On the exceptional spectrum of the asymmetric quantum Rabi model

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The asymmetric quantum Rabi model

The Hamiltonian

$$H = \Delta\,\sigma_z + \omega\, a^\dagger a + g\,\sigma_x (a^\dagger + a) + \epsilon\,\sigma_x$$

where

- σ_x and σ_z describe a two-level system with level splitting Δ
- $a^\dagger$and a create and destroy the bosonic mode with $[a, a^\dagger] = 1$ and frequency ω
- g is the coupling between the two systems
- $\epsilon\,\sigma_x$ allows tunnelling between the two atomic states

Rabi I I 1936 *Phys. Rev.* **49** 324

Outline

- The spectrum has a regular part and an exceptional part
- The exceptional part of the spectrum can be found via several different approaches
- Our main result is to connect these approaches explicitly via mapping to an equivalent problem

Bargmann realisation

Using

$$a^\dagger \to z, \quad a \to \frac{d}{dz}$$

the Hamiltonian is

$$H = \Delta\,\sigma_z + \epsilon\,\sigma_x + \omega\,z\frac{d}{dz} + g\,\sigma_x\left(z + \frac{d}{dz}\right)$$

and

$$H\left(\begin{array}{c}\psi_+(z)\\ \psi_-(z)\end{array}\right) = E\left(\begin{array}{c}\psi_+(z)\\ \psi_-(z)\end{array}\right)$$

gives rise to

$$\begin{aligned}(\omega z + g)\frac{d\psi_+}{dz} + (gz + \epsilon - E)\psi_+ + \Delta\psi_- &= 0\\ (\omega z - g)\frac{d\psi_-}{dz} - (gz + \epsilon + E)\psi_- + \Delta\psi_+ &= 0\end{aligned}$$

Substitute $\psi^1_\pm(z) = \mathrm{e}^{-gz/\omega}\phi^1_\pm(z)$ and eliminate $\phi^1_-(z)$ to find

$$(\omega z - g)(\omega z + g)\frac{d^2\phi^1_+(z)}{d^2 z}$$
$$+\left[-2g\omega z^2 + (\omega^2 - 2g^2 - 2E\omega)z + \frac{g}{\omega}(2g^2 - \omega^2 - 2\epsilon\omega)\right]\frac{d\phi^1_+(z)}{dz}$$
$$+\left[2g\left(\frac{g^2}{\omega} + E - \epsilon\right)z + E^2 - \Delta^2 - \epsilon^2 + \frac{2\epsilon g^2}{\omega} - \frac{g^4}{\omega^2}\right]\phi^1_+(z) = 0$$

A second set of solutions follow via $\psi^2_\pm(z) = \mathrm{e}^{gz/\omega}\phi^2_\pm(z)$

$$(\omega z - g)(\omega z + g)\frac{d^2\phi^2_-(z)}{d^2 z}$$
$$+\left[2g\omega z^2 + (\omega^2 - 2g^2 - 2E\omega)z - \frac{g}{\omega}(2g^2 - \omega^2 + 2\epsilon\omega)\right]\frac{d\phi^2_-(z)}{dz}$$
$$+\left[-2g\left(\frac{g^2}{\omega} + E + \epsilon\right)z + E^2 - \Delta^2 - \epsilon^2 - \frac{2\epsilon g^2}{\omega} - \frac{g^4}{\omega^2}\right]\phi^2_-(z) = 0$$

Spectrum of the AQRM

The spectrum has two parts:

- Regular spectrum

$$E_n^{\pm} + \frac{g^2}{\omega} - \epsilon \notin \omega\mathbb{N}$$

- Exceptional spectrum

$$E_n^{\pm} + \frac{g^2}{\omega} - \epsilon \in \omega\mathbb{N}$$

for certain ϵ, g, Δ

Braak's analytical solution when $\epsilon = 0$

The regular eigenvalues are found from the zeros of

$$G_{\pm}(x) = \sum_{m=0}^{\infty} K_m(x) \left(1 \mp \frac{\Delta}{x - m\omega}\right) \left(\frac{g}{\omega}\right)^m$$

where

$$m\, K_m(x) = \left[\frac{2g}{\omega} + \frac{1}{2g}\left(m\omega - x + \frac{\Delta^2}{x - m\omega}\right)\right] K_{m-1} - K_{m-2}$$

and $K_{-1} = 0, K_0 = 1$

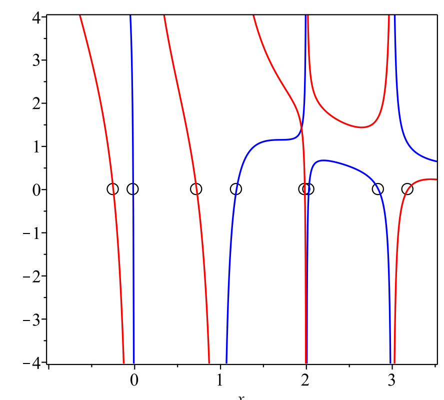
Braak D 2011 *Phys. Rev. Lett.* **107** 100401

