Symmetries and degeneracies of spin chains

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NUCLEAR PHYSICS B

NLIE for hole excited states in the sine-Gordon model with two boundaries

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Y-system for $Y = 0$ brane in planar AdS/CFT

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Gabor & Zoli

common threads:
integrability
&
boundaries

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The spectrum of tachyons in AdS/CFT

Zoltán Bajnok,[a] Nadav Drukker,[b] Árpád Hegedűs,[a] Rafael I. Nepomechie,[c] László Palla,[d] Christoph Sieg[e] and Ryo Suzuki[f]

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Pohang 2008


APCTP-IEU Focus Program on Solving AdS/CFT 2
Date: July 23 (Wed) - August 1(Fri), 2014
Place: APCTP Headquarters, Pohang, Korea

Pohang 2014

Natal 2012

$$\{1, 1, 10, 14, 14, 21, 30, 30\}$$

Degeneracies of a certain *anisotropic* integrable quantum spin chain

- A main aim of this talk is to explain such degeneracies
- Due to rich symmetries!

Outline

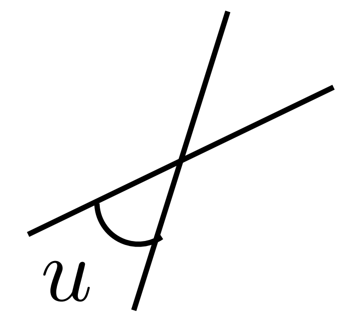
1. Introduction
2. Symmetries & degeneracies of integrable spin chains
3. Generalization

   1802.04864 + A. Retore

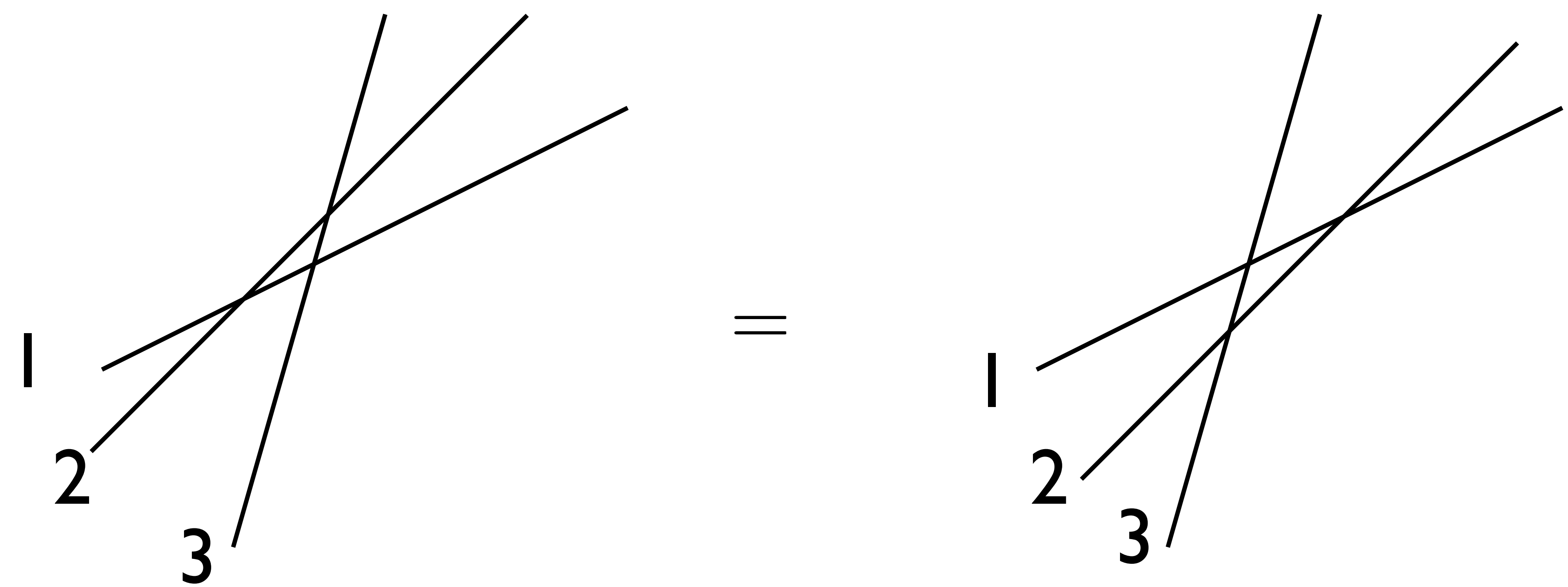
   1805.10144 + R. Pimenta
4. Conclusions

