Lax representations of sigma models on the half line

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Contents

Outline

Integrability for bulk models

- Classical model with infinite set of local conserved charges $\Longrightarrow$ Classical Integrable Field Theory
- If $\exists \mathrm{L}(z)$ Lax operator where $\mathrm{dL}(z) + \mathrm{L}(z) \wedge \mathrm{L}(z) = 0 \Longleftrightarrow$ E.O.M is satisfied $\Longrightarrow$ infinitely many conserved charges.

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PCM	✓	✓	✓
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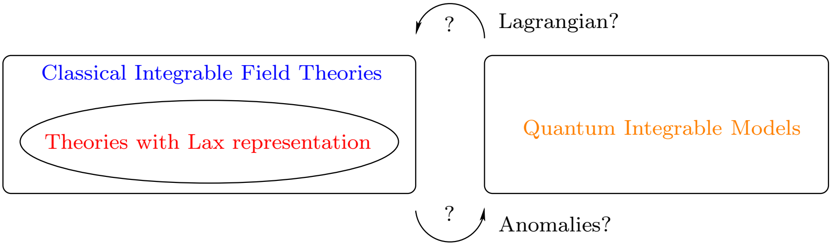

Sigma models

- The field of the sigma model $X : \Sigma \to \mathcal{M}$ where Σ is a 2 dimensional manifold e.g. $\mathbb{R}^2, \mathbb{R} \times S^1, \mathbb{R} \times (-\infty, 0], \mathbb{R} \times [0, \pi]$
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Principal Chiral Fields: $\mathcal{M} = G$ and $X(x) = g(x) \in G$ where G is a Lie group

- We can define two currents: $\mathrm{J}^R = g^{-1} \mathrm{d} g$ and $\mathrm{J}^L = g \mathrm{d} g^{-1} \Longrightarrow \mathrm{J}^L = -g \mathrm{J}^R g^{-1}$.
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- The E.O.M is $\mathrm{d} * \mathrm{J}^{L/R} = 0$.
- The PCMs have $G_L \times G_R$ global symmetry, the left/right group multiplication, $g(x) \to g_L g(x)$ and $g(x) \to g(x) g_R$.
- The transformation of the currents:

$$
\begin{aligned}
g_L : \quad & \mathrm{J}^L \to g_L \mathrm{J}^L g_L^{-1}, \quad & \mathrm{J}^R \to \mathrm{J}^R, \\
g_R : \quad & \mathrm{J}^L \to \mathrm{J}^L, \quad & \mathrm{J}^R \to g_R^{-1} \mathrm{J}^R g_R.
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- The $\mathrm{J}^{L/R}$ are the Noether currents of the left/right group multiplication symmetry.

Lax representation of PCM

- The E.O.M and the flatness condition is equivalent to the flatness condition of the Lax connection: $\mathrm{dL}(z) + \mathrm{L}(z) \wedge \mathrm{L}(z) = 0$ where

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- The conserved charges are generated by expanding the monodromy matrix in z, for instance at infinity:
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- If we expand around $z = 0$ we get an other infinite set of conserved non-local charges ($\{Q_r^L\}$) which is an other $\mathcal{Y}(\mathfrak{g})$ Yangian and the Q_0^L is the Noether charge of the left group multiplication therefore the PCMs have $\mathcal{Y}(\mathfrak{g})_L \oplus \mathcal{Y}(\mathfrak{g})_R$ symmetry.

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- Local conserved quantities are generated by Taylor expansion at $z = \pm 1$.

Outline

Integrable theories at the half line $(-\infty, 0]$

- At the boundary case we have a surface term: $\partial_\alpha J^\alpha = 0$ and $Q = \int_{-\infty}^{0} J_0 \mathrm{d}x^1$.
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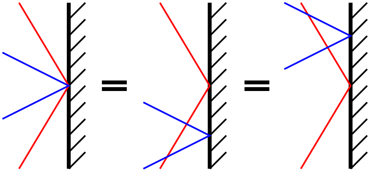
- The boundary monodromy matrix (double row monodromy matrix) contains the bulk monodromy matrix and the reflection matrix.
- Infinitely many conserved charge at the quantum level:
  - (i) no particle creation/annihilation at the boundary scattering
  - (ii) The n-particle scattering at the boundary factorizes into 1-particle boundary scattering and 2-particle bulk scattering uniquely (boundary Yang-Baxter Equation)
- We can define boundary IQFT with the quantum R-matrix bootstrap.



$$R_{12} = S_{12} R_1 S_{21} R_2 = R_2 S_{12} R_1 S_{21}$$

Symmetric spaces

- G and $H < G$ are Lie groups.
- If $\exists \alpha \in \mathrm{Aut}(G)$ where $\alpha^2 = \mathrm{id}$ and $\forall h \in H$ $\alpha(h) = h$ then $\mathcal{M} = G/H$ is a symmetric space.

