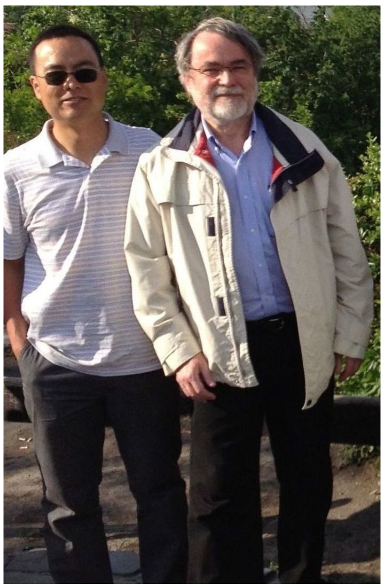
EISENHART-DUVAL LIFT:

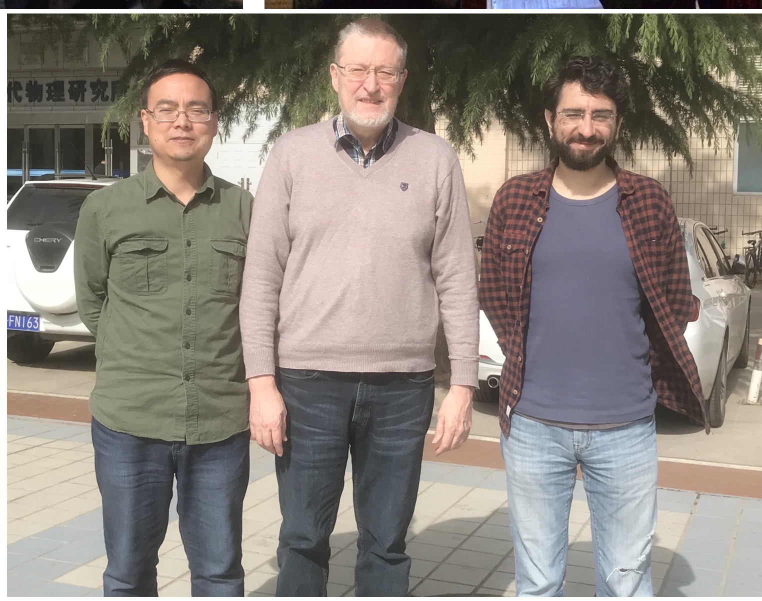
UNIFIED FRAMEWORK

for

GALILEI & CARROLL SYMMETRY

P-M Zhang, C. Duval, PH, G. Gibbons, M. Elbistan





PallaFest

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Abstract: While the usual Wigner-Inönü contraction $c \to \infty$ of the Poincaré group yields the Galilei group, another $c \to 0$ contraction yields the "Carroll group" of Lévy-Leblond. Both boost-invariant theories are conveniently unified within the "Eisenhart-Duval" framework. **Plane gravitational waves** carry a non-trivially implemented "distorted" Carroll symmetry with broken rotations.

Based on:

- C. Duval, G. W. Gibbons, PH and P. M. Zhang *"Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time,"* Class. Quant. Grav. **31** (2014) 085016 [arXiv:1402.0657 [gr-qc]]

- C. Duval, G. W. Gibbons, PH and P. M. Zhang *"Carroll symmetry of gravitational plane waves,"* Class. Quant. Grav. **34** (2017) 175003 [arXiv:1702.08284]

- P. M. Zhang, C. Duval, G. W. Gibbons and PH, "Velocity Memory Effect for Polarized Gravitational Waves," JCAP 05 (2018) 030 [arXiv:1802.09061 [gr-qc]].

- M. Elbistan, G. W. Gibbons, PH, P.-M. Zhang, *"Sturm-Liouville and Carroll : at the heart of the Memory Effect"* [arXiv:1803.09640 [gr-qc]]

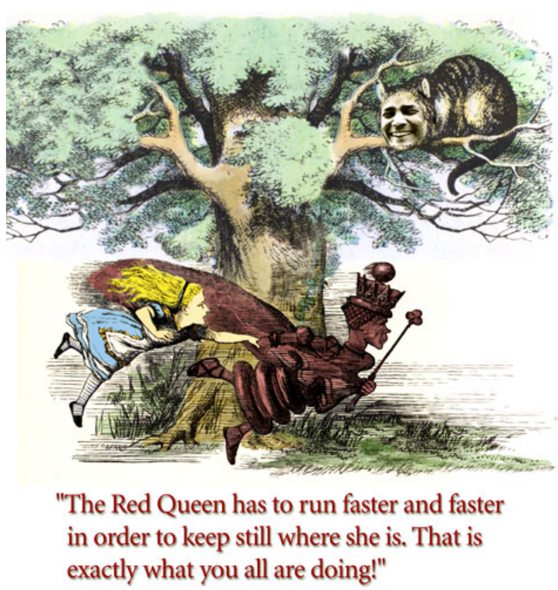
Carroll group

J. M. Lévy-Leblond Carroll group*
constructed as novel type of contraction of Poincaré group

"Une nouvelle limite non-relativiste du group de Poincaré," Ann. Inst. H. Poincaré **3** (1965) 1

V. D. Sen Gupta, "On an Analogue of the Galileo Group," Il Nuovo Cimento **44** (1966) 512

no motion - *no* physics - mathematical curiosity



* Lewis Carroll *Through the Looking Glass and what Alice Found There* (1871).

NEWTON-CARTAN STRUCTURE

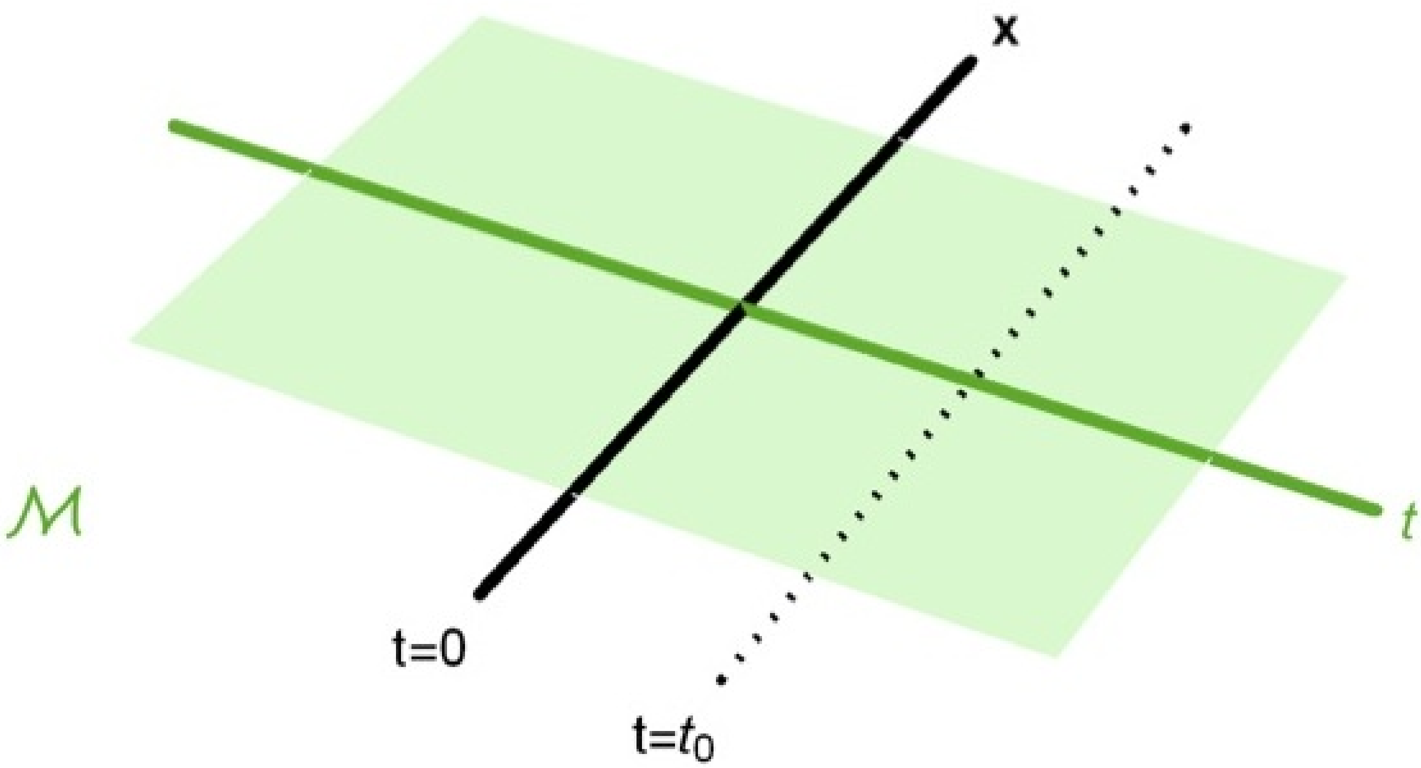


Fig. 1 : Galilean space-time, $\mathcal{M}$, described by $\begin{pmatrix} \boldsymbol{x} \\ t \end{pmatrix}$. Carries symmetric, contravariant non-negative [space-co-] "metric" tensor γ whose kernel is generated by dt. Projects onto absolute time.