1. Introduction

Key to quantum integrability:

R-matrix $R(u) \sim$ 

Solution of Yang-Baxter equation (YBE)



$$R_{12}(u-v)\,R_{13}(u)\,R_{23}(v) = R_{23}(v)\,R_{13}(u)\,R_{12}(u-v)$$

Classes of R-matrices:

- Rational invariant under Lie group G

Ex: $$R(u)=\left(\begin{array}{cc|cc} u+i & 0 & 0 & 0 \\ 0 & u & i & 0 \\ \hline 0 & i & u & 0 \\ 0 & 0 & 0 & u+i \end{array}\right)$$

$G = SU(2)$

"isotropic", XXX

- Trigonometric $\longleftrightarrow$ affine Lie algebra $\hat{g}$

Ex: $$R(u)=\left(\begin{array}{cc|cc} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh(u) & \sinh(\eta) & 0 \\ \hline 0 & \sinh(\eta) & \sinh(u) & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{array}\right)$$

$\hat{g} = A_1^{(1)}$

"anisotropic", XXZ

η : "anisotropy parameter"

- Elliptic

Ex: $$R(u)=\left(\begin{array}{cc|cc} \mathrm{sn}(u+\eta) & 0 & 0 & k\,\mathrm{sn}(u+\eta)\,\mathrm{sn}(u)\,\mathrm{sn}(\eta) \\ 0 & \mathrm{sn}(u) & \mathrm{sn}(\eta) & 0 \\ \hline 0 & \mathrm{sn}(\eta) & \mathrm{sn}(u) & 0 \\ k\,\mathrm{sn}(u+\eta)\,\mathrm{sn}(u)\,\mathrm{sn}(\eta) & 0 & 0 & \mathrm{sn}(u+\eta) \end{array}\right)$$

XYZ

Uses of R-matrices:

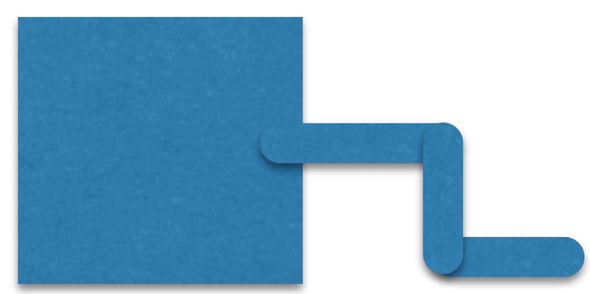
- S-matrices of integrable quantum field theories in 1+1 dimensions

  Ex: sine-Gordon model

- Building blocks of 1-dimensional integrable quantum spin chains

R-matrix

↓



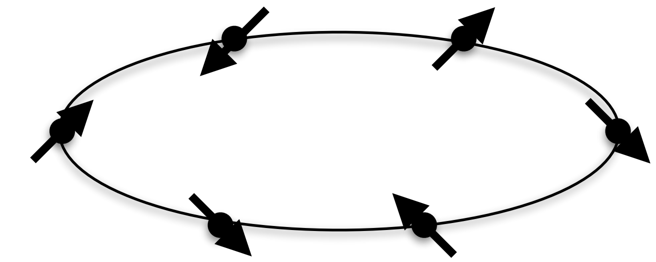
[Faddeev, … 1980's]

↓

integrable closed spin chain

Ex: Heisenberg (XXX) spin chain

XXX R-matrix $\longrightarrow$ Hamiltonian:



