Non-integrable kink scattering

Laci@70, 15/6/2018

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‘In Hungary the boards never move’

(Some of) Laci's travels: from Budapest...

Guess when? 12:24, 28 June 2002

Update:

13:57, 15 June 2018

. . . to Pohang. . .

20:56, 11 July 2008

. . . to Benasque

20:53, 16 July 2010

Plan

1. **Introduction – the sine-Gordon model on the full line**
2. **The half-line problem**
3. **First warm-up: ϕ^4 kinks and resonant scattering** (old stuff)
4. **Second warm-up: ϕ^4 kinks hitting boundaries**
5. **Back to boundary sine-Gordon**
6. **Conclusions**

1. The sine-Gordon model on the full line

The sine-Gordon model:

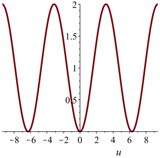
$$u_{tt} - u_{xx} + \sin(u) = 0\,.$$

Finiteness of total energy

$$E[u] = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2 + u_x^2 + 1 - \cos(u)\,dx$$

$\Rightarrow$ u_t^2, u_x^2 and $1-\cos(u)$ to tend to zero as $x \to \pm\infty$ (cf. Wojtek's talk)

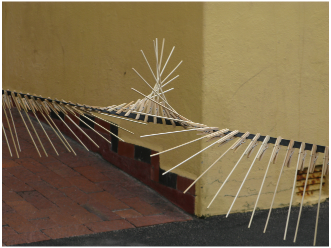
$1-\cos(u)$:



$\Rightarrow\ u(\pm\infty) \in 2\pi\mathbb{Z}$

Initial conditions such that $u(-\infty,0) \neq u(+\infty,0)$ lead to *topological solitons* or *kinks*.

If u is viewed as an angular variable with period 2π, they owe their stability to a non-trivial winding of u as x varies from $-\infty$ to $+\infty$:



These can be placed in sequence along the line, made to move, and scatter against each other. The integrabilty of the sine-Gordon equation means that the time-evolution of solutions is surprisingly simple. In particular the kinks and their reversely-wound friends called antikinks preserve their shapes when they scatter - they are true (integrable) *solitons*. There are also infinitely long-lived kink-antikink bound states called *breathers*, so the space of states is rather rich.

A mechanical model for the scattering of sine-Gordon kinks:

A mechanical model for the scattering of sine-Gordon kinks (2018 version):

A mechanical model for the scattering of sine-Gordon kinks (2018 version):

2. The half-line problem

What about on a half-line, $-\infty < x \leq 0$?

To make the initial value problem well-posed, a boundary condition needs to be imposed at $x = 0$. An interesting question: can this be done in such a way that the half-line theory is still integrable?

For the cases with no additional boundary degrees of freedom, the full two-parameter set of boundary conditions compatible with integrability was found by Ghoshal and Zamolodchikov (1994):

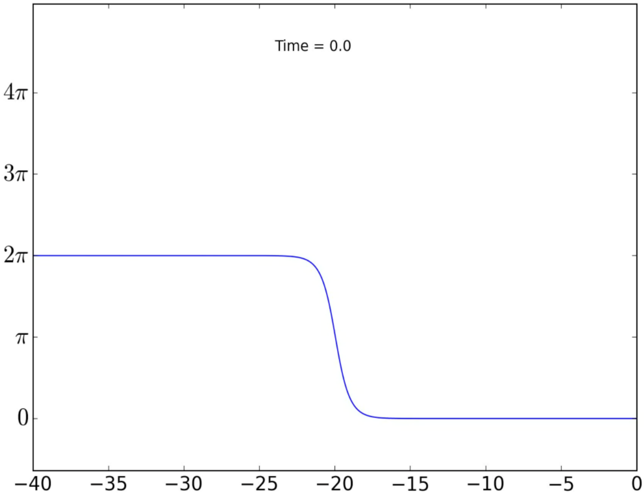
$$\left[u_x + 4K \sin\left(\frac{u - \widehat{u}}{2}\right)\right]\Big|_{x=0} = 0$$

where K, $\widehat{u} \in \mathbb{R}$. (See also MacIntyre (1995).)

The special cases of Dirichlet ($u|_{x=0} = 0$) and Neumann ($u_x|_{x=0} = 0$) were already known to be integrable. GZ found their more-general set by a consideration of the lowest-spin extra sine-Gordon conserved charge.

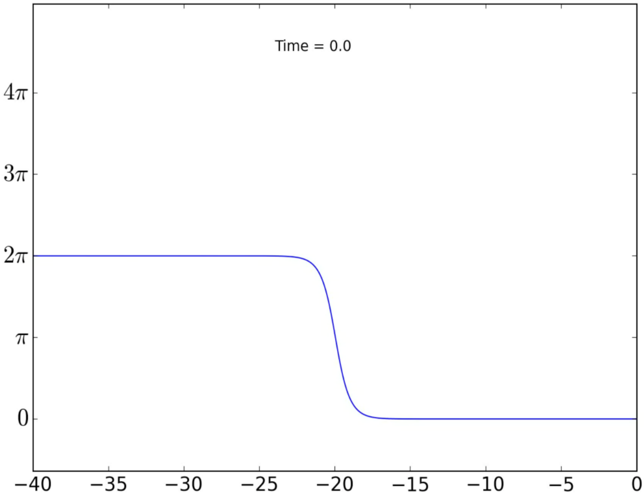
The conservation laws constrain scattering off the boundary to be as simple as in the full-line theory: kinks and antikinks reflect perfectly, as either kinks or antikinks:

Sine-Gordon boundary scattering: $u|_{x=0} = 0$ (Dirichlet):



antikink $\rightarrow$ antikink

Sine-Gordon boundary scattering: $u_x|_{x=0} = 0$ (Neumann):



antikink $\rightarrow$ kink

...but the real world is not integrable!

Imposing non-integrable boundary conditions while leaving the bulk unchanged would be a 'minimal' way to break integrability (just at one point – what harm could that possibly do?).

A natural choice which also interpolates between Dirichlet and Neumann is the (homogeneous) Robin condition.

Instead of the $\widehat{u} = 0$ GZ condition

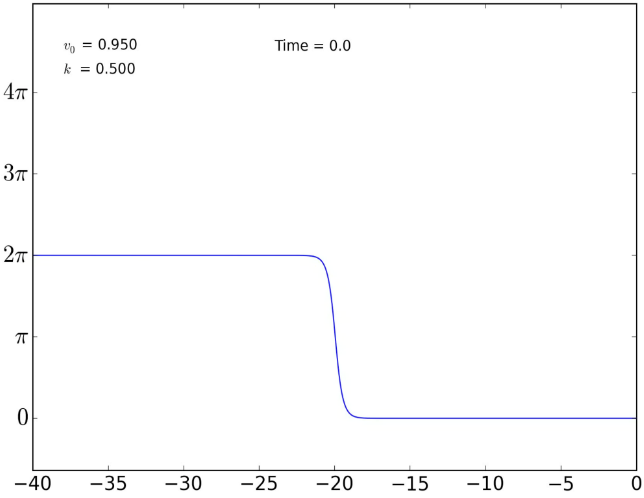
$$\left[u_x + 4K \sin\left(\frac{u}{2}\right)\right]\Big|_{x=0} = 0,$$