Regular spectrum

Specifically

$$G_{\pm}(x_i^{\pm}) = 0 \quad , \quad E_i^{\pm} = x_i^{\mp} - \frac{g^2}{\omega}$$

Example: $g = 0.9, \omega = 1, \Delta = 0.6$



Exceptional spectrum from Braak's solution

The constraint

$$K_n(n\omega) = 0$$

defines the g, Δ for which Juddian solutions $E = n - \frac{g^2}{\omega}$ exist

Example $N = 1$

$$K_1(\omega) = 0 \quad \Rightarrow \quad \Delta^2 + 4g^2 = \omega^2$$

Judd B R 1979 *J. Phys. C* **12** 1685

Alternative solution

The spectrum may also be found from the zeros of the Wronskian

$$W = f_1(z)f_2'(z) - f_1'(z)f_2(z)$$

where

$$f_1(z) = e^{-gz}\, HC(2\alpha, \beta - 1, -\alpha, \eta + (1 - 2E)g^2, \frac{g - z}{2g})$$

and

$$f_2(z) = \frac{\Delta}{E + g^2} e^{gz}\, HC(2\alpha, , \beta - 1, \alpha, \eta - (3 + 2E)g^2, \frac{g + z}{2g})$$

where

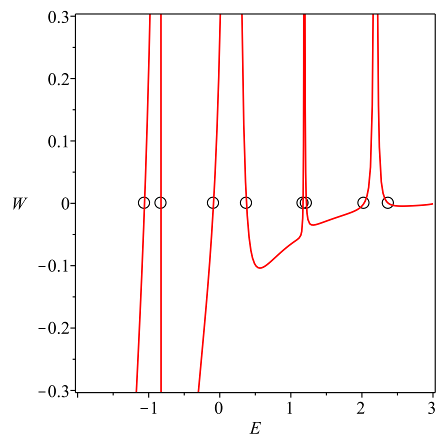
$$\alpha = 2g^2 \quad , \quad \beta = -E - g^2 \quad , \quad \eta = \tfrac{1}{2}(-3g^4 + E^2 - E - 2\Delta^2 + 1)$$

Zhong H, Xie Q, Guan X-W, Batchelor M T, Gao K and Lee C 2014 *J. Phys. A* **47** 045301

Example

$g = 0.9, \omega = 1, \Delta = 0.6$

Regular spectrum



Truncation of the confluent Heun function

The confluent Heun function $HC(a, b, c, d, e, x)$ becomes a polynomial

$$HC(a, b, c, d, e, x) = \sum_{m=0}^{n} h_m \, x^m$$

if

$$2d = -a(2n + d + b + 2)$$

and

$$h_{n+1} = 0$$

The exceptional spectrum of the QRM

For the QRM solutions the Heun functions are polynomials when

$$E = n_1 - \frac{g^2}{\omega} \quad , \quad h^1_{n_1+1} = 0$$

and

$$E = n_2 - \frac{g^2}{\omega} + \omega \quad , \quad h^2_{n_2+1} = 0$$

Example

Setting $n_1 = 1, n_2 = 0$ we have

$$E = \omega - \frac{g^2}{\omega}$$

and the Juddian constraint

$$\Delta^2 + 4g^2 = \omega^2$$

Constraint polynomials for the AQRM ($\epsilon \neq 0$)

The polynomials $P_k(x, y)$ of degree k are defined via

$$
\begin{aligned}
P_k(x, y) &= \left[kx + y - k^2\omega^2 - 2k\epsilon\,\omega\right] P_{k-1}(x, y) \\
&\quad - k(k-1)(n-k+1)x\omega^2 P_{k-2}(x, y)
\end{aligned}
$$

with $P_0(x, y) = 1$ and $P_1(x, y) = x + y - \omega^2 - 2\epsilon\omega$

The zeros of the **constraint polynomials**

$$P_n((2g)^2, \Delta^2) = 0$$

define the Juddian solutions with

$$E = n\omega - \frac{g^2}{\omega} + \epsilon$$

Exceptional spectrum and Gaudin-type BAE ($\epsilon \neq 0$)