N spin $\frac{1}{2}$

$$H^{(1)} = \sum_{n=1}^{N} \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}\,, \quad \vec{\sigma}_{N+1} \equiv \vec{\sigma}_1$$

- quantum many-body model
- integrable:

$$H^{(2)} = \sum_{n=1}^{N} \vec{\sigma}_n \cdot (\vec{\sigma}_{n+1} \times \vec{\sigma}_{n+2})$$

$$H^{(3)} = \dots$$

$$\vdots$$

$$\boxed{\left[H^{(j)}, H^{(k)}\right] = 0}$$

$\Rightarrow$ solvable by Bethe ansatz

- many applications

2. Symmetries and degeneracies of integrable spin chains

- Rational [easy!]

R-matrix with symmetry G

$\Rightarrow$ spin chain has same symmetry G

Ex: Heisenberg (XXX) spin chain has symmetry $SU(2)$

Degeneracies in the spectrum are given by group theory

Ex: N=4

Clebsch-Gordan $\Rightarrow$ $(\underline{\tfrac{1}{2}} \otimes \underline{\tfrac{1}{2}}) \otimes (\underline{\tfrac{1}{2}} \otimes \underline{\tfrac{1}{2}}) = (\underline{0} \oplus \underline{1}) \otimes (\underline{0} \oplus \underline{1})$

$$= 2 \cdot \underline{0} \oplus 3 \cdot \underline{1} \oplus \underline{2}$$

degeneracies $\{1, 1, 3, 3, 3, 5\}$

spin $s \longrightarrow 2s+1$

2. Symmetries and degeneracies of integrable spin chains

- Trigonometric [not so easy!]

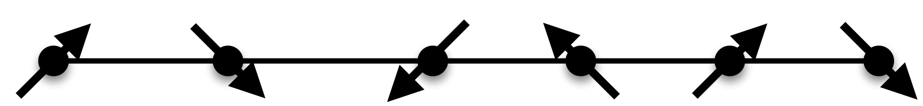
$\hat{g}$ R-matrix

$\Rightarrow$

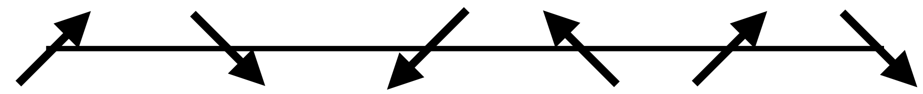
periodic spin chain does *not* have
any nice symmetry or degeneracy

Periodic boundary conditions break symmetry

To realize symmetry, we consider *open* spin chain



We insist that boundary conditions should preserve integrability

Example:  N spin $\frac{1}{2}$

$$H=\sum_{n=1}^{N-1}\left[\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y+\tfrac{1}{2}(q+q^{-1})\sigma_n^z\sigma_{n+1}^z\right]-\tfrac{1}{2}(q-q^{-1})\Big(\sigma_1^z-\sigma_N^z\Big)$$

[Pasquier, Saleur 1990]

- anisotropy parameter q
- integrable
- quantum group symmetry

Quantum Group (QG) symmetry

$$[H\,, S^z] = 0 = \left[H\,, S^{\pm}\right]$$

$$S^z = \sum_{k=1}^{N} S_k^z\,, \qquad S_k^z = \tfrac{1}{2}\sigma_k^z$$

"coproducts"

$$S^{\pm} = \sum_{k=1}^{N} q^{-(S_1^z+\cdots+S_{k-1}^z)}\; S_k^{\pm}\; q^{(S_{k+1}^z+\cdots+S_N^z)}\,, \qquad S_k^{\pm} = \tfrac{1}{2}(\sigma_k^x \pm i\sigma_k^y)$$

$$[S^z, S^{\pm}] = \pm S^{\pm}, \qquad [S^+, S^-] = [2S^z]_q, \qquad [x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}}$$

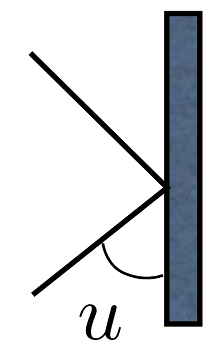
$U_q sl(2)$ $\qquad$ $\rightarrow 2S^z$ for $q \rightarrow 1$

For generic q, the degeneracies are the same as for isotropic!