Galilei boosts

$$\begin{cases} \boldsymbol{x}' &= \boldsymbol{x} + \boldsymbol{b}\,t \\ t' &= t \end{cases} \tag{1}$$

act as

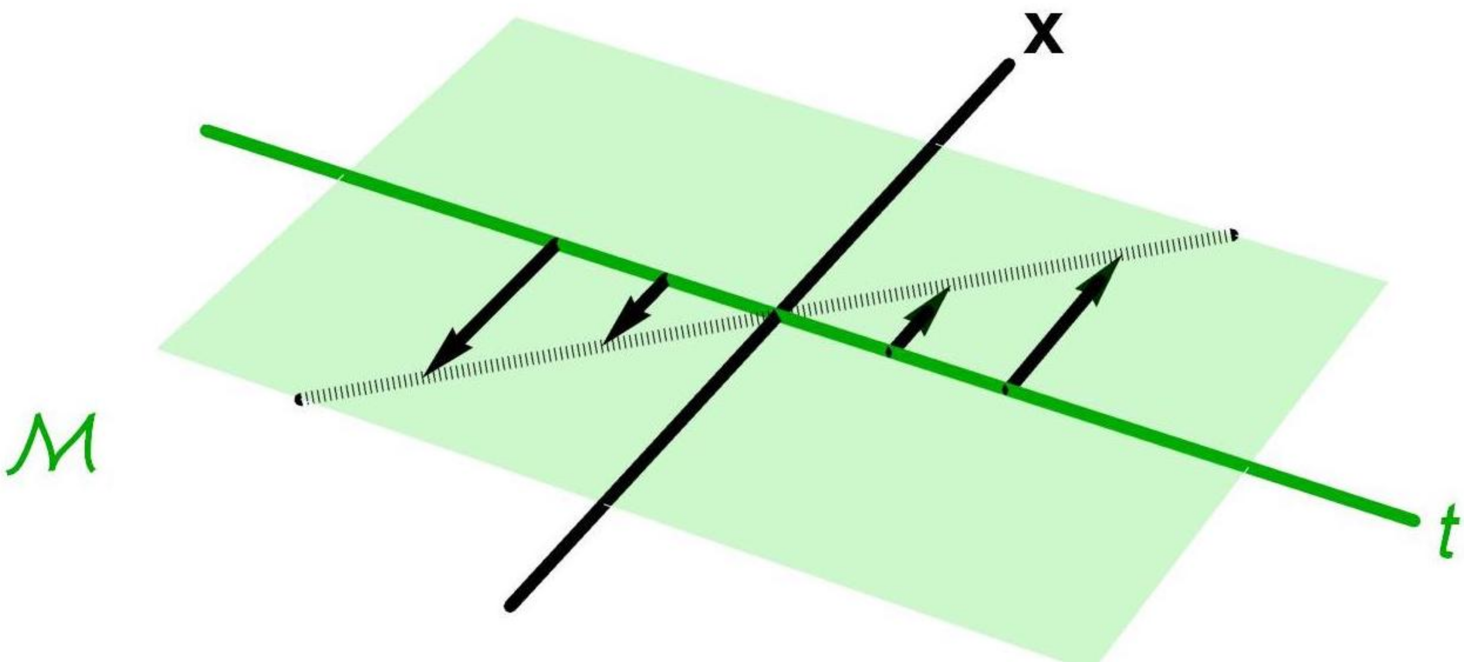


Fig.2 Galilei boost acting on Galilei space-time

CARROLL STRUCTURE

Lévy-Leblond 1965 :

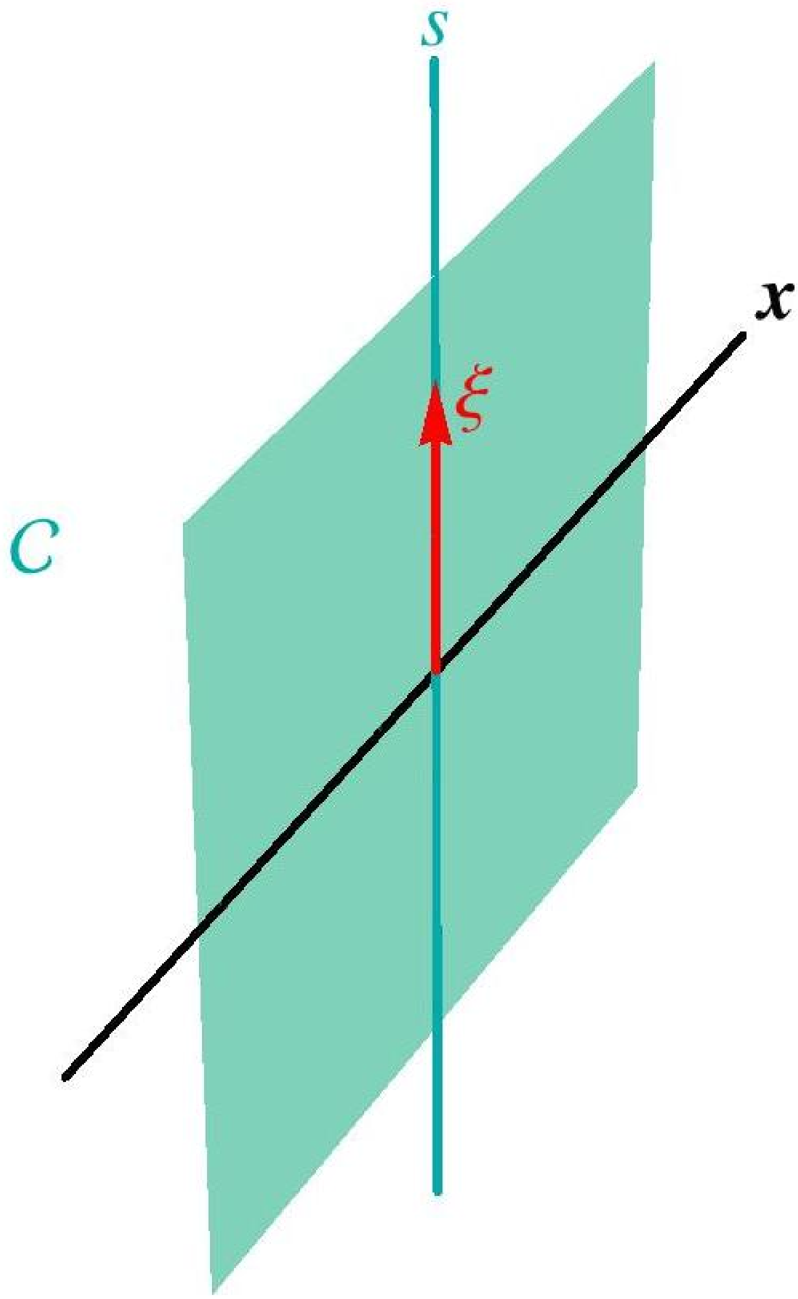


Fig.2 : Carroll space-time, $\mathcal{C}$ described by $\begin{pmatrix} \boldsymbol{x} \\ s \end{pmatrix}$, is endowed with vector $\boldsymbol{\xi}$ which generates kernel of (singular) [space-] "metric" $\bar{G} = \delta_{AB}\, dx^A dx^B$.

Carroll group Carr$(d+1)$ $\sim$

"Carrollian boosts"

$$\begin{cases} \boldsymbol{x}' &= \boldsymbol{x} \\ s' &= s - \boldsymbol{b}\cdot\boldsymbol{x} \end{cases} \tag{2}$$

NB: In NR QM wave fct including a phase factor,

$$\psi'(x,t) = e^{i(\mathbf{b}\cdot\boldsymbol{x}-\frac{1}{2}\mathbf{b}^2 t)}\psi(\boldsymbol{x}-\mathbf{b}t,t) \tag{3}$$

infinitesimally (for $t=0$):

$$\widehat{\boldsymbol{\beta}}\,\psi(\boldsymbol{x},0) = (-\boldsymbol{\beta}\cdot\boldsymbol{x})\,\psi(\boldsymbol{x},0) \tag{4}$$

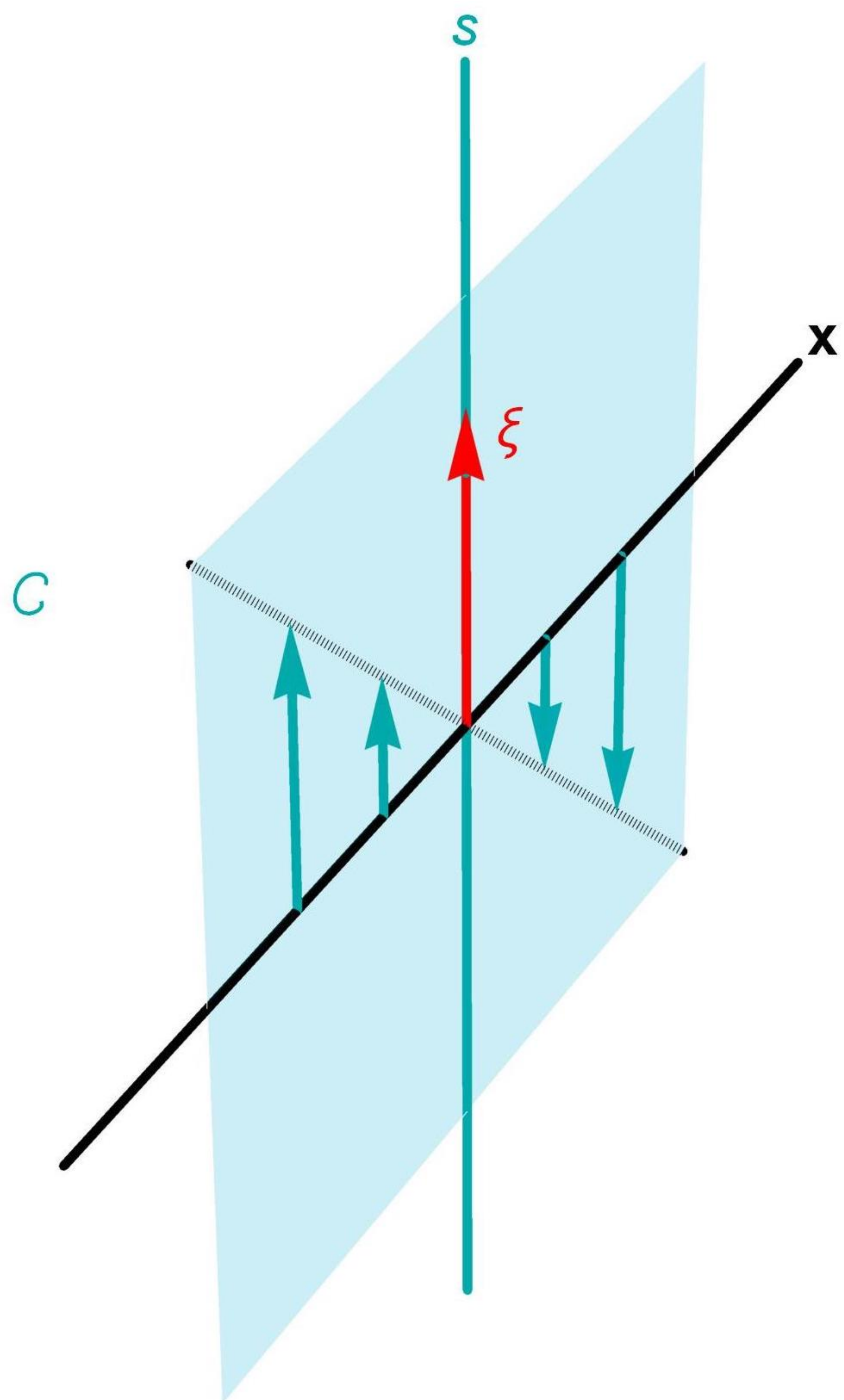


Fig.3 Carroll boost acts on flat **Carroll** space-time

Carroll group represented by $(d+2) \times (d+2)$ matrices

$$\begin{pmatrix} R & 0 & \mathbf{c} \\ -\mathbf{b}^T R & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \tag{5}$$

where $R \in \mathrm{O}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, $f \in \mathbb{R}$. Acts on $\begin{pmatrix} \boldsymbol{x} \\ s \\ 1 \end{pmatrix}$ affinely by matrix action. Carroll Lie algebra $\mathfrak{carr}(d+1)$

$$Z = \begin{pmatrix} \omega & 0 & \boldsymbol{\gamma} \\ -\boldsymbol{\beta}^T & 0 & \varphi \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

$\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$, and $\varphi \in \mathbb{R}$ acts on Carroll space-time as

$$X = (\omega^A_B x^B + \gamma^A)\frac{\partial}{\partial x^A} + \left(\varphi \boxed{-\beta_A x^A}\right)\frac{\partial}{\partial s}, \tag{7}$$

where $\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$, and $\varphi \in \mathbb{R}$.