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- For most of these cases α is an inner automorphism $\alpha(g) = UgU^{-1}$ where

$$U_s = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & -\mathbb{I}_m \end{pmatrix}, \quad \text{or} \quad U_a = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}.$$

- For $\mathrm{SU}(n)/\mathrm{SO}(n)$ and $\mathrm{SU}(2n)/\mathrm{Sp}(n)$ αs are outer automorphisms: $\alpha(g) = \bar{g}$ and $\alpha(g) = U_a \bar{g} U_a^{-1}$.

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- We have an Lie-algebra endomorphism: ad_M and we can denote $\mathfrak{h} = \mathrm{Ker}(\mathrm{ad}_M)$ and $\mathfrak{f} = \mathrm{Im}(\mathrm{ad}_M)$. Therefore $J_1 \in \mathfrak{f}$ and the residual symmetry is $G \times G \to G \times H$.

Double row monodromy matrices

- We use the one and the double row monodromy matrix:

$$T(z) = \mathcal{P}\overleftarrow{\exp}\left(-\int_{-\infty}^{0} L_1(z)\mathrm{d}x^1\right) \qquad \Omega(z) = T(-z)^{-1}\kappa(z)T(z)$$

- The time derivative of the double row monodromy matrix:

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$$T(z) = \mathcal{P}\overleftarrow{\exp}\left(-\int_{-\infty}^{0} L_1(z)\mathrm{d}x^1\right) \qquad \Omega(z) = T(-z)^{-1}\kappa(z)T(z)$$

- The time derivative of the double row monodromy matrix:

$$\partial_0\Omega(z) = T(-z)^{-1}\Big(L_0(-z,0)\kappa(z) + \dot{\kappa}(z) - \kappa(z)L_0(z,0)\Big)T(z)$$

- Therefore the monodromy matrix is time independent for all z if

$$\kappa(z)L_0(z) - L_0(-z)\kappa(z) = \dot{\kappa}(z)$$

- This equation (reflection equation) is equivalent to with the boundary EOM just like the flatness condition $\mathrm{dL} + \mathrm{L} \wedge \mathrm{L} = 0$ is equivalent with the bulk EOM.

Reflection matrices for PCM

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$$\tilde{R}(\theta) = \tilde{R}_0(\theta) \left[R^L(\theta) \otimes R^R(\theta) \right]$$

then the bYBE is also factorized ($\theta_{ij} = \theta_i - \theta_j$, $\vartheta_{ij} = \theta_i + \theta_j$):

$$S_{12}(\theta_{12}) R_1^{L/R}(\theta_1) S_{21}(\theta)(\vartheta_{12}) R_2^{L/R}(\theta_2) = R_2^{L/R}(\theta_2) S_{12}(\vartheta_{12}) R_1^{L/R}(\theta_1) S_{21}(\theta)(\theta_{12})$$

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- The solutions are classified and they depend on what is the residual symmetry algebra $\mathfrak{h}$.

$\mathfrak{g} = \mathfrak{su}(n)$

$\mathfrak{h} = \mathfrak{su}(k) \oplus \mathfrak{su}(n-k) \oplus \mathfrak{u}(1)$

$$R \sim \begin{pmatrix} \frac{\alpha+\theta}{\alpha-\theta}\mathbb{I}_k & 0 \\ 0 & \mathbb{I}_{n-k} \end{pmatrix}$$

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This case belongs to a representation changing reflection where a particle goes to its anti-particle.