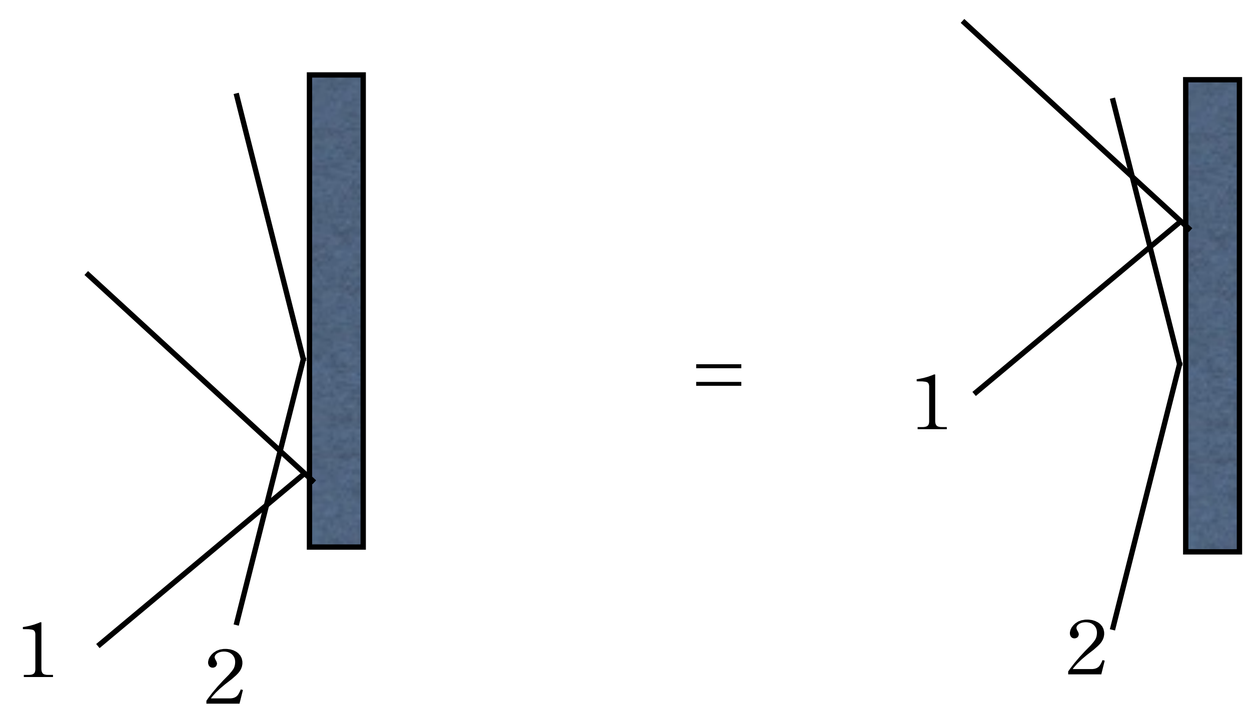
3. Generalization

Key to *boundary* quantum integrability:

K-matrix (reflection matrix) $K^R(u) \sim$

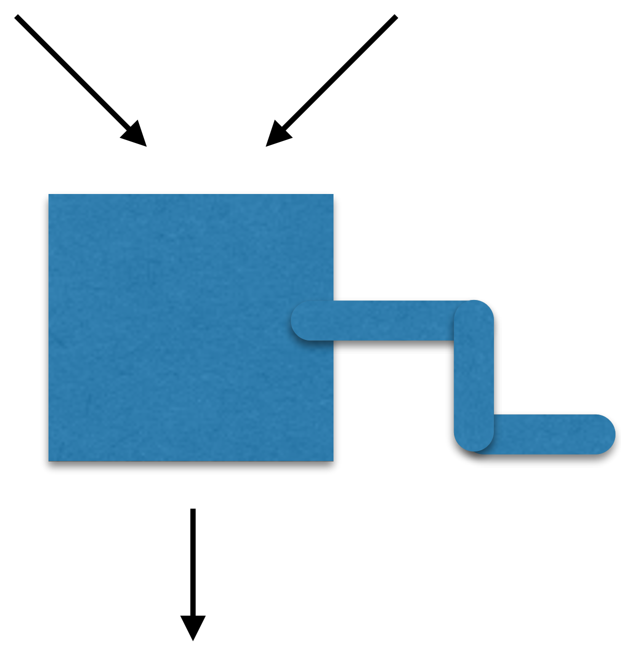


Solution of boundary Yang-Baxter equation (BYBE)



$$R_{12}(u-v)\, K_1^R(u)\; R_{21}(u+v)\, K_2^R(v) = K_2^R(v)\, R_{12}(u+v)\, K_1^R(u)\, R_{21}(u-v)$$

R-matrix K-matrix



[Sklyanin 1988]

integrable *open* spin chain

Let's consider integrable open spin chains constructed with:

- "trigonometric" R-matrix $\longleftrightarrow$ affine Lie algebra $\hat{g}$

[Jimbo 1986, Kuniba 1991]

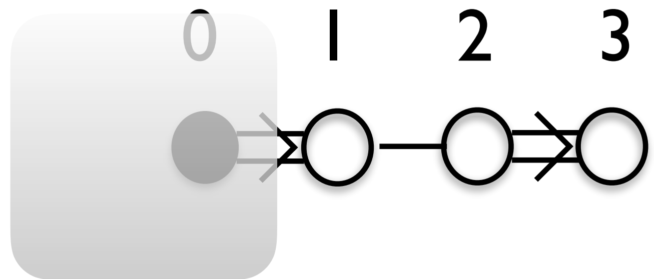
- diagonal K-matrix, depending on discrete parameter p

[Lima-Santos 2006]

> These spin chains have QG symmetry corresponding to removing p^{th} node of Dynkin diagram of $\hat{g}$

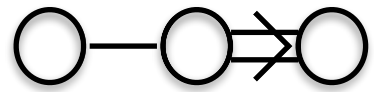
[RIN, Retore 2018]