N.B. : *Galilei Lie algebra* $\mathfrak{gal} \equiv \mathfrak{gal}(d+1)$

$$X = (\omega^A_B x^B + \boxed{\beta^A t} + \gamma^A)\frac{\partial}{\partial x^A} + \epsilon\frac{\partial}{\partial t} \tag{8}$$

where $\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$ and $\epsilon \in \mathbb{R}$.

Unification: Bargmann manifolds

A *Bargmann manifold** is

(i) a $(d+2)$-dim manif B

(ii) endowed with metric G of signature $(d+1,1)$

(iii) carries nowhere vanishing, complete, null "vertical" vector ξ, parallel-transported by Levi-Civita connection, ∇.

L. P. Eisenhart, "Dynamical trajectories and geodesics", Annals. Math. **30** 591-606 (1928).

J. B. Kogut and D. E. Soper, "Quantum Electrodynamics in the Infinite Momentum Frame," Phys. Rev. D **1** (1970) 2901.

C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, "Bargmann Structures and Newton-Cartan Theory," Phys. Rev. D **31** (1985) 1841.

* Introduced by Duval et al as geometrical structure underlying Bargmann [$\equiv$ centrally extended Galilei] group.

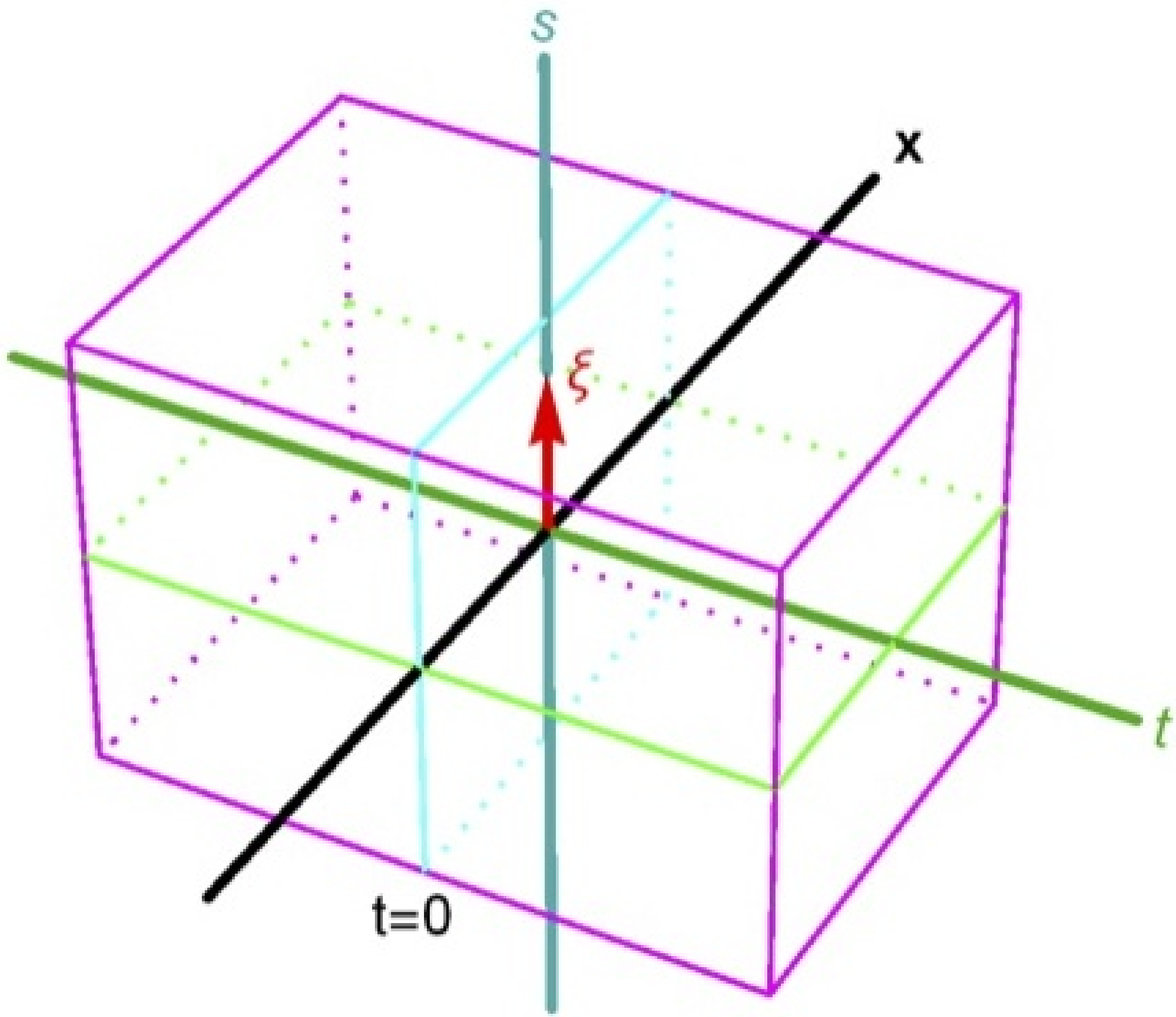


Fig. 4 : Bargmann space : $(d+1,1)$ dim manifold with Lorentz metric & coordinates $(\boldsymbol{x}, t, s)$, endowed with covariantly constant null vector $\xi = \partial_s$.

Flat Bargmann structure $\sim$ Minkowski space :

$$B = \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ t \\ s \end{pmatrix} \right\}, \tag{9}$$

$$G = \delta_{AB}\, dx^A dx^B + 2dtds, \tag{10}$$

$$\boldsymbol{\xi} = \partial_s\,. \tag{11}$$

Both s & t light-cone (null), coords. t has dimension of time, coordinate s has that of action/mass.

• Factoring out "vertical" translations along ξ, $(d+1)$-dim quotient acquires Newton-Cartan structure

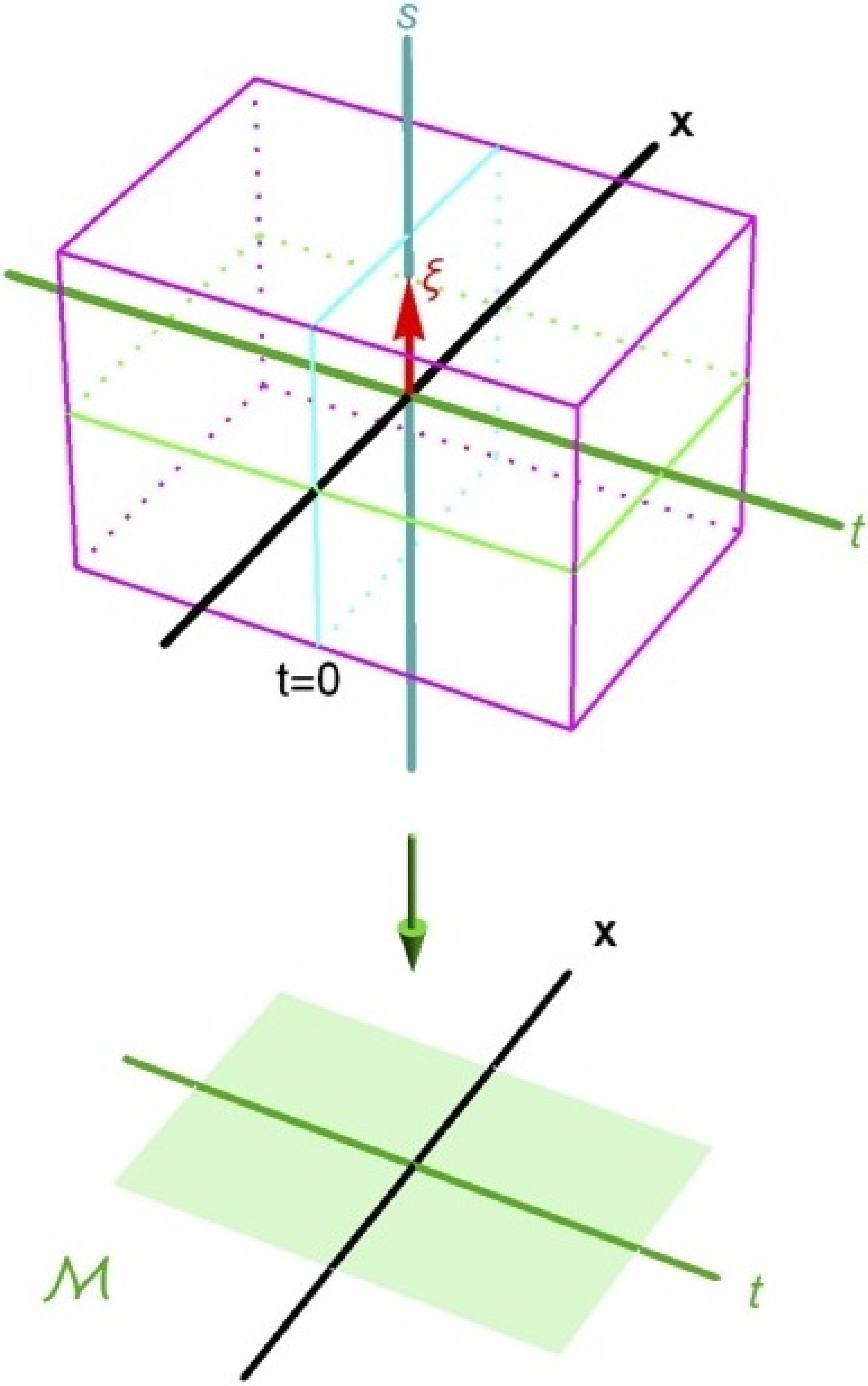


Fig.5 : Bargmann projects to Galilean space-time.