Setting

$$\psi_+^1(z) = \mathrm{e}^{-gz/\omega} \prod_{i=1}^{n} (z - z_i)$$

the z_i satisfy the algebraic equations

$$\sum_{j \neq i}^{n} \frac{2\omega}{z_i - z_j} = \frac{n\omega^2 + 2\epsilon\omega}{\omega z_i - g} + \frac{n\omega^2 - \omega^2}{\omega z_i + g} + 2g$$

and the system parameters obey

$$\Delta^2 + 2ng^2 + 2\omega g \sum_{i=1}^{n} z_i = 0$$

The energy of these states is

$$E = n\omega - \frac{g^2}{\omega} + \epsilon$$

Li Z-M and Batchelor M T 2015 *J. Phys. A* **48** 454005

Second set of solutions

Setting

$$\psi_{-}^{2}(z) = \mathrm{e}^{gz/\omega} \prod_{i=1}^{n} -(z - z_i)$$

the $\{z_k\}$ satisfy the algebraic equations

$$\sum_{j \neq i}^{n} \frac{2\omega}{z_i - z_j} = \frac{n\omega^2 - \omega^2}{\omega z_i - g} + \frac{n\omega^2 - 2\epsilon\omega}{\omega z_i + g} - 2g$$

and the system parameters obey

$$\Delta^2 + 2ng^2 - 2\omega g \sum_{i=1}^{n} z_i = 0$$

with energy

$$E = n\omega - \frac{g^2}{\omega} - \epsilon$$

Quasi-exactly solvable hyperbolic potentials

The Schrödinger equation

$$-\frac{d^2\Psi(x)}{dx^2} + V(x)\Psi(x) = \mathcal{E}\Psi(x)$$

with

$$V(x; A, B, C, \gamma) = M(M-1-B-C+\frac{A\gamma}{2}\cosh x) + \frac{1}{4}(B+C+1)^2$$
$$+\frac{A^2\gamma^2}{16}\sinh^2 x + \frac{A\gamma}{4}(C-B) - \frac{A\gamma}{4}(B+C)\cosh x$$
$$+\frac{(2B+1)(2B+3)}{8(\cosh x - 1)} - \frac{(2C+1)(2C+3)}{8(\cosh x + 1)}$$

has a quasi-exactly solvable set of solutions when $M \in \mathbb{N}$

Dunning C, Hibberd K E and Links J 2008 *J. Phys. A* **41** 315211

QES solutions

The wavefunction

$$\Psi(x) = (\cosh x - 1)^{-(B/2+1/4)}(\cosh x + 1)^{-(C/2+1/4)} \times \exp\left(\frac{A\gamma}{4}\cosh x\right)\prod_{j=1}^{M}\left(\frac{\gamma}{2}\cosh x + v_j\right)$$

is a QES solution whenever

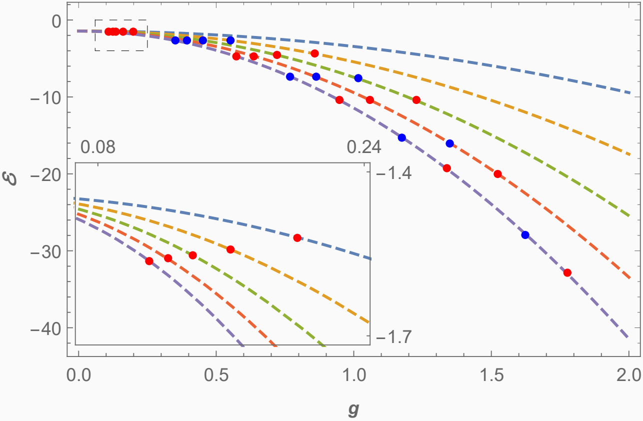
$$A + \frac{B}{v_j + \frac{1}{2}\gamma} + \frac{C}{v_j - \frac{1}{2}\gamma} = \sum_{k\neq j}^{M}\frac{2}{v_j - v_k}$$

with

$$\mathcal{E} = A\sum_{j=1}^{M} v_j$$

Spectrum $\mathcal{E}$ of the hyperbolic Schrödinger equations

$\Delta = 1.2$, $\omega = 1$ and $\epsilon = 0.3$.