we linearise and impose the Robin condition

$$[u_x + 2ku]\big|_{x=0} = 0.$$

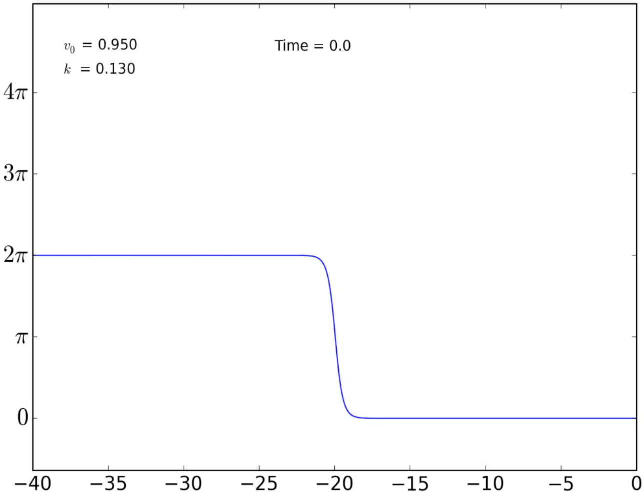
$k \to \infty$ is Dirichlet; $k = 0$ is Neumann. Away from these limits, the Robin boundary does not interact nicely with the higher sG conserved charges and boundary scattering becomes much more complicated.

Sine-Gordon boundary scattering (1/5): $(u_x + u)|_{x=0} = 0$, $v_0 = 0.95$



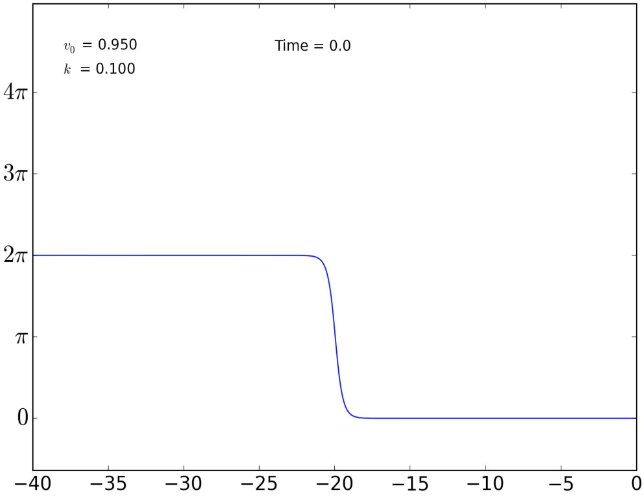
$k = 0.5$ (nearly Dirichlet): antikink $\rightarrow$?

Sine-Gordon boundary scattering (2/5): $(u_x + 0.26\,u)|_{x=0} = 0,\ v_0 = 0.95$



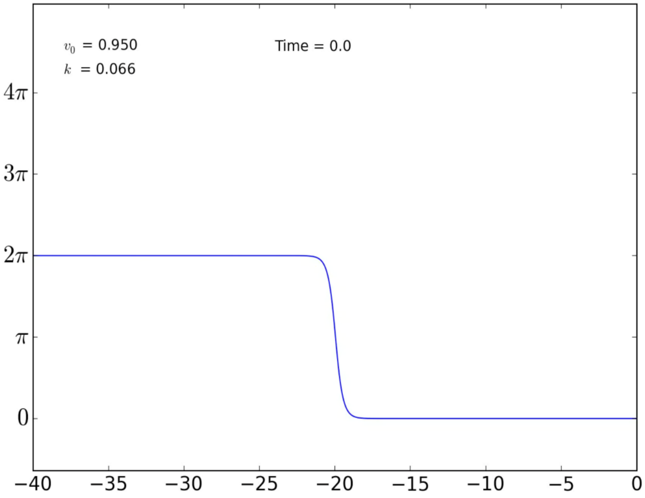
$k = 0.13$: antikink $\to$?

Sine-Gordon boundary scattering (3/5): $(u_x + 0.2\,u)|_{x=0} = 0,\ v_0 = 0.95$



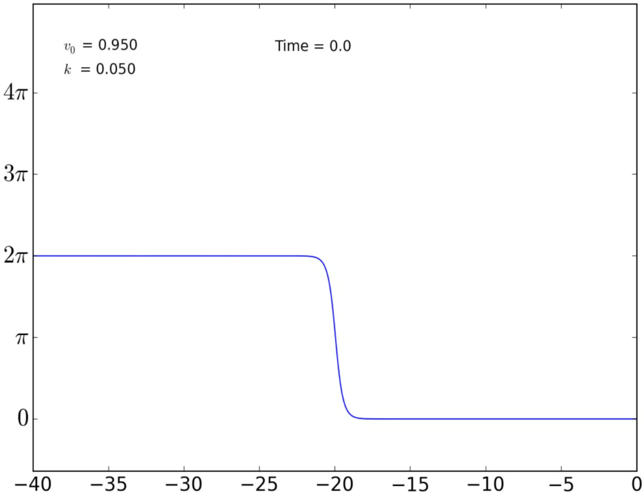
$k = 0.1$: antikink $\to$?

Sine-Gordon boundary scattering (4/5): $(u_x + 0.132\,u)|_{x=0} = 0$, $v_0 = 0.95$



$k = 0.066$: antikink $\rightarrow$?

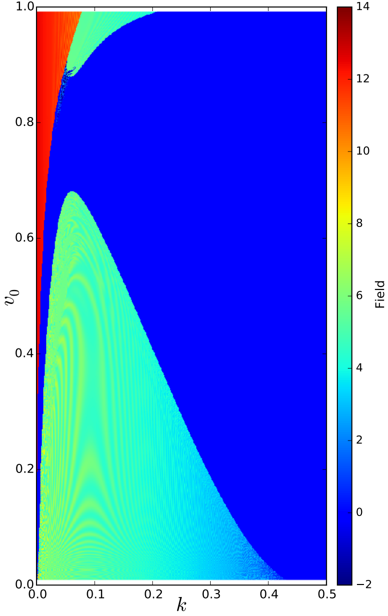
Sine-Gordon boundary scattering (5/5): $(u_x + 0.1\,u)|_{x=0} = 0$, $v_0 = 0.95$



$k = 0.05$ (nearly Neumann): antikink $\to$?

These plots were all for one specific initial velocity, $v_0 = 0.95$. What about the overall picture for general k and v_0?

Robin boundary scattering: phase diagram



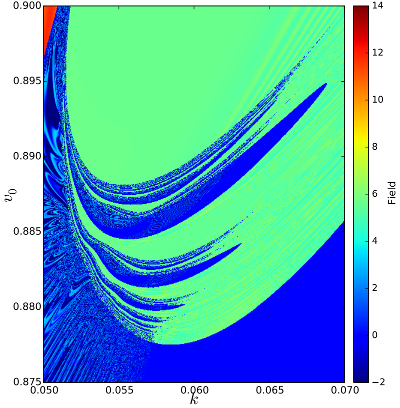
A snapshot of u_l, the late-time field value at $x = 0$ for the scattering of an initial sine-Gordon antikink with velocity v_0 and on a Robin boundary with parameter k.

Roughly speaking:

Emitted kink $\Rightarrow u_l \approx 4\pi$ (red);
Emitted antikink $\Rightarrow u_l \approx 0$ (blue);
Neither/both $\Rightarrow u_l \approx 2\pi$ (light green).

The blur near the top left hides even more complexity...

Robin boundary scattering: zoomed-in phase diagram



A zoomed-in snapshot near the top left of the previous slide.

