

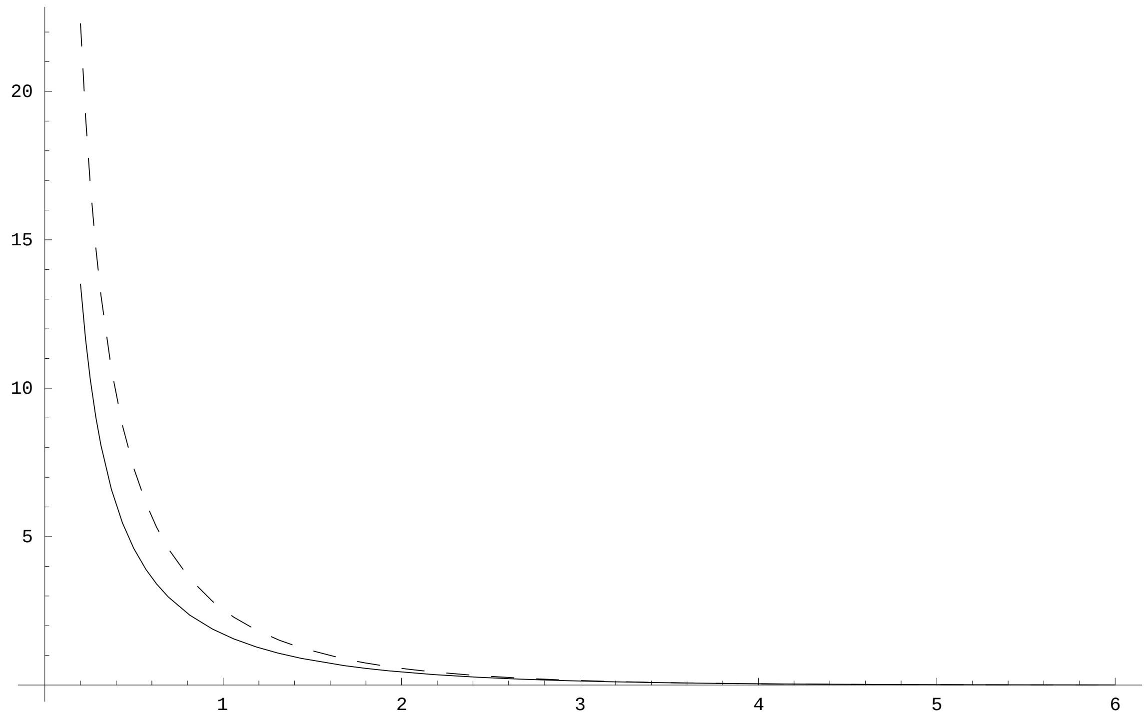


Figure 3: Comparison of the integrated effective potential $A(x, x)$ (solid) and the corresponding zero-momentum $A_o(x, x)$ (dashed) for $p = 3$.