■ Restriction to $t = \text{const}$ turns off $dtds$ in metric (10), leaving singular "metric" $\delta_{AB}\, dx^A dx^B$ $\rightsquigarrow$ admits flat Carroll structure embedded into Bargmann space $\mathcal{C} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ 0 \\ s \end{pmatrix} \right\}$

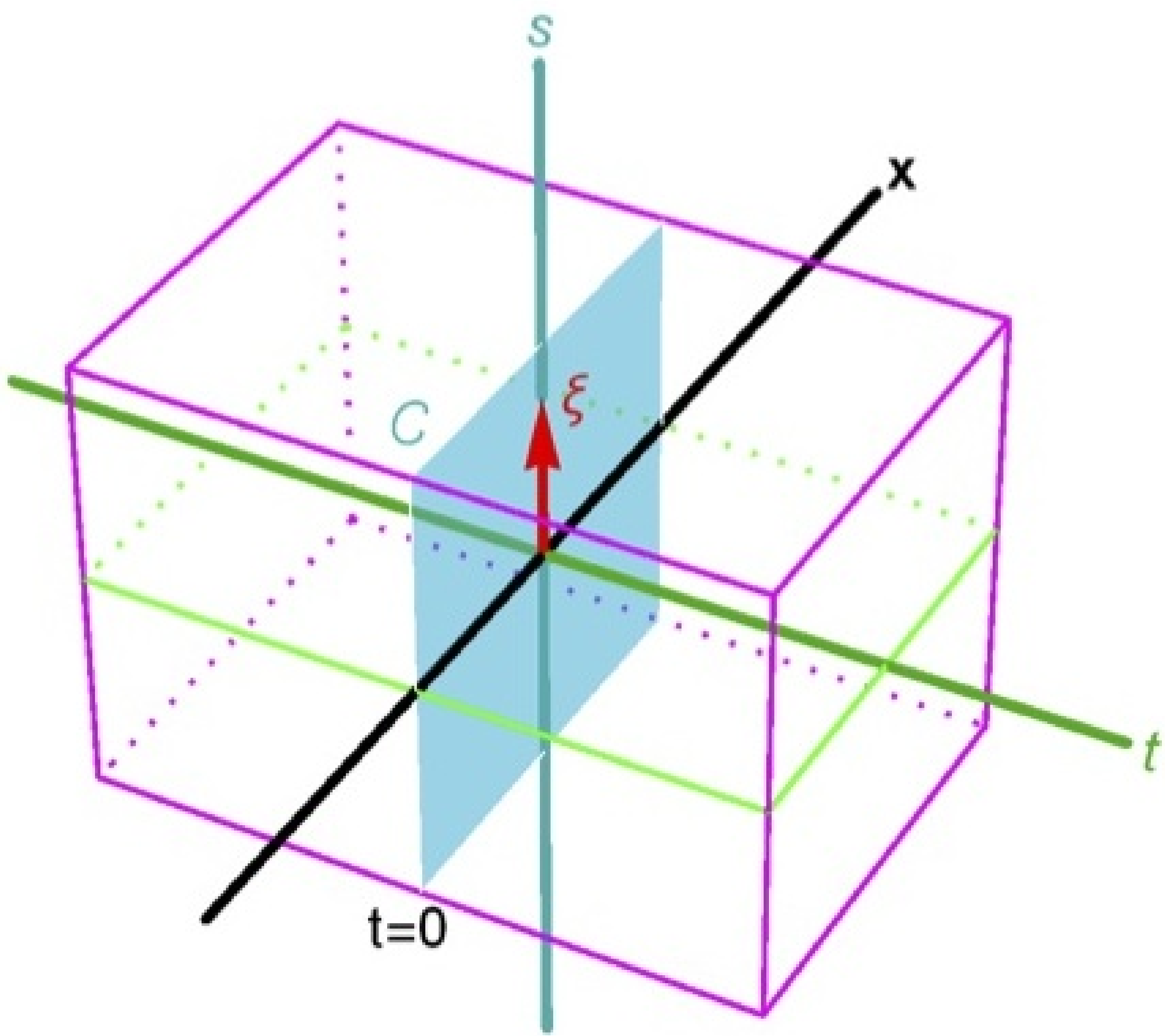


Fig.6 : $t = \text{const}$ slice is "Carroll space-time" $\mathcal{C}$ embedded into Bargmann space.

Symmetries

ξ-preserving isometries of Bargmann :

$$a = \begin{pmatrix} R & \mathbf{b} & 0 & \mathbf{c} \\ 0 & 1 & 0 & e \\ -\mathbf{b}^T R & -\frac{1}{2}\mathbf{b}^2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (12)$$

where $R \in \mathrm{O}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, and $e, f \in \mathbb{R}$ form centrally extended Galilei [$\equiv$ Bargmann] group Barg $\equiv \mathrm{Barg}(d+1)$. Boost :

$$\begin{pmatrix} \boldsymbol{x} \\ t \\ s \end{pmatrix} \to \begin{pmatrix} \boldsymbol{x} + \mathbf{b}t \\ t \\ s - \mathbf{b}\cdot\boldsymbol{x} + \frac{1}{2}\mathbf{b}^2 t \end{pmatrix} \qquad (13)$$

N.B. : lifting ordinary wave fct to equivariant ($\equiv \partial_s \Psi = im\Psi$) on B-space, Galilei boost action (3) is Bargmann action. Affine action on $\begin{pmatrix} \boldsymbol{x} \\ t \\ s \\ 1 \end{pmatrix} \rightsquigarrow$ Bargmann algebra $\mathfrak{barg} \equiv \mathfrak{barg}(d+1)$

$$\boxed{(\omega^A_B\, x^B + \beta^A\, t + \gamma^A)\frac{\partial}{\partial x^A} + \varepsilon \frac{\partial}{\partial t} + (\varphi - \beta_A\, x^A)\,\frac{\partial}{\partial s}} \qquad (14)$$

where $\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\varepsilon, \varphi \in \mathbb{R}$.

Seen before: restriction of Bargmann space to $t = 0$ is Carroll manifold $\mathcal{C}$ left invariant by restriction of Bargmann action (14) with $e = 0 \rightsquigarrow$ action of Carr, embedded into Bargmann group,

$$\begin{pmatrix} R & 0 & \mathbf{c} \\ -\mathbf{b}^T R & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \hookrightarrow \begin{pmatrix} R & \mathbf{b} & 0 & \mathbf{c} \\ 0 & 1 & 0 & 0 \\ -\mathbf{b}^T R & -\frac{1}{2}\mathbf{b}^2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (15)$$

where $R \in \mathrm{O}(d)$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, $f \in \mathbb{R}$.

Carr$(d+1)$: $e = 0$ subgroup of Barg(d+1).

Infinitesimally:

$$(\omega^A_B\, x^B + \gamma^A)\frac{\partial}{\partial x^A} + (\varphi - \beta_A\, x^A)\,\frac{\partial}{\partial s} \qquad (16)$$

$\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\varphi \in \mathbb{R}$ (seen before).

N.B. : for $t = t_0$ Carroll boost acts as

$$v \to v - \mathbf{b}.\boldsymbol{x} - \tfrac{1}{2}\mathbf{b}^2\, t_0 \qquad (17)$$

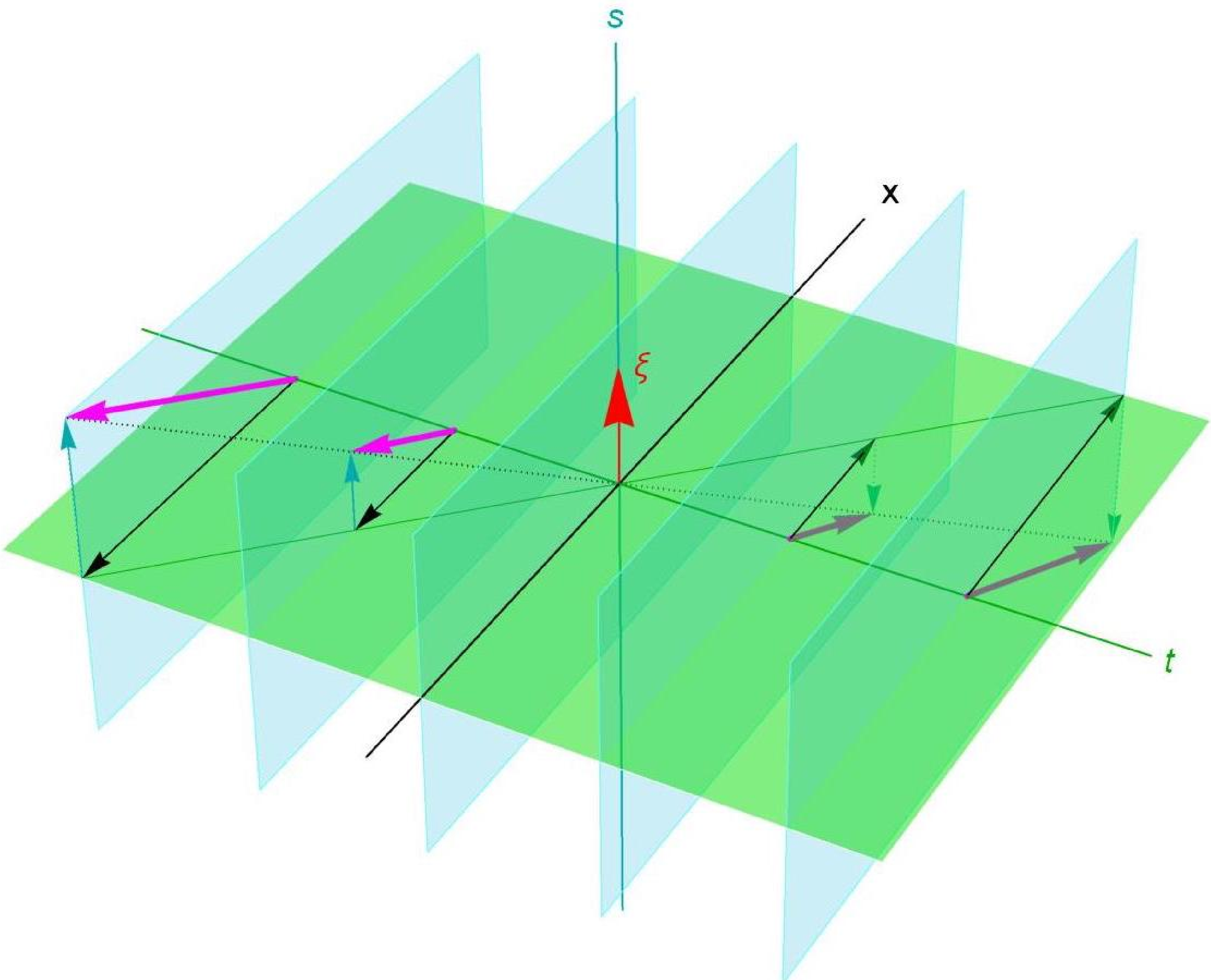


Fig.7 Boost acting on flat Bargmann [≡ Minkowski] space

Plane gravitational waves

In Brinkmann coordinates

$$ds^2 = d\boldsymbol{X}^2 + 2dU dV - K(U, \boldsymbol{X})\, dU^2 \qquad (18)$$

U and V light-cone coords, $\boldsymbol{X} = (X_1, X_2) \sim$ transverse plane. Vacuum Einstein eqn satisfied with

$$K(U, \boldsymbol{X}) = \mathcal{A}(U)\Big(X_1^2 - X_2^2\Big) + 2\mathcal{B}(U) X_1 X_2. \qquad (19)$$

Clue: (18) Bargmann space $\sim$ anisotropic oscillator.