Dark blue bands correspond an antikink being emitted; in light green areas breathers, or maybe kink-antikink pairs, are emitted. In between these areas are indeterminate regions where a very slight change in the initial parameters can cause an antikink to be produced or not.

Questions

- How to disentangle the general final state? What is its soliton content?

- More generally, what is the reason for the complicated, almost fractal, structures observed in some parts of the phase diagram?

For the first question, the 'direct' part of the inverse scattering method allowed us (PED + Robert Parini) to make progress (later, if time).

For the second, it turns out that despite being bulk-integrable, the story is particularly complicated for sine-Gordon due to the variety of stable excitations in the bulk theory – not just kinks and antikinks, but also breathers. So I'll illustrate the basic mechanisms first via in-some-senses simpler examples where bulk integrability is also lost: the ϕ^4 theory and some generalisations, which are of independent interest (coming next).

3. First warmup: ϕ^4 kinks and resonant scattering

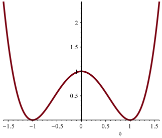
Switch attention to a scalar field $\phi(x,t)$ with energy and Lagrangian densities $\mathcal{E} = \mathcal{T} + \mathcal{V}$ and $\mathcal{L} = \mathcal{T} - \mathcal{V}$, where

$$\mathcal{T} = \tfrac{1}{2}\phi_t^2 \qquad \text{and} \qquad \mathcal{V} = \tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}(\phi^2-1)^2 \ .$$

and equation of motion $\phi_{tt} - \phi_{xx} + 2\phi(\phi^2-1) = 0$.

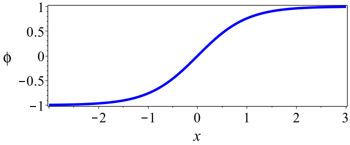
- Total energy $= E[\phi] = \int_{-\infty}^{\infty} \mathcal{E}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \phi_t^2 + \phi_x^2 + (\phi^2-1)^2\, dx\,.$
- For $E[\phi]$ to be finite, ϕ_t^2, ϕ_x^2 and $(\phi^2-1)^2$ must tend to zero as $x \to \pm\infty$.

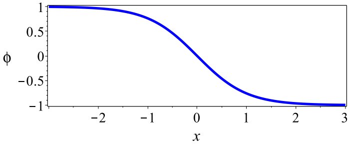
$(\phi^2-1)^2$:



- Hence instead of the infinitely many vacua of sine-Gordon, the theory has just two, $\phi = \pm 1$, and finite energy $\Rightarrow \phi(\pm\infty) \in \{\pm 1\}$.

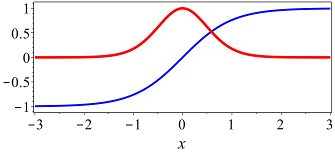
The minimal energy configurations with $\phi(-\infty) \neq \phi(\infty)$ are the topological solitons of the model, the (static) *kinks* and *antikinks*

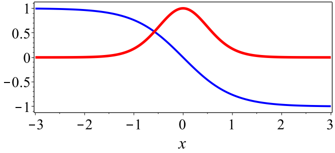




$$\phi_K(x) = \tanh(x - x_0) \quad , \quad \phi_{\bar{K}}(x) = -\tanh(x - x_1)$$

with energy densities localised near to $x = x_0$ or $x = x_1$:



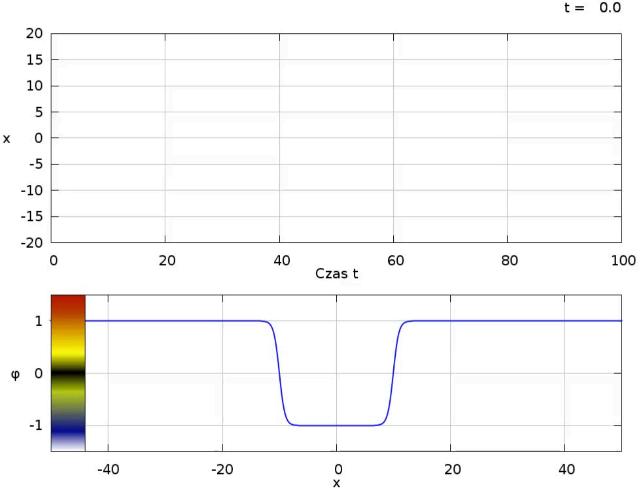


$$\mathcal{E}[\phi_K] = \text{sech}^4(x - x_0) \quad , \quad \mathcal{E}[\phi_{\bar{K}}] = \text{sech}^4(x - x_1)$$

The kink and antikink have rest mass $4/3$, and *attract* each other with an asymptotic force $F \sim 32e^{-2R}$, where $R = |x_1 - x_0|$.

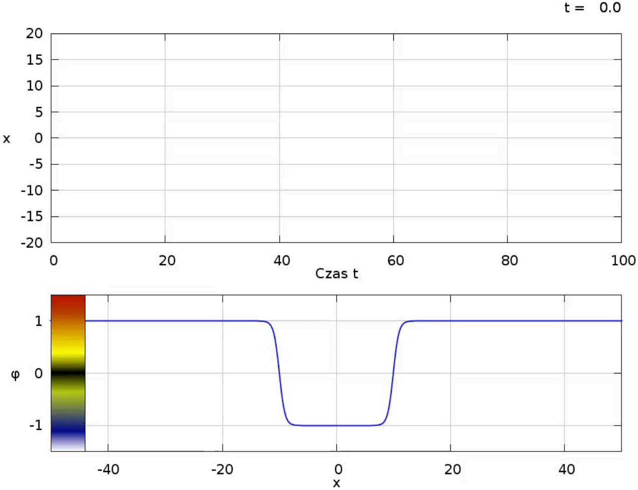
If a K and $\bar{K}$ are oppositely-boosted and scattered, then for large enough initial velocities they bounce off each other. However, the theory is not integrable, and so some energy is lost from the translational modes in the process. . .

ϕ^4 kink scattering: $v_i = 0.27$



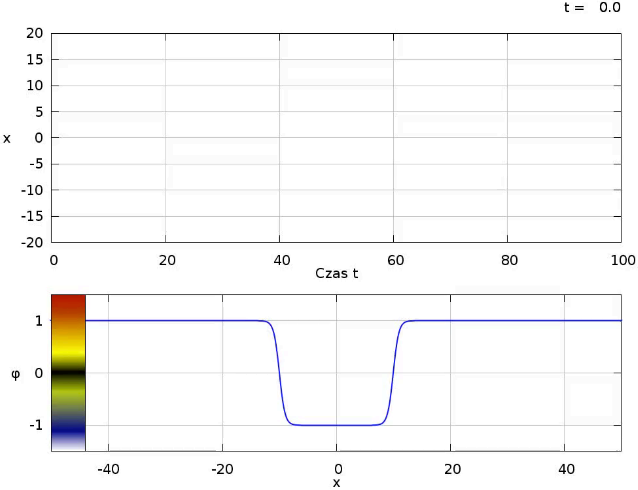
If the initial velocity is reduced below some critical value v_c, one would expect there to be so little energy left in the translational modes after the collision that the kink and antikink can no longer overcome the attractive force between them and separate, and are instead trapped:

ϕ^4 kink scattering: $v_i = \cancel{0.27}\ 0.24$



However there is a surprise waiting if the velocity is reduced further:

ϕ^4 kink scattering: $v_i = \cancel{0.27}\ \cancel{0.24}\ 0.225$

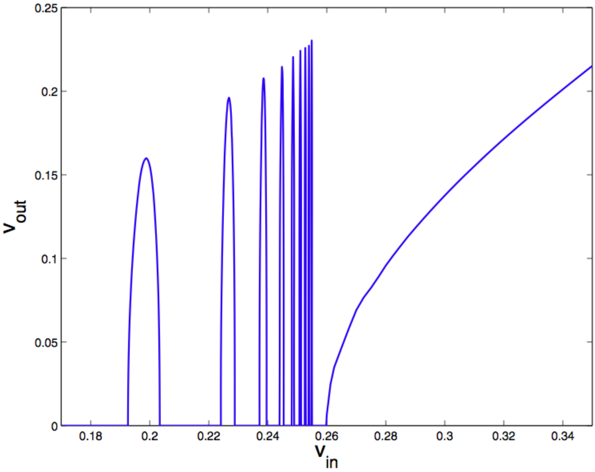


Thus there is at least one ‘escape window’: a range of velocities below the first trapping velocity v_c within which the kink and antikink are again able to separate.