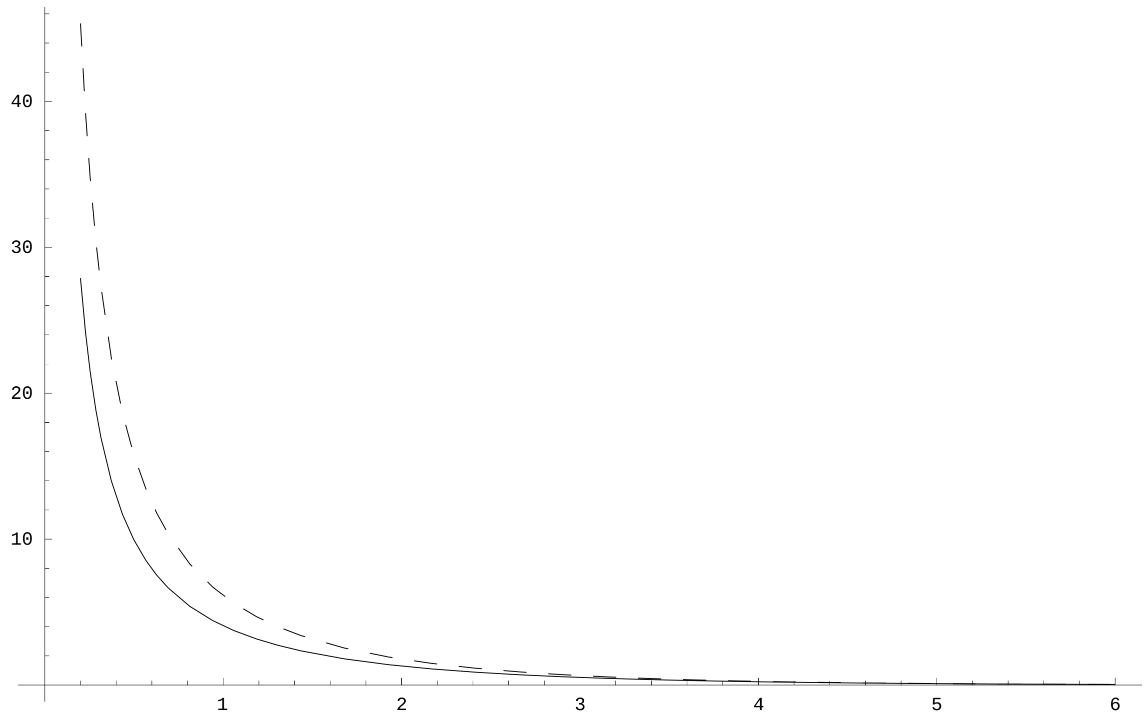


Figure 4: Comparison of the integrated effective potential $A(x,x)$ (solid) and the corresponding zero-momentum $A_o(x,x)$ (dashed) for $p=4$.