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O(n) sigma models

- $\mathcal{M} = \mathrm{SO}(n)/\mathrm{SO}(n-1) \equiv S^{n-1}$ and $X = \mathbf{n} \in \mathbb{R}^n$ where $\mathbf{n} \cdot \mathbf{n} = 1$.
- The Lagrangian is $\mathcal{L} = \frac{1}{2}\left[\partial_\alpha \mathbf{n} \cdot \partial^\alpha \mathbf{n} + \sigma(\mathbf{n} \cdot \mathbf{n} - 1)\right]$.
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- We can define an O(n) group element: $h = \mathbb{I} - 2\mathbf{n} \otimes \mathbf{n}$ which is satisfy $h^T h = \mathbb{I}$ and $h = h^T$ identities.
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Some comments on the quantum classification

- For all possible solutions G/H is a symmetric space.
- We can choose $R^L \neq R^R$ reflection matrices which have $H_L \times H_R$ residual symmetry where $H_L \neq H_R$ for PCMs and U($n/2$) symmetric R-matrices for O(n) sigma models.
- The so far known integrable boundary conditions with Lax description have $H_L \times H_R$ symmetry where $H_L = H_R$ for PCMs and O(k) $\times$ O($n-k$) for O(n) sigma models.
- We can see from the classification that some Rs contain parameters. The residual symmetries of these cases are not semi-simples.
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Questions

- Are there Lax descriptions of the solutions with $H_L \times H_R$ symmetries where $H_L \neq H_R$ for PCMs?
- Are there Lax descriptions of the solutions with U($n/2$) symmetries for O(n) sigma models?
- Are there Lax descriptions of the solutions which have free parameters?

Outline

The new $\kappa(z)$ for PCMs

- The ansatz is the following:

$$\kappa(z) = k(z)\left(\mathbb{I} + zM + z^2 N\right), \qquad \text{where } M \in \mathfrak{g} \text{ and } k(z) \in \mathbb{R}.$$

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- Which leads to the following equation system:

$$\begin{aligned} z^1 : &\qquad [M, J_0^R] - 2J_1^R = 0 \\ z^2 : &\qquad [N, J_0^R] - \{M, J_1^R\} = 0 \\ z^3 : &\qquad \{N, J_1^R\} = 0 \end{aligned}$$

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$$Q_R = \int_{-\infty}^{0} J_0^R \mathrm{d}x^1 \quad \Longrightarrow \quad \dot{Q}_R = J_1^R\Big|_{x=0} \Longrightarrow \Pi_{\mathfrak{h}}(Q_R) \text{ is conserved.}$$

Some comments for these new $\kappa(z)$s

- If $N \neq 0$ then it defines an automorphism which leads to a $\mathbb{Z}_2$ graded decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f}$.
- If $N = 0$ then $M^2 \sim \mathbb{I}$ so M defines the decomposition : $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f}$.
- The BCs on $\mathfrak{h}$ and $\mathfrak{f}$: $\Pi_{\mathfrak{h}}(J_1^R) = 0$ and $\Pi_{\mathfrak{f}}(J_1^R) = \beta(\Pi_{\mathfrak{f}}(J_0^R))$ where $\beta : \mathfrak{f} \to \mathfrak{f}$ is an invertible linear map.
- The $M \in \mathfrak{h}$ and $[M, \mathfrak{h}] = 0$ therefore $\mathfrak{h}$ has non-trivial center so it is not semi-simple: $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{h}_s$ where $\mathfrak{u}(1)$ is generated by M.
- We have seen that this boundary condition belongs this boundary Lagrangian:

$$L_b = \frac{1}{4} \mathrm{Tr} \left[M J_0^R \right].$$

- The model has $G \times H$ symmetry and one free parameter.
- The right conserved charges are:

$$Q_R = \int_{-\infty}^{0} J_0^R \mathrm{d}x^1 \quad \Longrightarrow \quad \dot{Q}_R = J_1^R \Big|_{x=0} \Longrightarrow \Pi_{\mathfrak{h}}(Q_R) \text{ is conserved.}$$

- The left conserved charges are ($\mathrm{J}^L = -g \mathrm{J}^R g^{-1}$):

$$Q_L = \int_{-\infty}^{0} J_0^L - \frac{1}{2} g M g^{-1} \delta(x) \mathrm{d}x \quad \Longrightarrow$$

$$\dot{Q}_L = \left(J_1^L - \frac{1}{2} g [J_0^R, M] g^{-1} \right) \Big|_{x=0} = \left(-g J_1^R g^{-1} + \frac{1}{2} g [M, J_0^R] g^{-1} \right) \Big|_{x=0} = 0.$$

The solutions

- $\mathfrak{g} = \mathfrak{su}(n)$ and $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{su}(m) \oplus \mathfrak{su}(k)$ where $n = m + k$.