This was first observed in the 1970s by, among others, Ablowitz, Kruskal and Ladik. A theoretical explanation was found by Campbell and collaborators in the 1980s and elaborated by many others since; see for example Goodman and Haberman (2005).

The full picture is surprisingly rich. There is an initial sequence of ‘two-bounce’ windows:

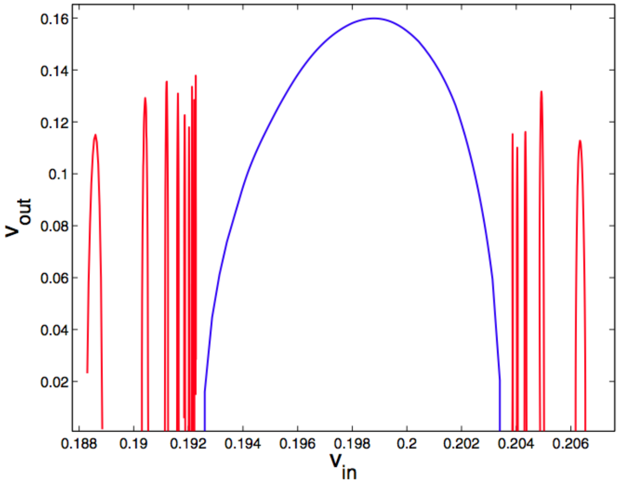
ϕ^4 kink scattering: the first windows



(From Goodman and Haberman, 2005)

However at the edges of each of these windows there are sequences of further ‘baby windows’:

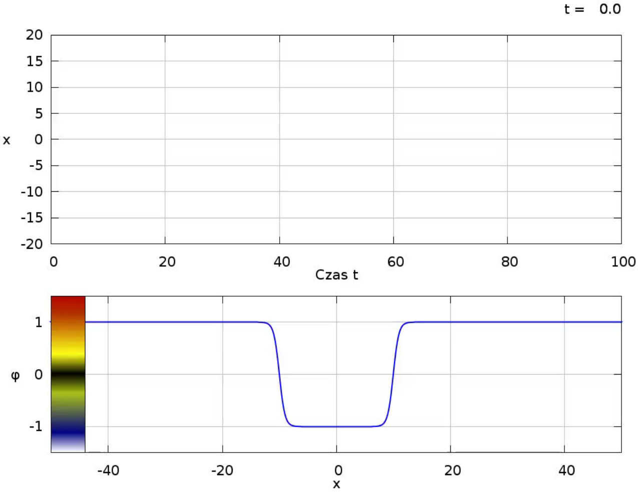
ϕ^4 kink scattering: baby windows



(From Goodman and Haberman, 2005)

Inside these windows the kinks bounce *three* times before re-separating:

Inside a three-bounce window: $v_i = 0.24385$



. . . and then at the edges of each three-bounce window there are sequences of four-bounce windows, and so on.

Theoretical treatment

The key point is that the ϕ^4 kink has an internal 'wobble' mode.

Take a small oscillation about a single kink $\phi_K(x) = \tanh(x)$:

$$\phi(x,t) = \phi_K(x) + \eta(x,t)$$

The e.o.m. for ϕ, $\phi_{tt} - \phi_{xx} + 2\phi(\phi^2 - 1) = 0$, implies for (small) η

$$\eta_{tt} - \eta_{xx} + (6\phi_K^2 - 2)\eta = 0$$

or, if $\eta(x,t) = e^{i\omega t}\chi(x)$,

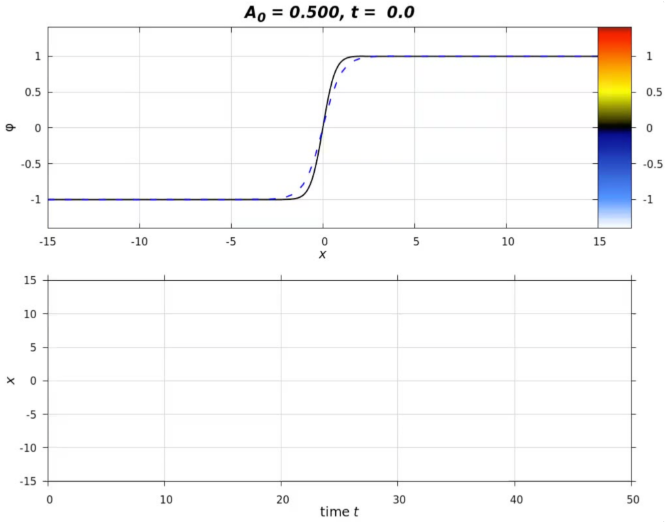
$$-\frac{d^2}{dx^2}\chi - 6\,\mathrm{sech}^2(x)\chi = (\omega^2 - 4)\chi$$

an eigenvalue problem with *two* eigenvalues ('bound states'), $\omega = 0$, $\sqrt{3}$.

The first is the translational mode; the second is the wobble (absent for sine-Gordon kinks) with period $2\pi/\sqrt{3} \approx 3.63$.

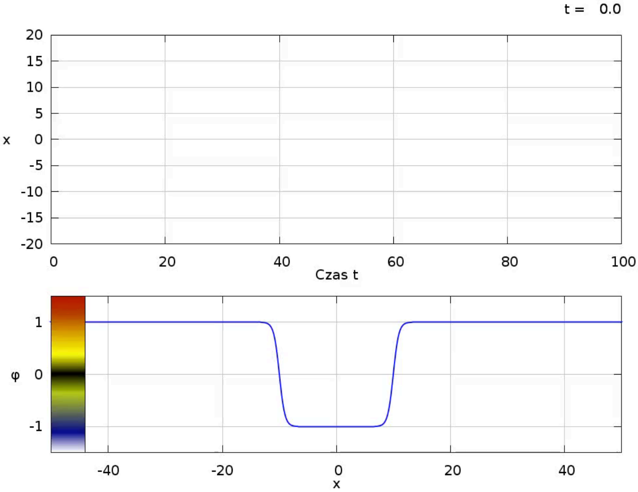
This wobble seen if we start from a distorted kink...

The basic ϕ^4 kink wobble:



$(\phi(x,0) = \tanh(x) + A_0 \tanh(x)/\cosh(x),\ \phi_t(x,0) = 0)$

. . . and it is also excited in kink-antikink scattering: ($v_i = 0.27$ again)



Most of the lost translational energy has been 'parked' in the wobble mode.