Circles indicate QES eigenvalues $n = 1 \ldots 5$

AQRM to hyperbolic Schrödinger equation

The z_i satisfy

$$\sum_{j \neq i}^{n} \frac{2\omega}{z_i - z_j} = \frac{n\omega^2 + 2\epsilon\omega}{\omega z_i - g} + \frac{n\omega^2 - \omega^2}{\omega z_i + g} + 2g$$

and the v_j satisfy

$$A + \frac{B}{v_j + \frac{1}{2}\gamma} + \frac{C}{v_j - \frac{1}{2}\gamma} = \sum_{k \neq j}^{M} \frac{2}{v_j - v_k}$$

AQRM to hyperbolic Schrödinger equation

The BAE match if

$$A_+ = -2g/\omega,\ B_+ = n + 2\epsilon/\omega,\ C_+ = n - 1,\ \gamma = 2g/\omega$$

or

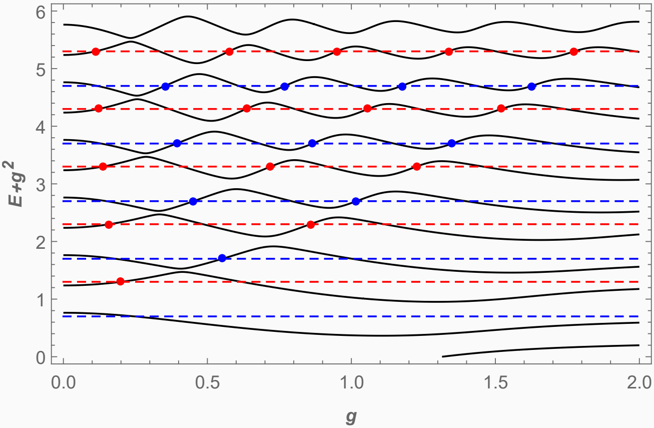
$$A_- = 2g/\omega,\ B_- = n - 1,\ C_- = n - 2\epsilon/\omega,\ \gamma = 2g/\omega$$

with in both cases

$$\mathcal{E} = -\Delta^2/\omega^2 - 2n\,g^2/\omega^2$$

Rescaled $E + g^2$ spectrum of the AQRM

$\Delta = 1.2$, $\omega = 1$ and $\epsilon = 0.3$.



Blue lines denote $E + g^2 = n - \epsilon$ and red lines $E + g^2 = n + \epsilon$

Circles indicate exceptional eigenvalues for $n = 1 \ldots 5$

Squaring the circle

The Heun polynomial solution is of the form

$$\sum_{m=0}^{n} h_m \, x^m$$

whereas the BAE solution is of the form

$$\phi_+^1(z) = \prod_{i=1}^{n} (z - z_i)$$

Suppose we set

$$\phi_+^1(z) = \sum_{k=0}^{\infty} R_k(n, \epsilon, \omega, \Delta) \, z^k$$

Then we find a 4-term recursion relation for the coefficients R_k and ϕ_+^1 does not obviously truncate to a polynomial

A final variable transformation to another ODE

Set

$$z = -\frac{g}{\omega}\frac{u+1}{u-1}, \qquad y(u) = \omega(u-1)^{-n} f(u)$$

in

$$(\omega z - g)(\omega z + g)\frac{d^2\phi_+^1(z)}{d^2 z}$$
$$+ \left[-2g\omega z^2 + (\omega^2 - 2g^2 - 2E\omega)z + \frac{g}{\omega}(2g^2 - \omega^2 - 2\epsilon\omega)\right]\frac{d\phi_+^1(z)}{dz}$$
$$+ \left[2g\left(\frac{g^2}{\omega} + E - \epsilon\right)z + E^2 - \Delta^2 - \epsilon^2 + \frac{2\epsilon g^2}{\omega} - \frac{g^4}{\omega^2}\right]\phi_+^1(z) = 0$$

we find

$$u(u-1)^2\omega^2 f''(u)$$
$$+\left((1-n)\omega^2 u^2 + (2\epsilon\omega + (2n-1)\omega^2 - 4g^2)u - \omega(2\epsilon + n\omega)\right)f'(u)$$
$$-\Delta^2 f(u) = 0$$

QES solution via a 3-term recurrence relation

Suppose

$$f(u) = \sum_{k=0}^{\infty} Q_k(n, \epsilon, \omega, \Delta)\, u^k$$

Then Q_k must satisfy

$$\begin{aligned}
&\omega(k+1)(2\epsilon + n\omega - k\omega)Q_{k+1} \\
&\qquad = -Q_k(\omega^2(2k^2 - 2kn - k) - 2k\epsilon\omega + 4kg^2 + \Delta^2) \\
&\qquad\quad + Q_{k-1}(1-k)\omega^2(n-k+1)
\end{aligned}$$

with initial condition $Q_{-1} = 0$ and Q_0.