$$M = i\frac{2\lambda}{k-m}\begin{pmatrix} -k\mathbb{I}_{m\times m} & \mathbb{O}_{m\times k} \\ \mathbb{O}_{k\times m} & m\mathbb{I}_{k\times k} \end{pmatrix}, \qquad N = \lambda^2 \frac{n}{k-m}\begin{pmatrix} -\mathbb{I}_{m\times m} & \mathbb{O}_{m\times k} \\ \mathbb{O}_{k\times m} & \mathbb{I}_{k\times k} \end{pmatrix},$$

$$\kappa(z) = \begin{pmatrix} \frac{1+iz\lambda}{1-iz\lambda}\mathbb{I}_{m\times m} & \mathbb{O}_{m\times k} \\ \mathbb{O}_{k\times m} & \mathbb{I}_{k\times k} \end{pmatrix} \in \mathrm{U}(n).$$

- $\mathfrak{g} = \mathfrak{so}(2n)$ or $\mathfrak{g} = \mathfrak{sp}(n)$ and $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{su}(n)$

$$M = \lambda\begin{pmatrix} \mathbb{O}_{n\times n} & -\mathbb{I}_{n\times n} \\ \mathbb{I}_{n\times n} & \mathbb{O}_{n\times n} \end{pmatrix}.$$

  Because of $M^2 = -\lambda^2\mathbb{I}$ than $N = 0$. The matrix κ is the following:

$$\kappa(z) = \frac{1}{\sqrt{1+\lambda^2 z^2}}\begin{pmatrix} \mathbb{O}_{n\times n} & -\lambda\mathbb{I}_{n\times n} \\ \lambda\mathbb{I}_{n\times n} & \mathbb{O}_{n\times n} \end{pmatrix}.$$

  We can check that $\kappa(z) \in \mathrm{SO}(2n)$ and $\kappa(z) \in \mathrm{Sp}(n)$ too.

The solutions

- $\mathfrak{g} = \mathfrak{so}(n)$ and $\mathfrak{h} = \mathfrak{so}(2) \oplus \mathfrak{so}(n-2)$

$$M = 2\lambda \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad N = \lambda^2 \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\kappa(z) = \begin{pmatrix} A(z) & -B(z) & 0 & 0 & \cdots \\ B(z) & A(z) & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathrm{SO}(n),$$

where

$$A(z) = \frac{1-\lambda^2 z^2}{1+\lambda^2 z^2},$$

$$B(z) = \frac{2\lambda z}{1+\lambda^2 z^2}.$$

Equivalence between the O(4) sigma model and the SU(2) *PCM*

- Since $SO(4) \cong SU(2) \times SU(2)$ the SU(2) principal and the SO(4) sigma models are equivalent.
- Using $g_{\alpha\dot{\alpha}} = n_{\alpha\dot{\alpha}} = \sigma^{i}_{\alpha\dot{\alpha}} n_i$ basis transformation $\longrightarrow g \in SU(2)$.

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$$\mathrm{L}^{SO(4)}(z) = \mathrm{L}^L(z) \otimes \mathbb{I} + \mathbb{I} \otimes \bar{\mathrm{L}}^R(z) \qquad \Longrightarrow \qquad T^{SO(4)}(z) = T^L(z) \otimes \bar{T}^R(z).$$

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$$\kappa^{SO(4)}(z) \sim \left(\mathbb{I}_2 + \frac{1}{z} g M_R g^\dagger\right) \otimes \left(\mathbb{I}_2 + z\bar{M}_R\right) = \mathbb{I}_4 + zM + \frac{1}{z} hMh + \frac{1}{2}\{M, hMh\},$$

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$$\hat{J}_1 = \frac{1}{2}[M + hMh, \hat{J}_0], \qquad L_b = -\frac{1}{8}\mathrm{Tr}\left[M\hat{J}_0\right].$$

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- The above BC and κ can be generalized for the $\mathrm{SO}(2N)$ sigma models with any $N > 2$ and the residual symmetry is $\mathrm{U}(N)$.

Conclusions

- We have determined new κ matrices for the principal models whose residual symmetry is $G \times H$ or $H \times G$.
- We have seen that if the center of the residual symmetry is one dimensional then the boundary condition and the κ matrix contain one free parameter.
- The $\mathrm{SO}(4) \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ case can be used to determine the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_R$ symmetric κ matrices for $\mathrm{SO}(4)$ sigma models.
- This can be generalized for $\mathrm{SO}(2N)$ sigma models with $\mathrm{U}(N)$ symmetric boundary condition which are also new solutions.

Open questions

- Are there Lax descriptions for cases when the residual symmetry is $G \times H$ but the H is semi-simple?
- Are there Lax descriptions for general $H_L \times H_R$ with two free parameters?