For initial velocities v just below v_c, the kink and antikink separate after collision but do not quite have the necessary escape velocity to overcome the attractive force between them.

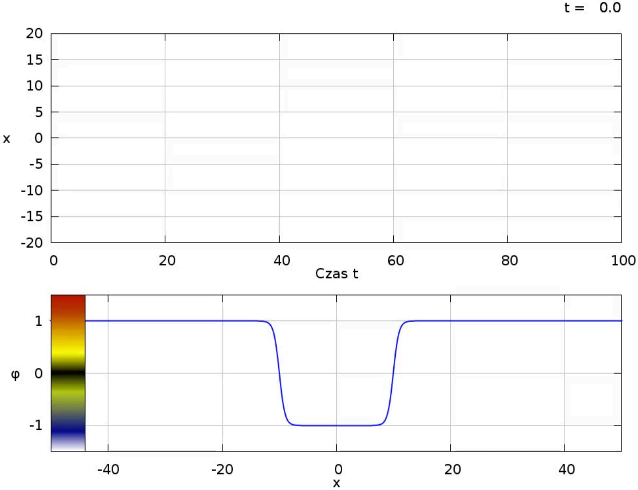
However if on recollision the situation is an approximate **time-reversal** of the initial impact, then the energy stored in the wobbles can be returned to the translational modes and escape is possible at the second attempt.

Note we do **not** need to solve for the nonlinear dynamics of the collision to see that this must work – the argument only uses time-reversal invariance of the equations of motion.

This might happen after one, two or more periods of the internal mode. It might also happen only after two recollisions, or three, and so on, explaining the nested structure of escape windows.

A more quantitative theory can be developed from these ideas but the correctness of the scenario can be seen on re-examining the two-bounce movie. . .

Two-bounce scattering revisited:



Key features:

To generate the 'fractal' structures we needed

- An attractive force putting the kink and antikink at risk of mutual capture;
- A 'battery': an energy storage mechanism with some periodicity (here, the wobble of the kink) so that this energy could be returned after an integer number of periods, perhaps after multiple recollisions.

This turns out to be rather a general mechanism, observed in many nonintegrable theories. In some cases the energy may be stored *between* the kink and antikink rather than on each one separately (first example: ϕ^6 theory), or in a quasinormal mode (a leaky battery). It can also be seen when firing kinks at boundaries. . .

4. Second warmup: ϕ^4 kinks hitting boundaries

Now put the ϕ^4 theory on a half line $-\infty < x < 0$, with a boundary magnetic field H placed at $x = 0$, so the energy is now

$$E[\phi] = \int_{-\infty}^{0} \tfrac{1}{2}\phi_t^2 + \tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}(\phi^2 - 1)^2 \, dx - H\,\phi(0,t)$$

$$\text{Boundary condition:} \quad \phi_x|_{x=0} = H$$

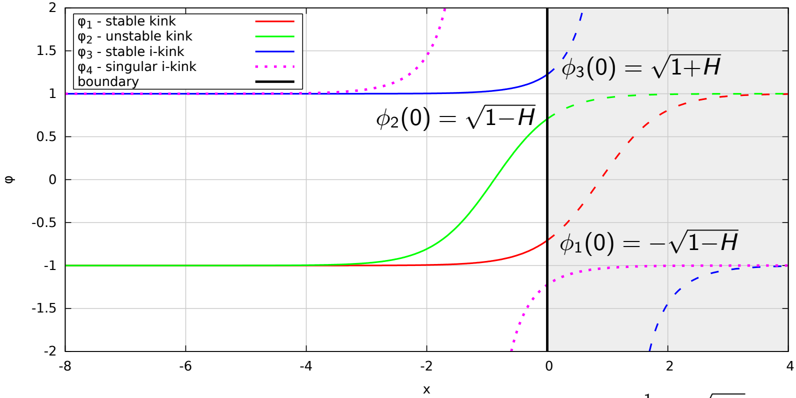
$$\text{Boundary energy:} \quad -H\phi(0,t)$$

Static kink and antikink solutions on full line are as before:

$$\phi_K(x) = \tanh(x - x_0) \quad , \quad \phi_{\bar{K}}(x) = -\tanh(x - x_0)$$

On a half line, use these to find the ground state by adjusting x_0 so that the boundary condition is satisfied at $x = 0$. Note: can also use singular solutions $\pm\coth(x - x_0)$ so long as the singularity is behind the boundary.

Static solutions for $0 < H < 1$



$$\phi_1 = \tanh(x - X_0)\,; \quad \phi_2 = \tanh(x + X_0)\,; \quad X_0 = \cosh^{-1}(1/\sqrt{|H|})$$

$$\phi_3 = -\coth(x - X_1)\,; \quad \phi_4 = -\coth(x + X_1)\,; \quad X_1 = \sinh^{-1}(1/\sqrt{|H|})$$

ϕ_3 is absolutely stable, ϕ_1 is metastable, and ϕ_2 is unstable.

In fact ϕ_2 is a saddle point between ϕ_1 and ϕ_3; at $H = 1$, ϕ_1 merges with ϕ_2 and the metastable state disappears from the spectrum.

(For $-1 < H < 0$, repeat the above with $\phi \to -\phi$.)

Energies

For these static solutions adapt the Bogomolnyi trick:

$$E[\phi] = \tfrac{1}{2}\int_{-\infty}^{0} \left((\phi_x)^2 + (\phi^2-1)^2\right) dx - H\phi|_{x=0}$$
$$= \tfrac{1}{2}\int_{-\infty}^{0} \left(\phi_x \pm (\phi^2-1)\right)^2 \, dx \mp \left[\tfrac{1}{3}\phi^3 - \phi\right]_{-\infty}^{0} - H\phi|_{x=0}$$

The integrated term on last line vanishes for kink/antikink profiles and the remainder rearranges to give

$$E[\phi_2] = \tfrac{2}{3} + \tfrac{2}{3}(1{-}H)^{3/2}$$

$$E[\phi_1] = \tfrac{2}{3} - \tfrac{2}{3}(1{-}H)^{3/2}$$

$$E[\phi_3] = \tfrac{2}{3} - \tfrac{2}{3}(1{+}H)^{3/2}$$

matching the earlier statement that ϕ_1 is metastable (a local minimum of the energy), ϕ_3 is absolutely stable (the global minimum) and ϕ_2 is unstable (a saddle point lying between ϕ_1 and ϕ_3).

Forces and scattering

At $t = 0$ we fire a single antikink, located at $x_0 < 0$, at the boundary, with a velocity v_i.

For $H > 0$ the initial boundary profile is ϕ_1, while for $H < 0$ it is $-\phi_3$.

At $t \approx |x_0|/v_i$ the antikink will hit the wall; but what happens next?

For $H = 0$ the Neumann boundary condition $\phi_x|_{x=0} = 0$ can be reflected onto the full line, so an antikink incident on the $H = 0$ boundary is trapped or reflected from that boundary with exactly the same 'phase diagram' as for full-line kink-antikink scattering.