Frame dependence

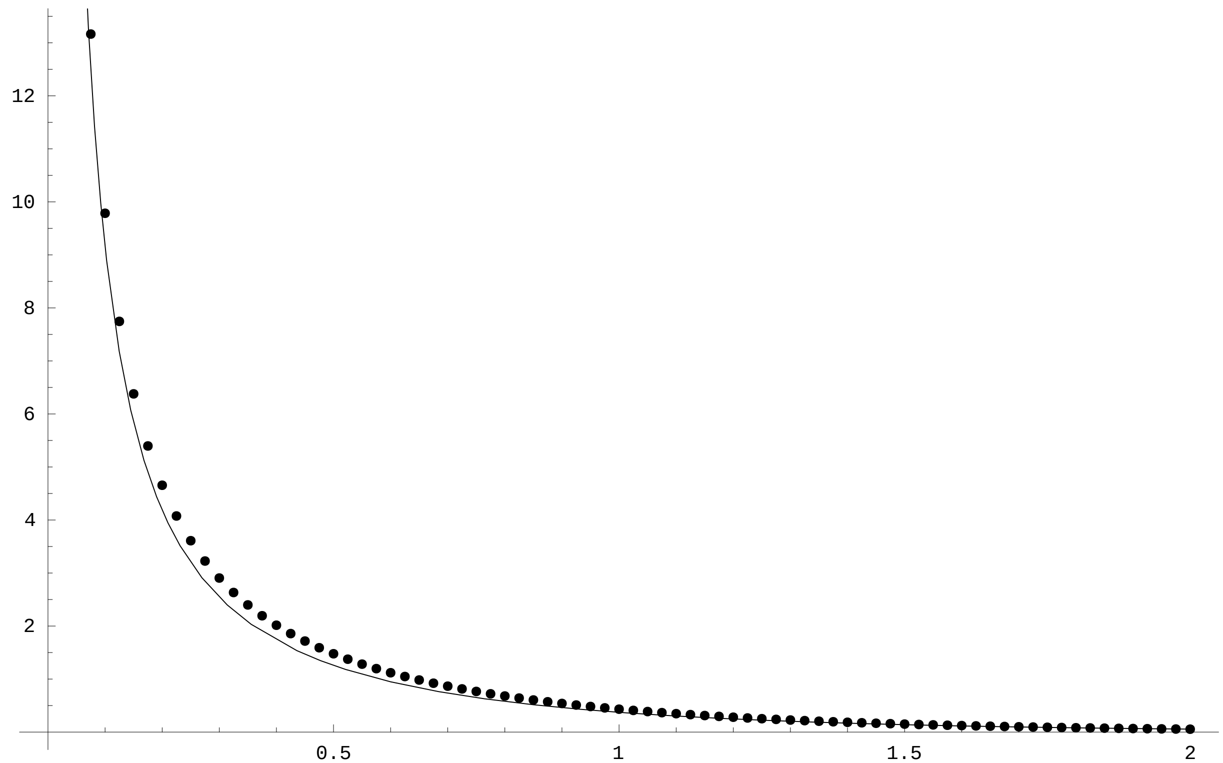


Figure 5: The integrated effective potential in the COM frame for $p = 2$ (dots). For comparison the analytically obtained LAB frame integrated effective potential $A(x, x)$ (solid) is also shown.