QES solution via a 3-term recurrence relation

When $k = n + 1$

$$\begin{aligned}\omega(k+1)(2\epsilon + n\omega - k\omega)Q_{k+1} \\ = -Q_k(\omega^2(2k^2 - 2kn - k) - 2k\epsilon\omega + 4kg^2 + \Delta^2) \\ +Q_{k-1}(1-k)\omega^2(n-k+1)\end{aligned}$$

we have

$$\omega(n+2)(2\epsilon - \omega)\, Q_{n+2} = \left((2\,\epsilon\,\omega - 4\,g^2 - \omega^2)(n+1) - \Delta^2\right) Q_{n+1}$$

and if

$$Q_{n+1}(n, \epsilon, \omega, \Delta) = 0$$

leads to $Q_{n+1+k} = 0$ for $k = 0, 1, \ldots$ and a polynomial solution

Explicit relations

By direct comparison, we explicitly connect the BAE and Juddian constraint polynomials

$$Q_{n+1}(n,\epsilon,\omega,\Delta) = \frac{(-1)^{n+1}\Delta^2}{\omega^{n+1}2^{n+1}(n+1)!\prod_{k=0}^{n}(\epsilon+k\omega/2)} P_n((2g)^2,\Delta^2).$$

$$Q_k = \frac{(-1)^n}{\omega} S_{n-k}\left(\frac{g-\omega z_1}{g+\omega z_1}, \frac{g-\omega z_2}{g+\omega z_2}, \ldots, \frac{g-\omega z_n}{g+\omega z_n}\right) \prod_{k=1}^{n}\left(\frac{g}{\omega}+z_k\right)$$

where $S_j(x_1,\ldots,x_n)$ is the j^{th} symmetric polynomial on n variables

Complete spectral equivalence with the hyperbolic potentials

The ODEs for ϕ_-^1 and ϕ_+^2 can be mapped to

$$
\begin{aligned}
V_+(x) \;=\;& \frac{\epsilon^2}{\omega^2} + \frac{g^2}{\omega^2}\left(1+\frac{2\epsilon}{\omega}\right) - \frac{g^2}{\omega^2}\left(1-\frac{2\epsilon}{\omega}\right)\cosh x + \frac{g^4}{\omega^4}\sinh^2 x \\
& + \left[\left(E + g^2/\omega + \omega/2\right)^2 + \epsilon^2 + \epsilon\omega\right]\mathrm{csch}^2 x \\
& + (2\epsilon/\omega + 1)(E\omega + g^2 + \omega^2/2)\coth x\,\mathrm{csch}\,x \\
V_-(x) \;=\;& \frac{\epsilon^2}{\omega^2} + \frac{g^2}{\omega^2}\left(1-\frac{2\epsilon}{\omega}\right) + \frac{g^2}{\omega^2}\left(1+\frac{2\epsilon}{\omega}\right)\cosh x + \frac{g^4}{\omega^4}\sinh^2 x \\
& + \left[\left(E + g^2/\omega + \omega/2\right)^2 + \epsilon^2 - \epsilon\omega\right]\mathrm{csch}^2 x \\
& + (2\epsilon/\omega - 1)(E\omega + g^2 + \omega^2/2)\coth x\,\mathrm{csch}\,x
\end{aligned}
$$

and

$$\mathcal{E}_\pm = -2Eg^2/\omega^3 - 2g^4/\omega^4 - \Delta^2/\omega^2 \pm 2g^2\epsilon/\omega^3$$

Conclusion

- Established a spectral equivalence between the exceptional spectrum of the AQRM and a set of QES hyperbolic potentials
- Explicitly connected the Gaudin-type BAE approach with that from the Juddian constraint polynomial approach
- Extended the spectral equivalence to the entire spectrum of the AQRM and the hyperbolic potentials

The hyperbolic potentials are of generalised Pöschl-teller type so it may be possible to explore such models via experimental realisations of the quantum Rabi model

Finite-Size Technology in Low-Dimensional Quantum Systems (VI), Seoul 2012