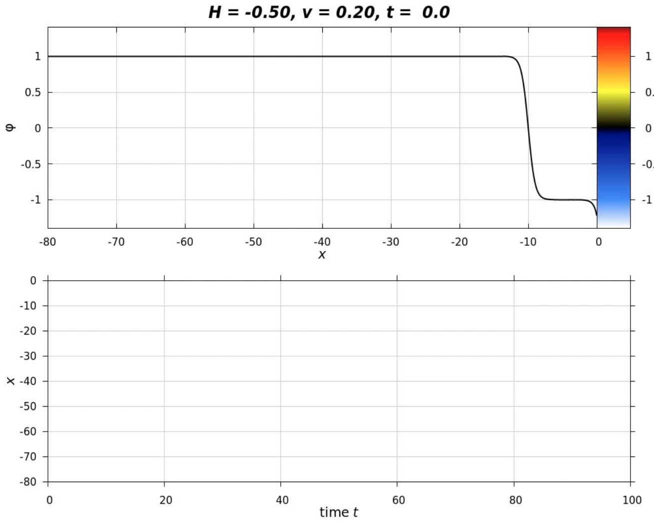
For $H \neq 0$ the picture distorts. An antikink at $x_0 < 0$ experiences an asymptotic force

$$F \sim 32 \left(\frac{1}{4} H + e^{2x_0} \right) e^{2x_0}$$

(To calculate F, use a modified method of images to fit the boundary condition with a full line antikink/(singular) kink solution, and then the bulk force law.)

For $H < 0$, F is *repulsive* at large distances, *unlike* for the bulk kink and antikink. This means that for small enough initial velocities scattering is almost perfectly elastic as the antikink stays far from the wall...

ϕ^4 boundary scattering: $H = -0.5$ and $v_i = 0.2$:



Critical velocity for $H < 0$

The initial energy is the sum of the energies of the moving antikink (which has rest mass 4/3) and the static $H < 0$ boundary, which is $-\phi_3$:

$$E_i(v_i) = \tfrac{4}{3}(1 - v_i^2)^{-1/2} + \tfrac{2}{3} - \tfrac{2}{3}(1{-}H)^{3/2}$$

The critical velocity v_{cr} is when the final state is 'at the top of the hill' at the $-\phi_2$ saddle point, with $E_i(v_{cr}) = E[-\phi_2]$:

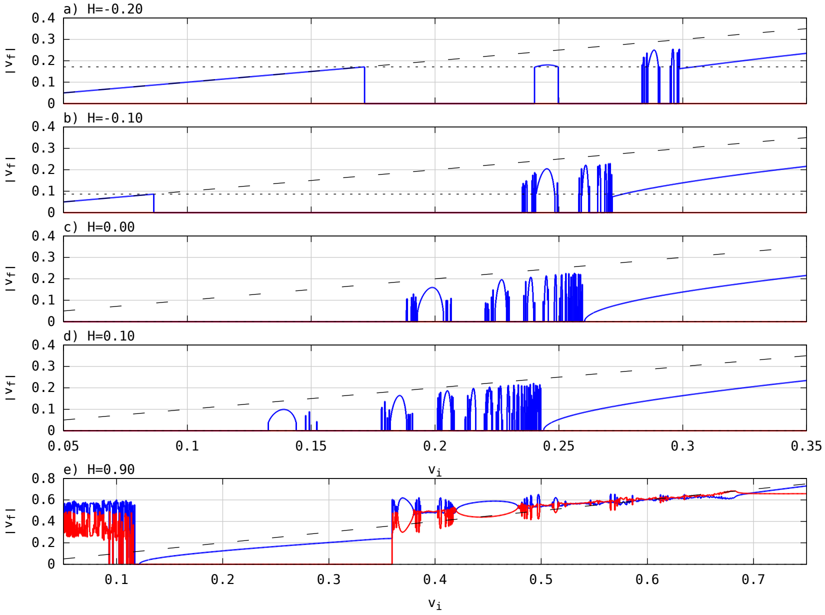
$H = -0.4$
$v_i = v_{cr}(H) = 0.3332\ldots$

Solving for v_{cr},

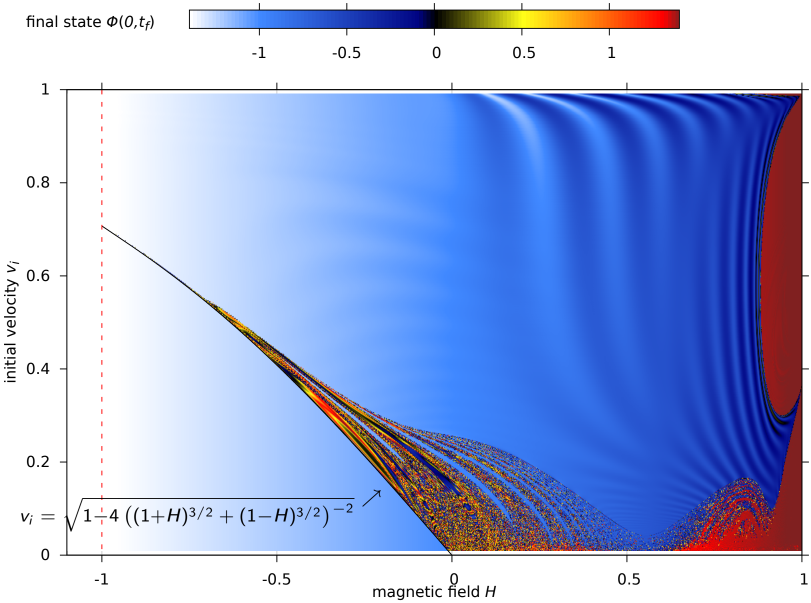
$$v_{cr}(H) = \sqrt{1 - 4\left((1{+}H)^{3/2} + (1{-}H)^{3/2}\right)^{-2}}$$

For $v_i > v_{cr}(H)$, the incoming antikink overcomes the energy barrier, nonlinear effects begin, and life gets complicated again. . .

ϕ^4 boundary scattering: escape windows



ϕ^4 boundary scattering: the phase diagram



Slow-then fast boundary decay

One further feature of the boundary ϕ^4 theory: for $0 < H < 1$ there is an oscillating boundary mode. Its H-dependent small-amplitude frequency ω_B can be found by linearising about the static solution.

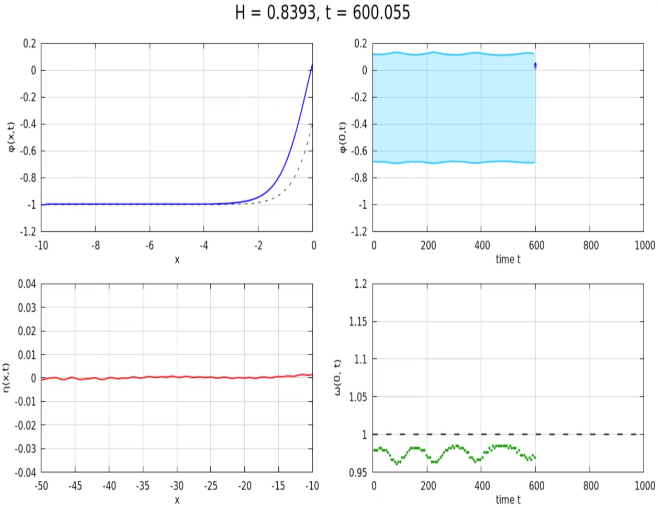
For H small, $2\omega_B > 2$, the lowest frequency for bulk radiation, and the second harmonic of the boundary mode can couple to bulk radiative modes, causing it to decay relatively rapidly.

For larger values of H, ω_B is reduced, and for $H \gtrapprox 0.925$, $2\omega_B < 2$ and it is only the *third* harmonic of the boundary oscillation that can couple to the bulk radiation, resulting in a much slower decay rate.