P. M. Zhang, P. A. Horvathy, K. Andrzejewski, J. Gonera and P. Kosinski, "Newton-Hooke type symmetry of anisotropic oscillators," Ann. Phys. **333** (2013) 335 [arXiv:1207.2875 [hep-th]].

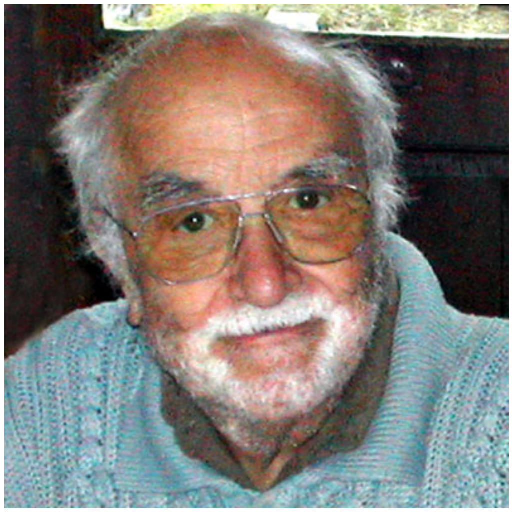
Isometries : Bondi et al 1959. 5-parameters. 1 "vertical" translation + 4 MYSTERIOUS (**not** written explicitely). Torre 2006 Gen. Rel. Grav. **38** (2006) 653 : isometries ∂_V +

$$S_i(U)\partial_i + \dot{S}_i(U) X_i\, \partial_V \qquad (20)$$

where S_i is solution of Sturm-Liouville eqn

$$\boxed{\ddot{S}_i = K_{ij}(U)\, S_j} \qquad (21)$$

(. . . that we can't solve in general . . .)

Souriau 1973  metric in BJR (Baldwin-Jeffery-Rosen) coords :

$$\boxed{ds^2 = \boxed{a_{ij}(u)}\, dx^i dx^j + 2du dv.} \tag{22}$$

Isometries : $u \to u$, completed with

$$\boxed{\boldsymbol{x} \to \boldsymbol{x} + H(u)\mathbf{b} + \mathbf{c},} \tag{23a}$$

$$\boxed{v \to v - \mathbf{b} \cdot \boldsymbol{x} - \tfrac{1}{2}\mathbf{b} \cdot H(u)\mathbf{b} + f} \tag{23b}$$

where $H = (H_{ij})$ is 2×2 matrix

$$\boxed{H(u) = \int_{u_0}^{u} a^{-1}(w)dw.} \tag{24}$$

$\mathbf{c} \in \mathbb{R}^2 \sim$ transverse-space transl, $f \sim$ null translat along v coord.

Group composition law: that of Carroll group with no rotations. $\mathbf{b} \in \mathbb{R}^2$ generates Carroll boost, implemented as in (23).

Flat case: $a_{ij} = \delta_{ij} \quad \Rightarrow$

$$H(u) = (u - u_0)\,\mathrm{Id} \tag{25}$$

choosing $u_0 = 0$

$$\boldsymbol{x} \to \boldsymbol{x} + u\,\mathbf{b}, \tag{26a}$$

$$u \to u, \tag{26b}$$

$$v \to v - \mathbf{b} \cdot \boldsymbol{x} - \tfrac{1}{2}\mathbf{b}^2\,u \tag{26c}$$

Galilei boosts lifted to flat Bargmann space.

(See again at the end)

Relation with Brinkmann-coords ?

1. Given B-profile $K(U)$, solve Sturm-Liouville

$$\boxed{\ddot{P}_{kj} = K_{kr} P_{rj}} \qquad (27)$$

for U-dept 2×2 matrix $P_{kj}(U)$.

2. Putting

$$X^i = P_{ij}\, x^j \qquad U = u \qquad (28a)$$

$$a_{ij}(u) = P_{ri} P_{rj}, \quad V = v - \frac{1}{4}\frac{da_{ij}}{du}\, x^i x^j \qquad (28b)$$

allows to present metric (18) in BJR form

$$ds^2 = \boxed{a_{ij}(u)}\, dx^i dx^j + 2du dv$$

cf. (22) *provided* also $P^\dagger \dot{P} = \dot{P}^\dagger P$.

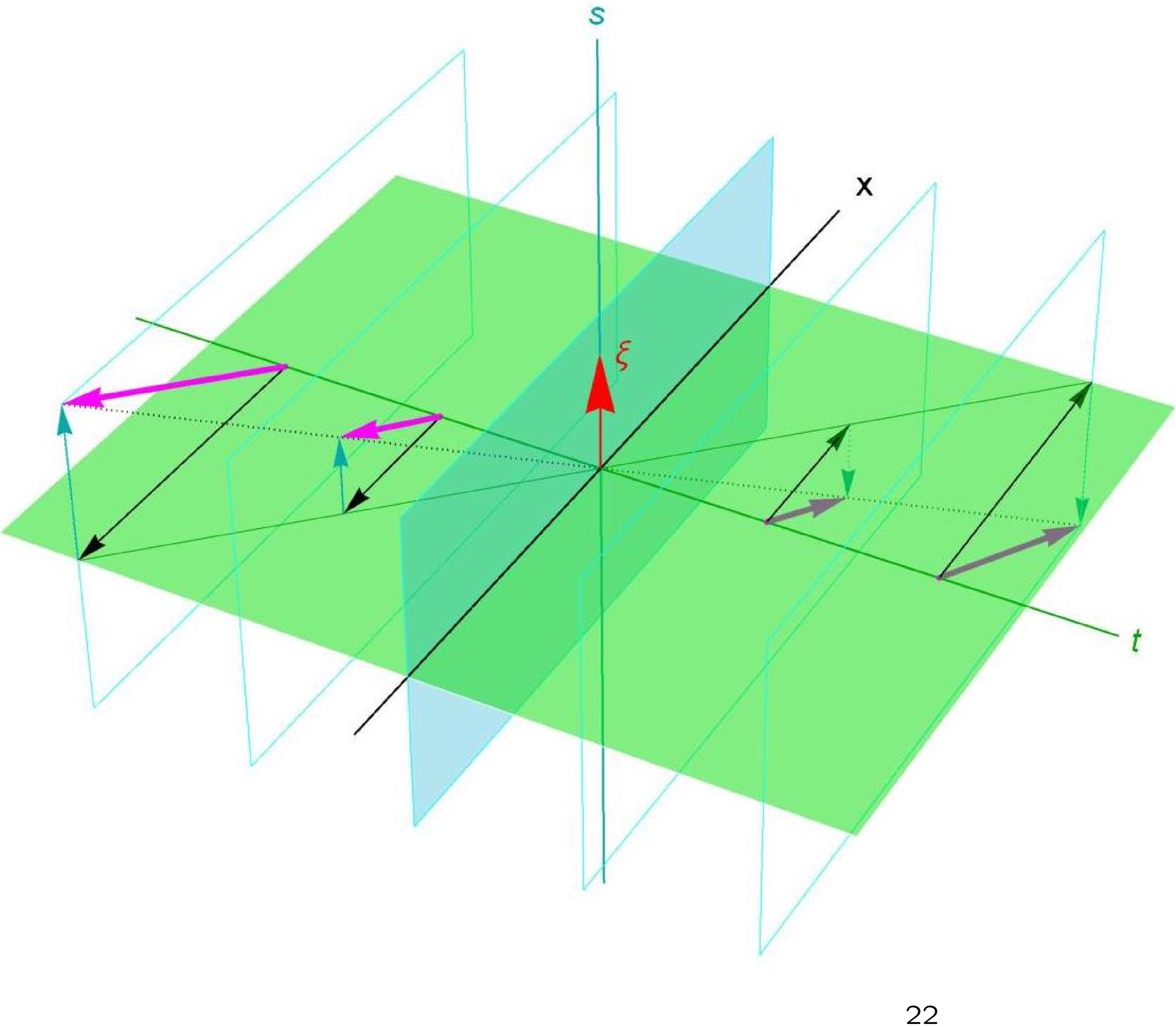
Quadratic "scalar potential" in B, $K_{ij} X^i X^j \, dU^2$ in (18), traded for "time"-dependent" transverse metric $a_{ij}(u)$ (while leaving $U = u$ unchanged).

EXAMPLES

0. Restriction of flat Minkowski space

$$d\boldsymbol{r}^2 + 2dt\,ds$$

to $t = 0$ is Carroll manifold, upon which restriction $e = 0$ of Bargmann group acts consistently with Carroll action.