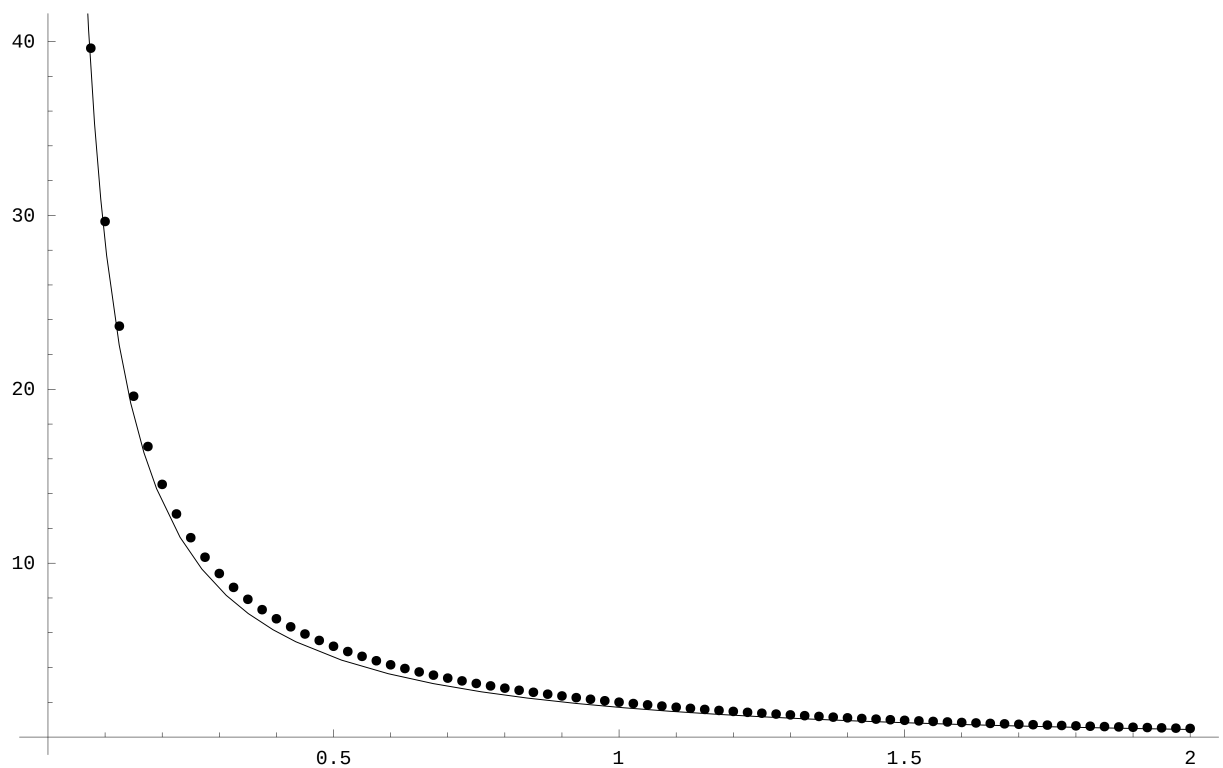


Figure 6: The integrated effective potential in the COM frame for $p = 3$ (dots). For comparison the analytically obtained LAB frame integrated effective potential $A(x, x)$ (solid) is also shown.

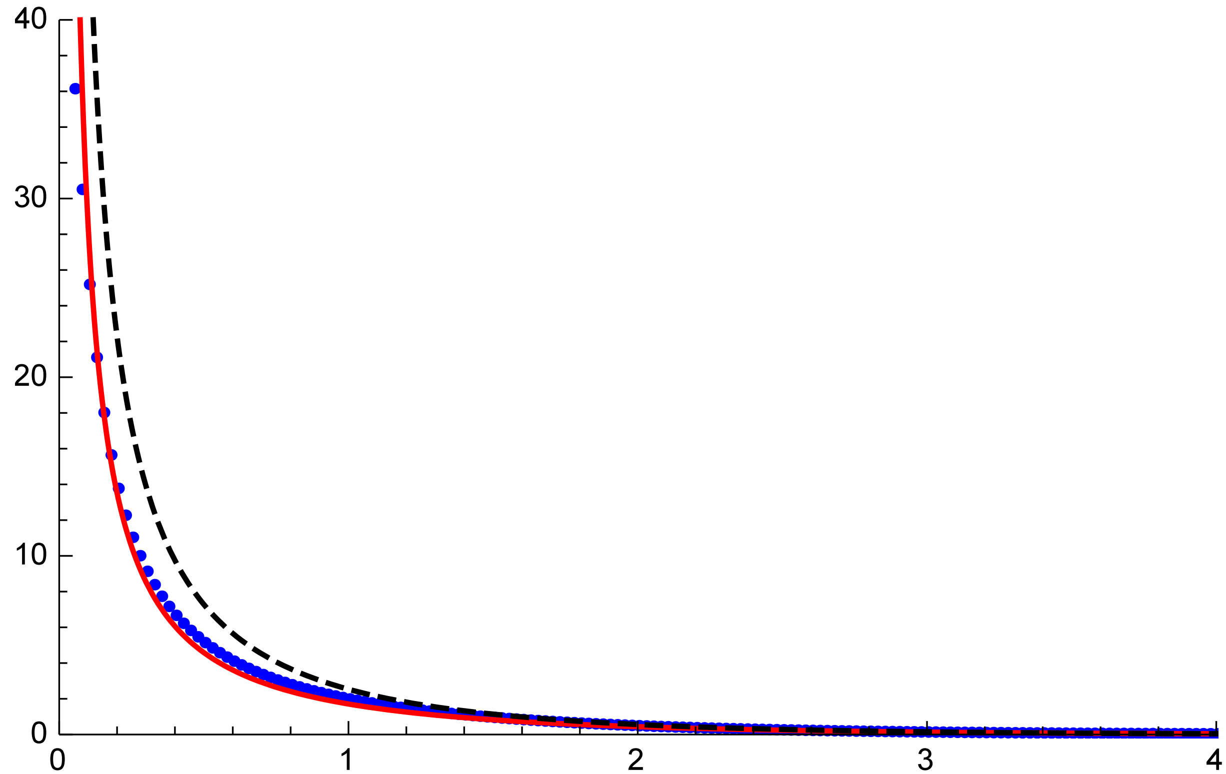


Figure 7: Comparison of integrated SG effective potentials for $p = 3$. The solid (red) line, the (blue) dots and the dashed (black) line are the LAB frame, the COM frame and the zero-momentum potential, respectively.

Thank you!