More interestingly, if one considers larger amplitudes at fixed H, nonlinear effects become relevant (cf. Peter F's talk), and the boundary mode frequency ω_B is also reduced (just like a real pendulum).

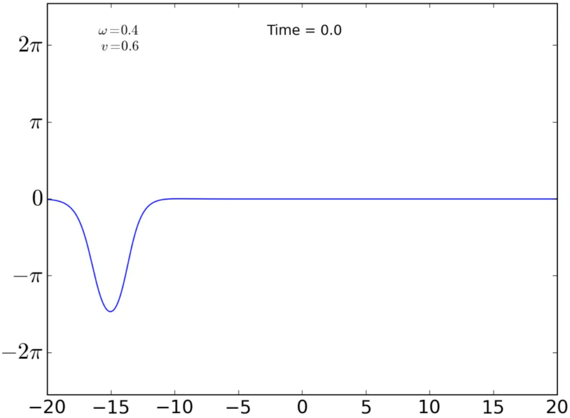
Suitably tuning H one can find a situation where a large-amplitude boundary mode has a decay channel forbidden to it, which only opens up once sufficient radiation has been emitted. This 'slow-then-fast' decay is illustrated in the following movie…

ϕ^4 boundary theory: slow-then-fast decay of the boundary mode



5. Back to boundary sine-Gordon

Reminder: in the bulk, $u_{tt} - u_{xx} + \sin(u) = 0$.
Since this model is bulk integrable,
(a) kinks and antikinks scatter with no loss of velocity; and
(b) kink-antikink bound states live forever, forming a further class of 'immortal' excitations: the breathers. Here's a moving one:



Now put the model on a half-line, $x < 0$, and break integrability by imposing a Robin boundary condition at $x = 0$.
The new setup:

$$\begin{aligned} u_{tt} - u_{xx} + \sin(u) &= 0 \qquad\qquad (x < 0); \\ u_x + 2ku &= 0 \qquad\qquad (x = 0). \end{aligned}$$

As before, we fire a kink or antikink at the boundary, and ask about what comes back.

If we wait long enough, all excitations will be far from the boundary, where integrability still holds. There is some sort of 'asymptotic integrability' at work, whereby integrability is only broken for a finite amount of time. This makes the model more interesting to study, but also adds greatly to the possible complexity of the final state, which might contain not only kinks and antikinks but also breathers.

But, it would be tedious to wait long enough for all the solitons and breathers to separate out. Fortunately we don't need to – once everything is far from the boundary (but still tangled up) we can use full-line integrabilty to extract the kink/antikink/breather content from the numerical data by computer.

Extracting the soliton content on a full line

...use ideas from inverse scattering theory...

The x part of the full-line Lax pair is

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} -\frac{i(\phi_x+\phi_t)}{4} & \lambda - \frac{e^{-i\phi}}{16\lambda} \\ \frac{e^{i\phi}}{16\lambda} - \lambda & \frac{i(\phi_x+\phi_t)}{4} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

An eigenfunction decaying at $x \to \pm\infty \Rightarrow$ an eigenvalue $\lambda \in \mathbb{C}$.

Eigenvalues are either on the positive imaginary axis (kinks or antikinks), or in symmetrically-placed pairs $(\lambda_n, -\lambda_n^*)$ (breathers).

Their velocities and (in the case of breathers) frequencies are

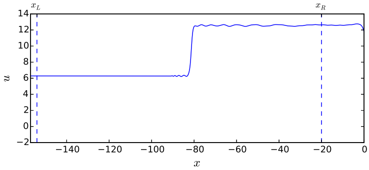
$$v = \frac{1 - 16\,|\lambda_n|^2}{1 + 16\,|\lambda_n|^2}\,, \qquad \omega = \frac{\mathrm{Re}[\lambda_n]}{|\lambda_n|}\,,$$

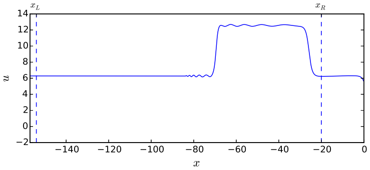
and their energies are

$$E_{\mathrm{soliton}} = \frac{1}{|\lambda_n|} + 16\,|\lambda_n|\,, \quad E_{\mathrm{breather}} = 2\,\mathrm{Im}[\lambda_n]\left(\frac{1}{|\lambda_n|^2} + 16\right).$$

Application to the boundary problem

Wait until all excitations have departed from the boundary region, and then patch boundary solution onto full line and compute scattering data to find soliton content of final state:





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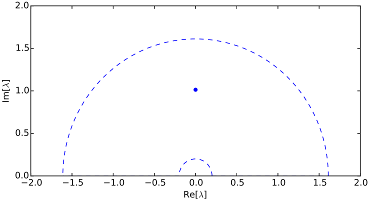

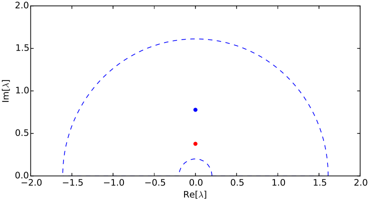

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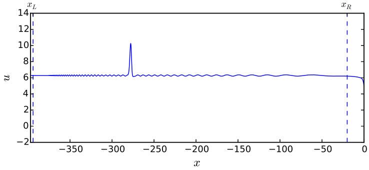
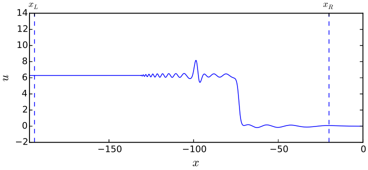
Kink

Kink and antikink

Application to the boundary problem (continued)





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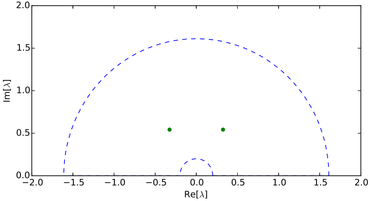

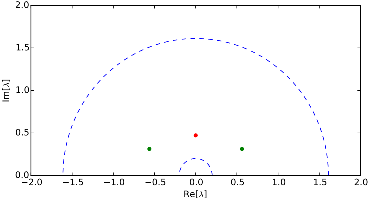

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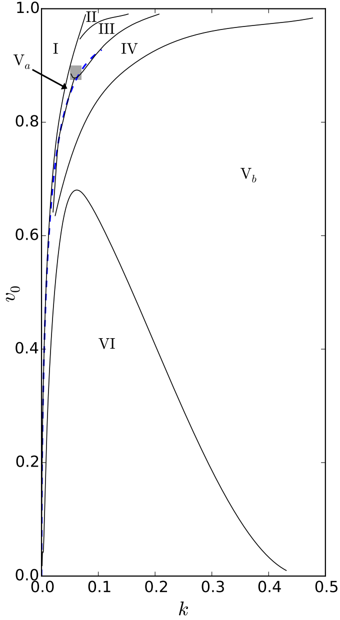
Breather

Antikink and breather

Implement this numerically by searching for zeros of the Wronskian $W(\lambda) = \mathrm{Det}(\psi_+, \psi_-)$ where $\psi_\pm$ decay as $x \to \pm\infty$ to find...