Linearly polarized "sudden burst" $\sim$ Gaussian profile ($\sim$ anisotropic oscillator with time-dependent frequency)

$$K_{ij}(u)X^iX^j = \frac{e^{-u^2}}{\sqrt{\pi}}\Big((X^1)^2 - (X^2)^2\Big). \qquad (29)$$

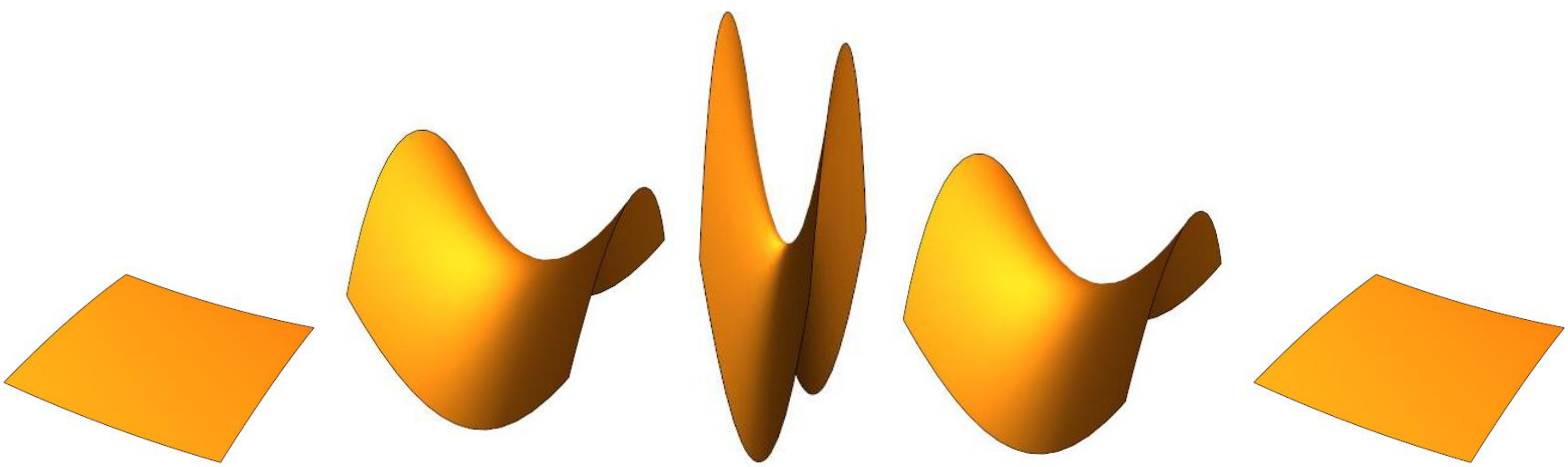


Fig.8 "Time" evolution of wave for "sudden burst" with Gaussian profile $\mathcal{A}(u) = \exp[-u^2]$.

Sandwich wave: $K(u) \neq 0$ only in "wave zone" $U_i < U < U_f$. Assumption : metric Minkowski in "before-zone" $U < U_i$ and flat in "after-zone" $U_f < U$.

Impulsive wave :

$$\mathcal{A}(U) = 2k\,\delta(U) \qquad (30)$$

$k \in \mathbb{R}$. Wave zone suppressed, $U_i = U_f = 0$. SL eqn. (27) solved by

$$\boxed{P(u) = 1 + u\,\theta(u)\,c_0} \qquad (31)$$

where $\theta(u)$ Heaviside, $c_0 = \frac{1}{2}\dot{a}(0+)$ initial "speed" of transverse metric. Can be chosen $c_0 = k\,\mathrm{diag}(1,-1)$.

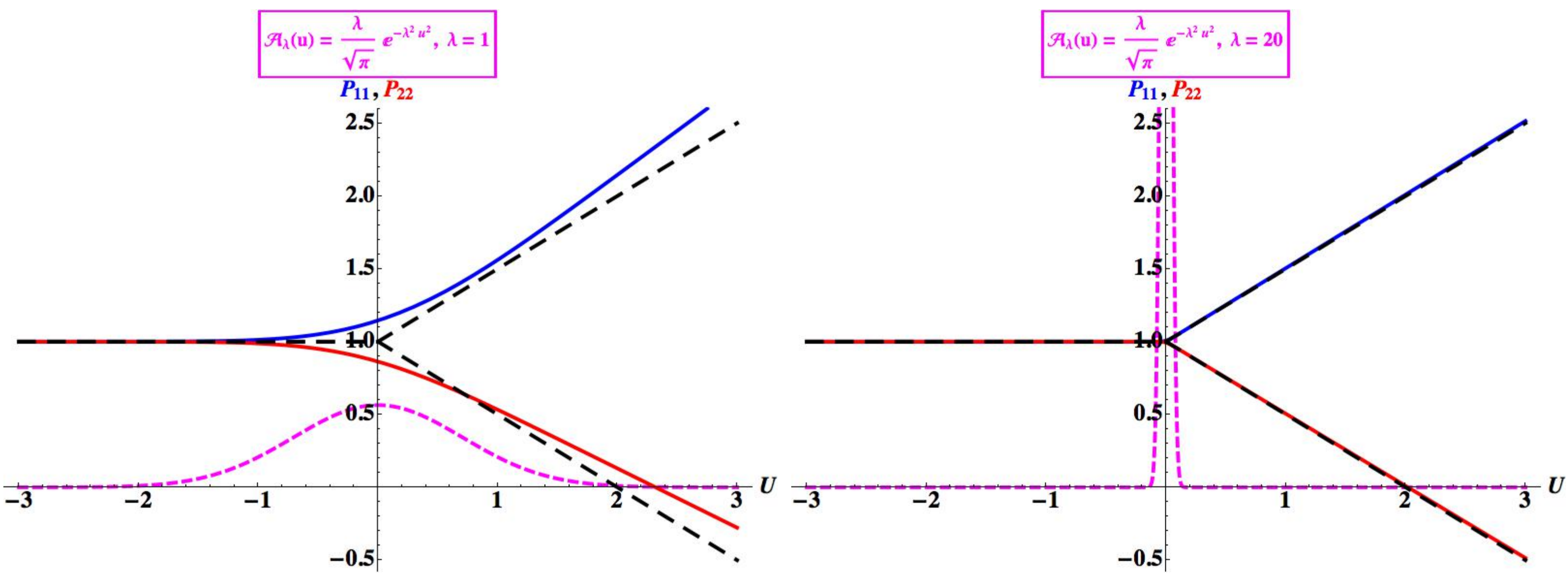


*Fig.9. Numerical solution of S-L eqn (27) for profile $\mathcal{A}_\lambda(U) = (\lambda/\sqrt{\pi})\,e^{-\lambda^2U^2}$ shows that components of diagonal matrix $P_\lambda(U)$ approach, for large λ, those of impulsive wave [in **dashed black**].*

$$a(u) = \begin{cases} 1 & \text{for} \quad u \leq 0, \\ (1 + u\,c_0)^2 & \text{for} \quad u \geq 0. \end{cases} \qquad (32)$$

More generally

$$\mathcal{A}_\lambda(U) = \frac{\lambda}{\sqrt{\pi}} e^{-\lambda^2 U^2}. \qquad (33)$$

Squeezing Gaussians to Dirac δ by letting $\lambda \to \infty$, components of $P_\lambda(u)$ and of transverse metric $a_\lambda(u) = P_\lambda^T(u) P_\lambda(u)$ tend to those of impulsive wave.

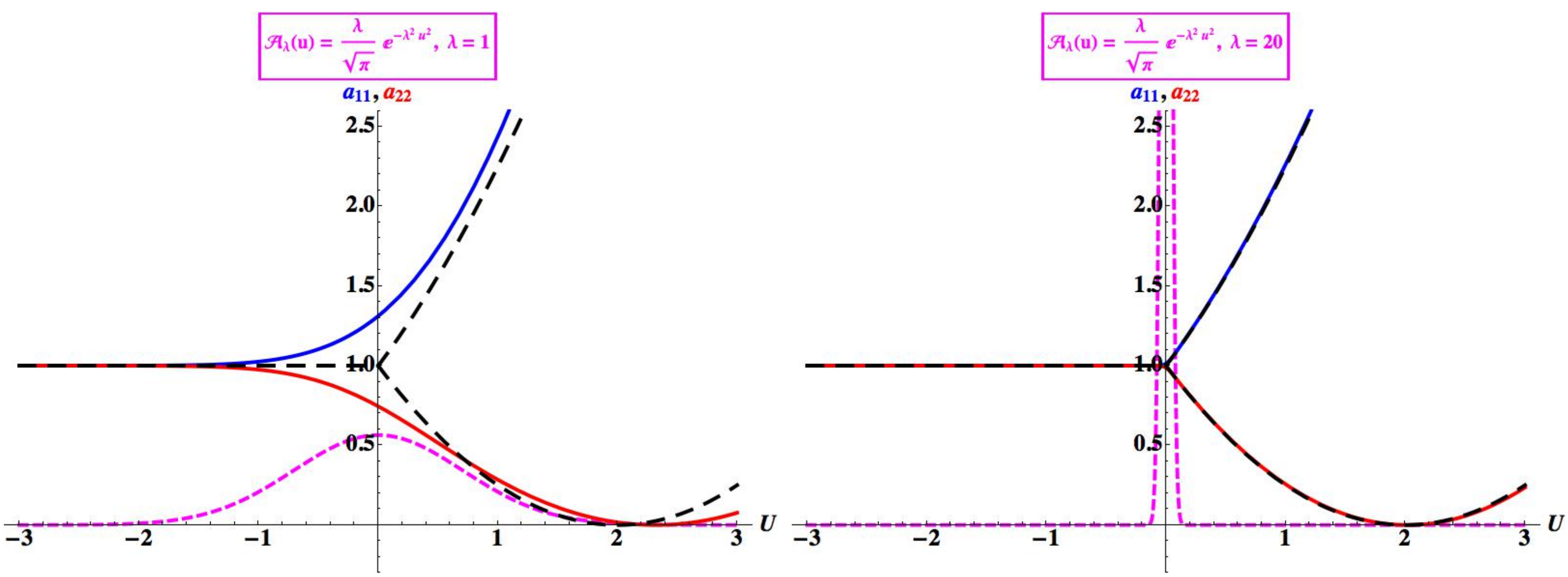


Fig.10. Squeezing Gaussians $\mathcal{A}_\lambda$ to Dirac δ, transverse metrics $a_\lambda(u)$ (in **red** *and* **blue***) tend to that of impulsive wave in BJR coordinates, depicted in* **dashed black** *lines.*

Carroll boost for impulsive GW

Boost implemented as $\boldsymbol{x} \to \boldsymbol{x} + H(u)\mathbf{b}$, $v \to v - \mathbf{b}\cdot\boldsymbol{x} - \frac{1}{2}\mathbf{b}\cdot H(u)\mathbf{b}$ cf. (23). For impulsive

$$H(u) = u\,P^{-1}(u) \tag{34}$$

$$P = \begin{cases} 1 & u \leq 0 \\ \mathrm{diag}(1+u/2, 1-u/2) & u \geq 0 \end{cases} \tag{35}$$

$$H = \mathrm{diag}(H_+, H_-) = \begin{pmatrix} \frac{u}{1+u/2} & \\ & \frac{u}{1-u/2} \end{pmatrix} \quad u \geq 0. \tag{36}$$