Robin boundary scattering: final state soliton content



Final states classified by kink, antikink and high energy breather content:

I: Kink

II: Kink and antikink

III: High-energy breather

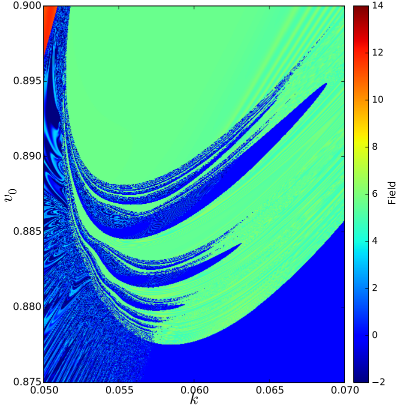
IV: High-energy breather and antikink

V_a & V_b: Antikink

VI: None of the above.

Note the match with the earlier snapshot!

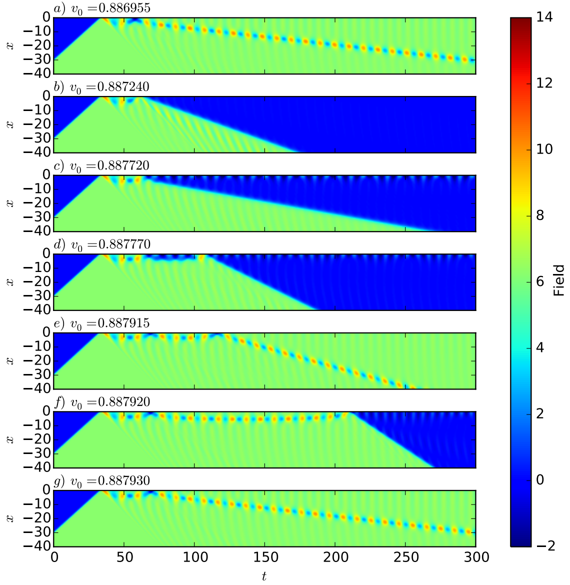
The zoomed-in phase diagram again



The zoomed-in snapshot shows the late-time values of the field at $x = 0$ for the shaded area on the previous slide. Dark blue bands correspond an antikink being emitted; in light green areas only breathers are emitted.

Sections taken at fixed k exhibit v_0-dependent windows, similar to those seen in the ϕ^4 theory.

Robin boundary scattering: the resonance mechanism



The key feature behind the ‘chaotic’ structure: even though the sG kink has no wobble mode, the breather *does* oscillate, and in some regimes it is both produced in the initial boundary collision, and also attracted back to the boundary afterwards. This is enough to get a resonance mechanism to work.

(Plots shown are for $k = 0.058$.)

This picture can be backed up by a variety of analytical results, such as calculations of the kink-boundary and breather-boundary forces:

• For an antikink located at $x_0 < 0$, park an image kink at $x_1 > 0$ to form a full-line configuration

$$u(x) = 4\arctan\left(e^{-(x-x_0)}\right) + 4\arctan\left(e^{(x-x_1)}\right)$$

For $|x_0|$ and $|x_1|$ both large the Robin boundary condition $(u_x + 2ku)|_{x=0} = 0$ becomes

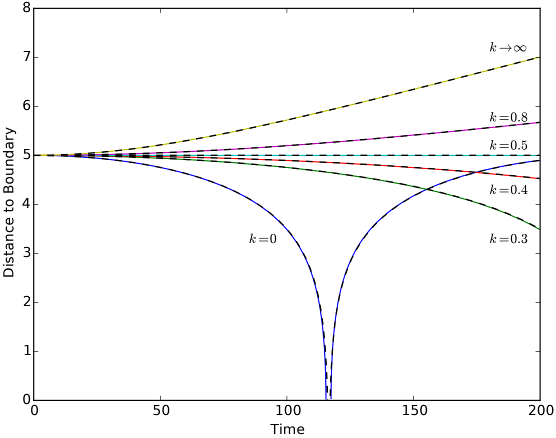
$$4(-e^{x_0} + e^{-x_1}) + 8k(e^{x_0} + e^{-x_1}) = 0\,.$$

Solving for e^{-x_1} and computing the force as for ϕ^4 yields

$$F = 32\,e^{-(x_1-x_0)} = 32\,\frac{1-2k}{1+2k}\,e^{2x_0}\,.$$

For $k > 1/2$ an image antikink should be used instead, but the final formula is unchanged, with the force now repulsive instead of attractive.

In the integrable Neumann and Dirichlet limits $k = 0$ and $k \to \infty$ this result matches the asymptotic behaviour of the corresponding exact solutions; it also agrees well at intermediate points, including the 'critical' value $k_c = 1/2$ at which the predicted force vanishes.



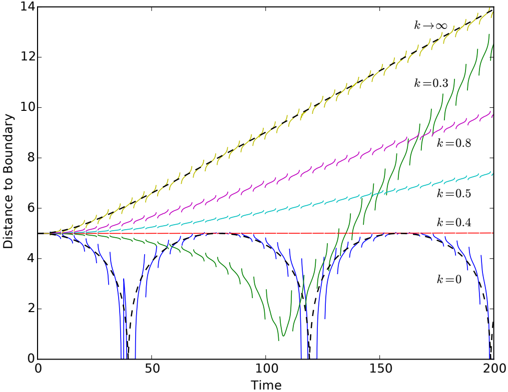
Antikink trajectories near a Robin boundary. (Coloured lines: numerical solutions of the full p.d.e.; dashed lines: predicitions from the formula for F.)

• For breathers the situation is more complicated as we don't have a static solution around which to expand.

The integrable Dirichlet and Neumann limits can be modelled on the full line by adding a symmetrically-placed image breather, exactly in phase with the 'real' breather for the Neumann boundary, and exactly out of phase for Dirichlet.

From the relevant exact two-breather solutions on the full line, it is known that two in-phase breathers feel an attractive force while two out-of-phase breathers experience a repulsive force. Hence a stationary breather is attracted by the $k = 0$ (Neumann) boundary, while for $k = \infty$ (Dirichlet) it is repelled.

Numerically we found that the general Robin boundary interpolates between these two limits with a breather-frequency dependent critical velocity at which the force vanishes tending to the value $k_c = 1/2$ from below as the frequency tends to zero. Recently there's been some analytic progress on this issue (nice idea by Peter Bowcock).



Trajectories of an initially-static breather with frequency 0.6 near to a Robin boundary. (Coloured lines: numercial solutions; Dashed lines: exact trajectories for the Dirichlet (top) and Neumann (bottom) limits.)

These results go most of the way to justifying the claimed resonance mechanism. However a fuller treatment would need some quantitative understanding of the initial bounce, which is still lacking...

6. Conclusions

- Classical boundary scattering in sine-Gordon is surprisingly rich once integrability is broken at the boundary.
- Many features of the 'phase diagram' still to be understood. Would like to develop an effective collective-coordinate description – this is hard as the boundary collision tends to excite many other modes, but at least while everything is far from boundary integrability may help.
- So far integrability was used rather stupidly, just to disentangle the final state. Is there more that can be done? (Maybe the so-called Fokas method can be applied.)
- It is natural to ask about the quantum theory, since the space of asymptotic *in* and *out* states should be the same as in the integrable case - this looks to be the ideal half-way-house to a full study of integrability breaking in QFT, where perhaps tools from quantum integrability can be used to study its breakdown.
- Finally, resonant scattering makes the final state extremely sensitive to initial conditions and poses many numerical challenges. It would be great to implement more-advanced schemes for the boundary problem...

Further reading

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P. Dorey & T. Romanczukiewicz,
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Happy birthday Laci!