Boost with $\mathbf{b} = (b_+, b_-)$ implemented as,

$$x_1 \to x_1 + \frac{u}{1+u/2}\,b_+ \tag{37a}$$

$$x_2 \to x_2 + \frac{u}{1-u/2}\,b_- \tag{37b}$$

$$v \to v - (x_1 b_+ + x_2 b_-) - \frac{1}{2}\left(\frac{u}{1+u/2}\,b_+^2 + \frac{u}{1-u/2}\,b_-^2\right) \tag{37c}$$

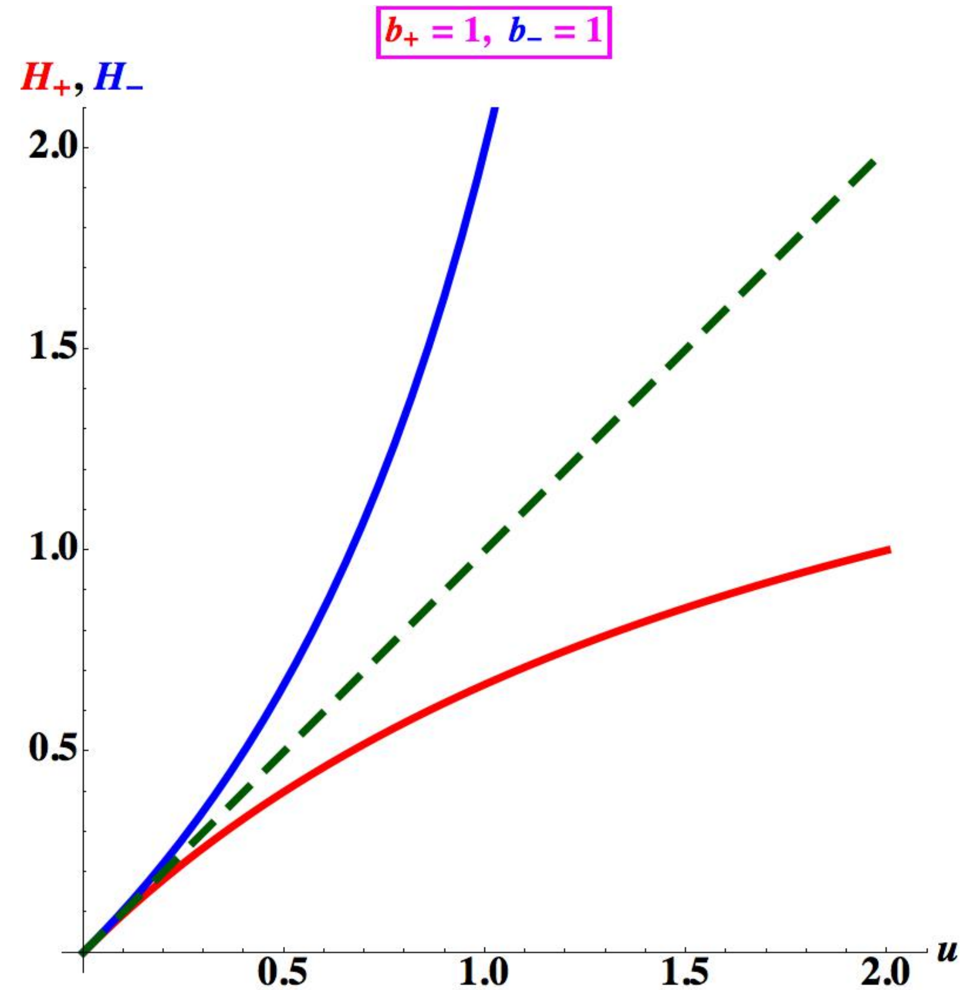


Fig.11 Boost acts on impulsive space-time according to $\boldsymbol{x} \to \boldsymbol{x} + H\mathbf{b}$, $v \to v - \mathbf{b}\cdot\boldsymbol{x} - \frac{1}{2}\mathbf{b}\cdot H(u)\mathbf{b}$. $H = \text{diag}(H_+, H_-)$ but components differ considerably from usual Galilei implementation $H_{Gal} = u\,\text{Id}$.

Polarized oscillating GW with Gaussian envelope

In Brinkmann (B) coordinates $(\boldsymbol{X}, U, V)$ profile of plane GW given by symmetric & traceless 2×2 matrix $H(U) = H_{ij}(U)$

$$\delta_{ij} dX^i dX^j + 2dUdV + H_{ij}(U) X^i X^j dU^2, \quad (38a)$$

$$H_{ij}(U) X^i X^j = \quad (38b)$$

$$\tfrac{1}{2}\mathcal{A}(U)\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{B}(U)\, X^1 X^2$$

$\mathcal{A}$ & $\mathcal{B}(U)$ amplitudes of $+$ and $\times$ polarization states.

(geodesic) eqns motion

$$\frac{d^2\boldsymbol{X}}{dU^2} - H(U)\,\boldsymbol{X} = 0, \quad H(U) = \tfrac{1}{2}\begin{pmatrix} \mathcal{A} & \mathcal{B}(U) \\ \mathcal{B}(U) & -\mathcal{A} \end{pmatrix}, \quad (39a)$$

$$\frac{d^2V}{dU^2} + \frac{1}{4}\frac{d\mathcal{A}}{dU}\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{A}\left(X^1\frac{dX^1}{dU} - X^2\frac{dX^2}{dU}\right)$$

$$+ \frac{1}{2}\frac{d\mathcal{B}}{dU} X^1 X^2 + \mathcal{B}\left(X^2\frac{dX^1}{dU} + X^1\frac{dX^2}{dU}\right) = 0. \quad (39b)$$

Circularly Polarized Waves

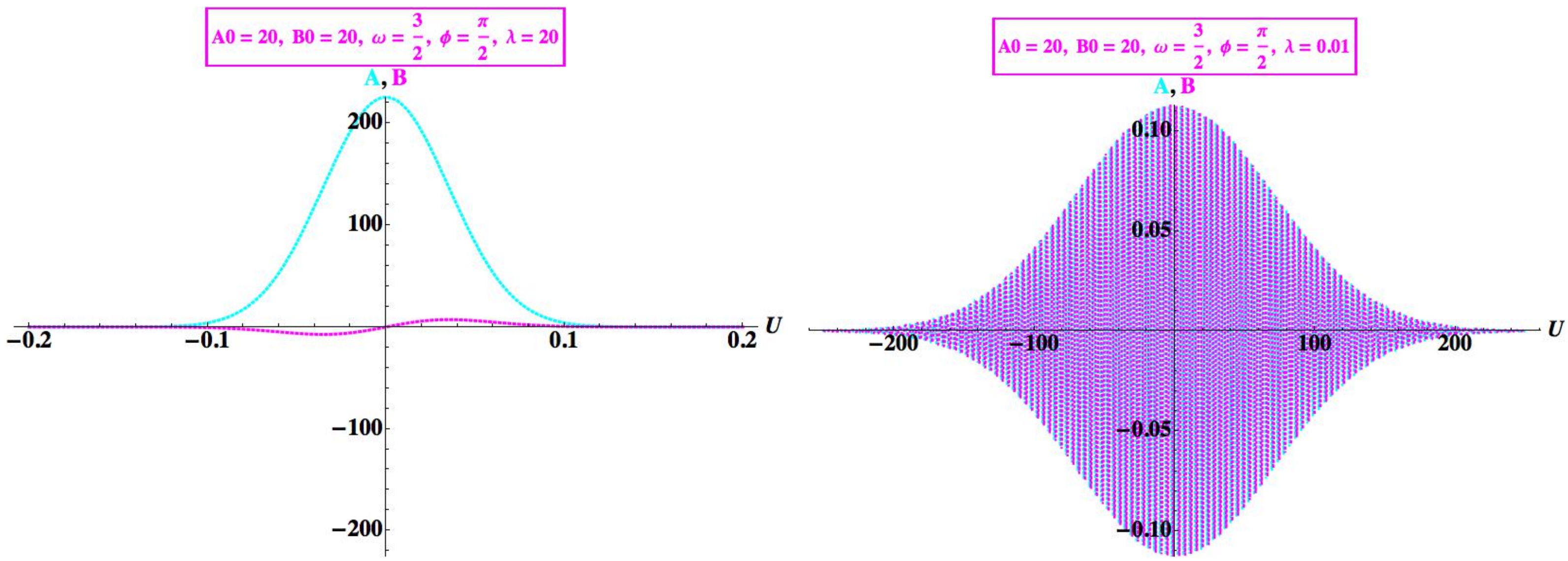
- If $\mathcal{A}(U) = 0$ or $\mathcal{B}(U) = 0$, wave linearly polarized.

- Polarized waves approximating the sandwich by Gaussian,

$$\mathcal{A}(U) = A_0 \frac{\lambda}{\sqrt{\pi}} e^{-\lambda^2 U^2} \cos(\omega U), \qquad (40a)$$

$$\mathcal{B}(U) = B_0 \frac{\lambda}{\sqrt{\pi}} e^{-\lambda^2 U^2} \sin(\omega U). \qquad (40b)$$

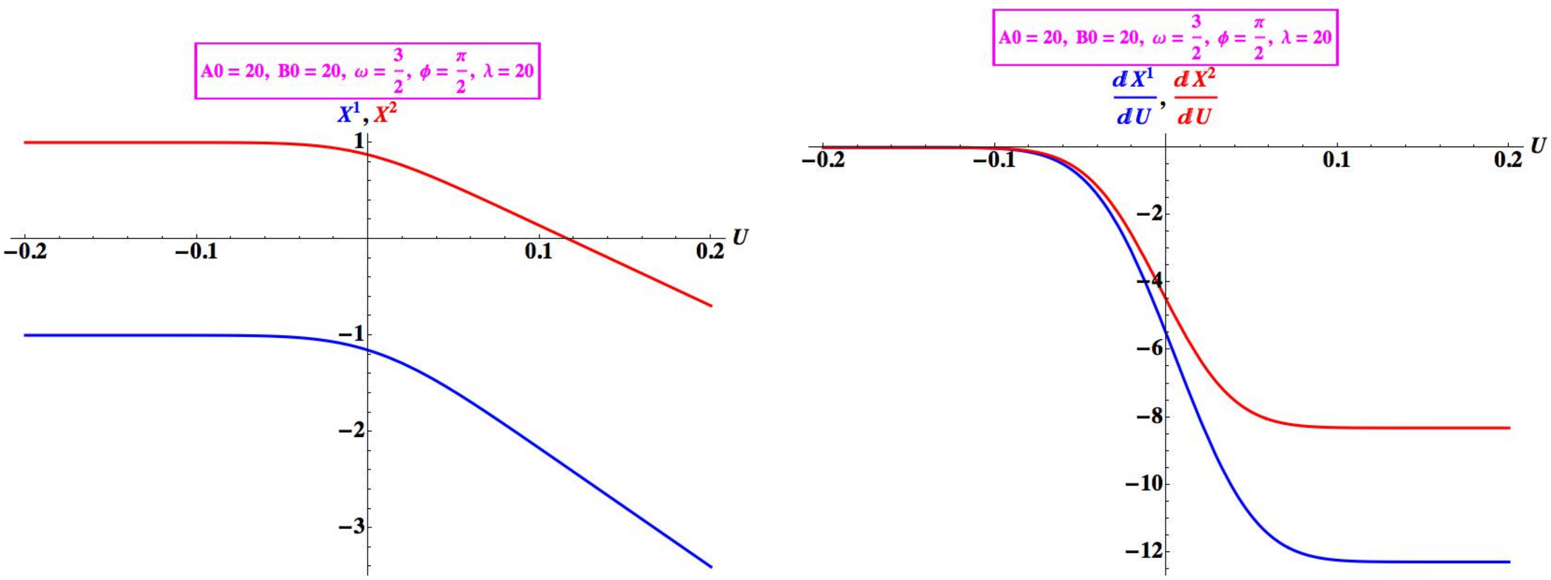
choose $A_0 = B_0$.



Profile of circularly polarized sandwich waves for large / small λ. For $\lambda \to \infty$ Gaussian profile approximates polarized impulsive wave; for $\lambda \to 0$ becomes weak but wide. (Note different scales).

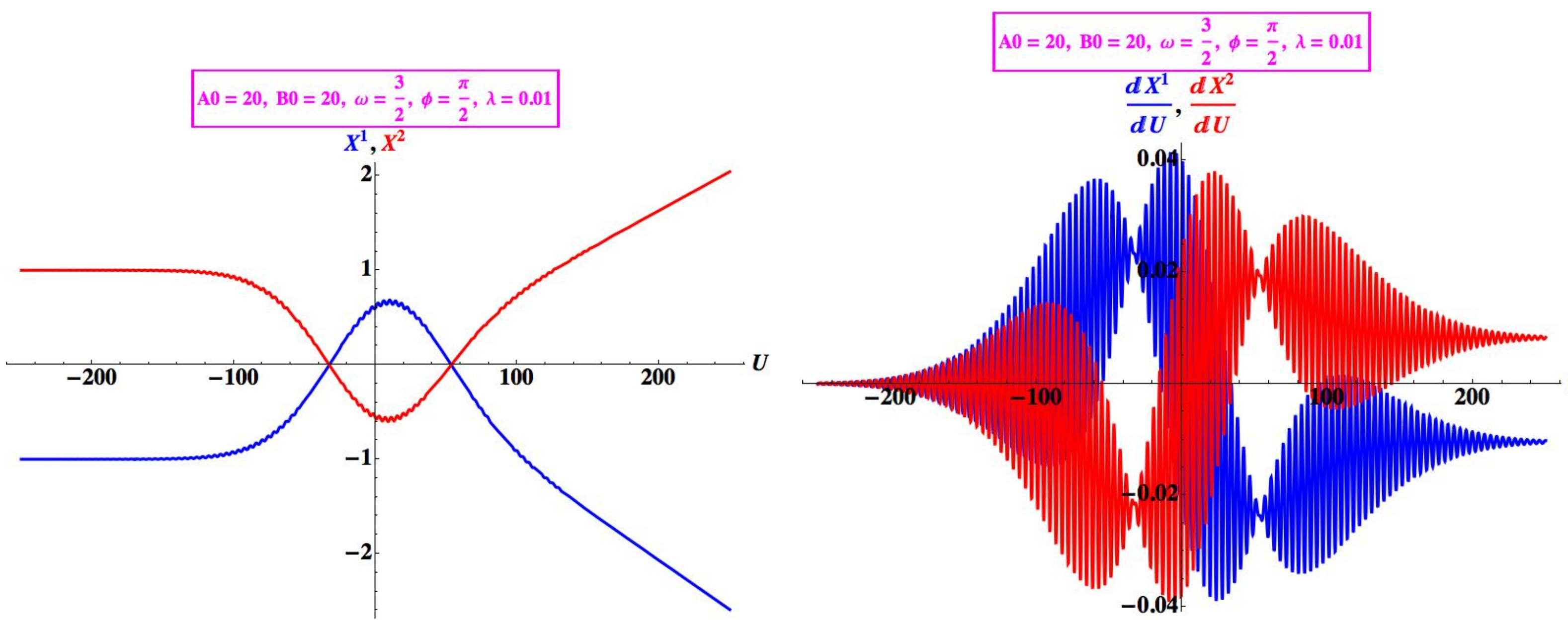
Numerically calculated trajectories and velocities hint at following behavior:

• In $\boxed{\text{large-}\lambda}$ regime Gaussian thin & high, motion roughly along straight lines with constant velocity, except in short ($\sim O(1/\lambda)$ transitory "inside" region, where trajectory sharply bent, velocity changes rapidly from zero to non-zero value — reminiscent of motion in impulsive wave.



Trajectories / velocities after passage of circularly polarized Gaussian sandwich wave (40) in impulsive limit $\lambda \to \infty$. blue and red colors refer to transverse components X^1 and X^2. Trajectory bent in (narrow) inside zone & then escapes with non-vanishing constant velocity in flat after-zone.

• In $\boxed{\textit{small-}\lambda}$ regime profile wide & low, with many oscillations in inside zone. Apart of fine "denting", motion "reasonably regular" in inside zone. Effect of "denting" important when velocity is plotted. Trajectories suffer weak rotation.



Trajectories / velocities for small λ, describing particle motion in wide but weak circularly polarized Gaussian sandwich GW (40). blue and red colors refer to transverse components X^1 and X^2. Particle initially at rest has complicated motion in inside zone however escapes with non-zero constant velocity in flat after-zone.

In outside zones, $U < U_i$ & $U > U_f$, everything smoothes out for both regimes: the trajectory (just like on the linearly polarized case) follows straight lines with constant velocity.

Large and small-λ regimes differ in inside zone: motion in after zone always simple.

boost implemented

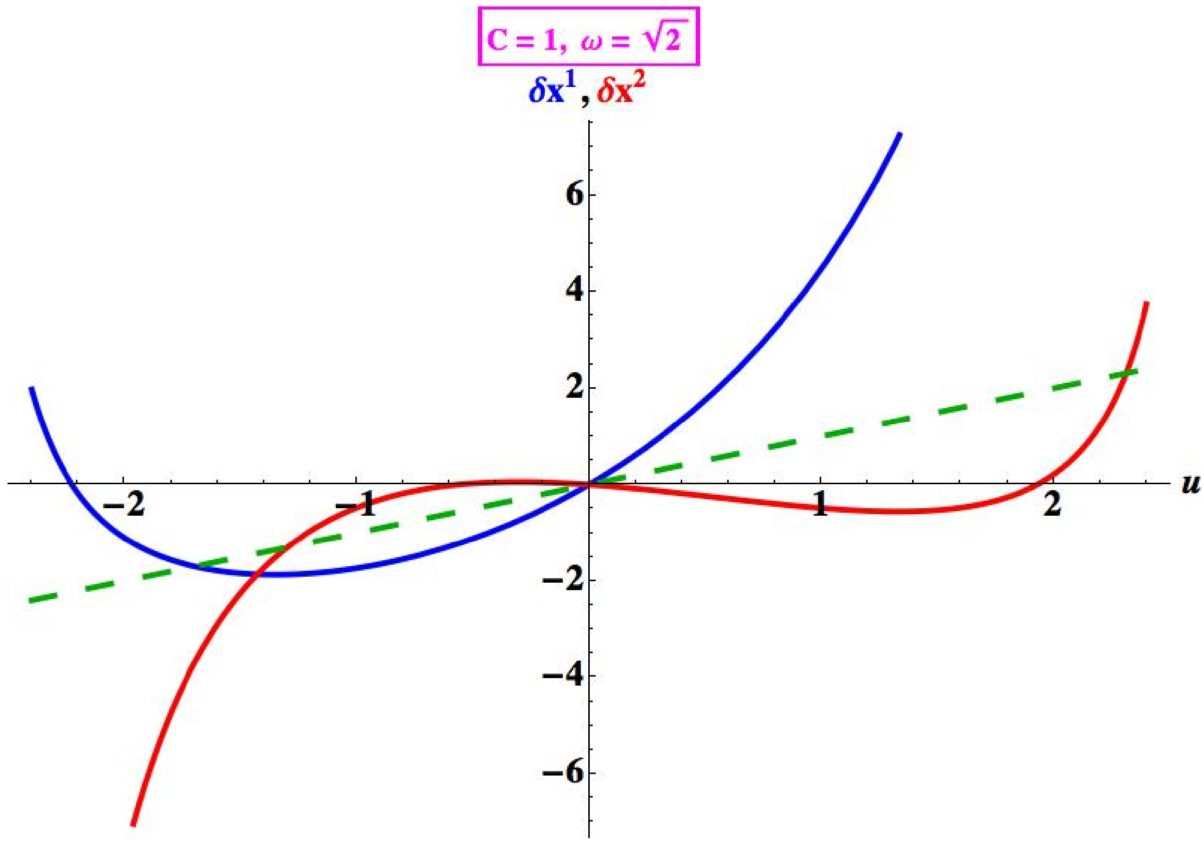


Fig.12 Boost acts on polarized oscillating GW with Gaussian envelope according to

$$\boldsymbol{x} \to \boldsymbol{x} + H\mathbf{b},\, v \to v - \mathbf{b}\cdot\boldsymbol{x} - \tfrac{1}{2}\mathbf{b}\cdot H(u)\mathbf{b}$$

Components differ considerably from **Galilei** implementation $\mathbf{H} = \mathbf{u}\,\mathrm{Id}$.