Ex: $\hat{g} = A_6^{(2)}$



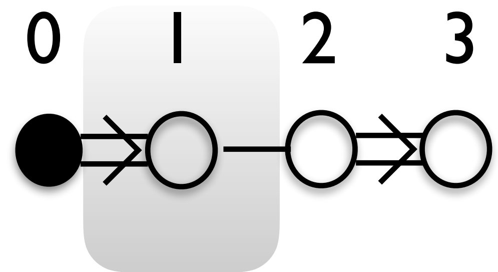
QG symmetry

$p = 0$



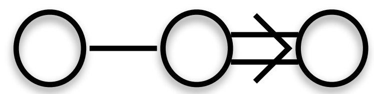
$U_q(B_3)$

Ex: $\hat{g} = A_6^{(2)}$



QG symmetry

$p = 0$

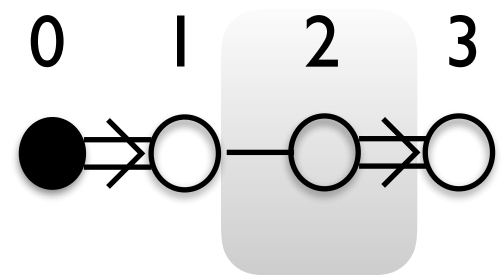


$U_q(B_3)$

$p = 1$

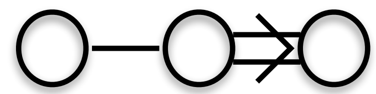
$U_q(B_2) \otimes U_q(C_1)$

Ex: $\hat{g} = A_6^{(2)}$



QG symmetry

$p = 0$



$U_q(B_3)$

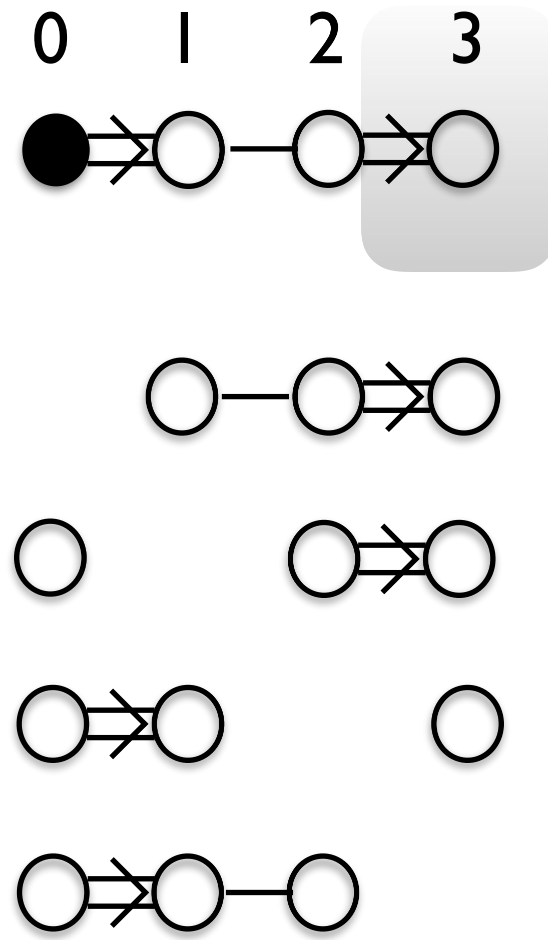
$p = 1$

$U_q(B_2) \otimes U_q(C_1)$

$p = 2$

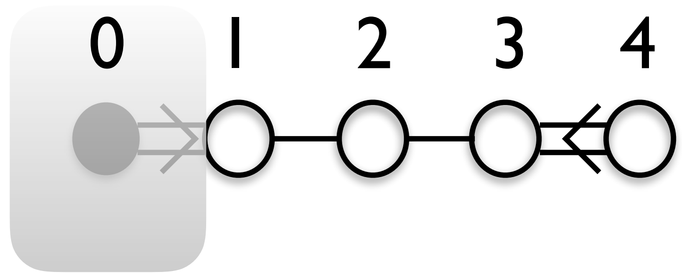
$U_q(B_1) \otimes U_q(C_2)$

Ex: $\hat{g} = A_6^{(2)}$



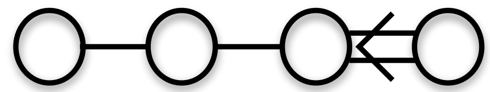
	QG symmetry
$p = 0$	$U_q(B_3)$
$p = 1$	$U_q(B_2) \otimes U_q(C_1)$
$p = 2$	$U_q(B_1) \otimes U_q(C_2)$
$p = 3$	$U_q(C_3)$

Ex: $\hat{g} = C_4^{(1)}$



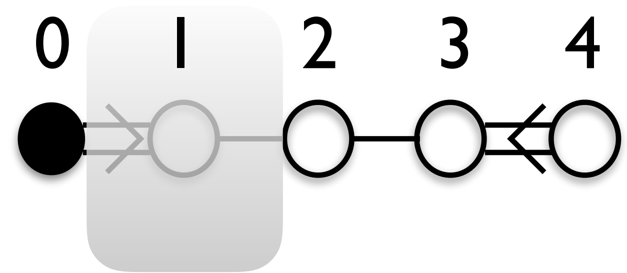
QG symmetry

$p = 0$



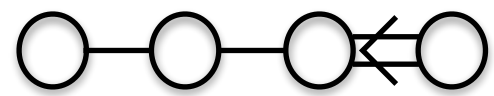
$U_q(C_4)$

Ex: $\hat{g} = C_4^{(1)}$



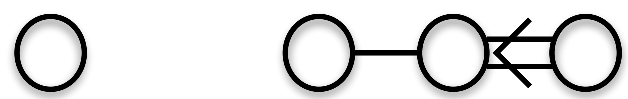
QG symmetry

$p = 0$



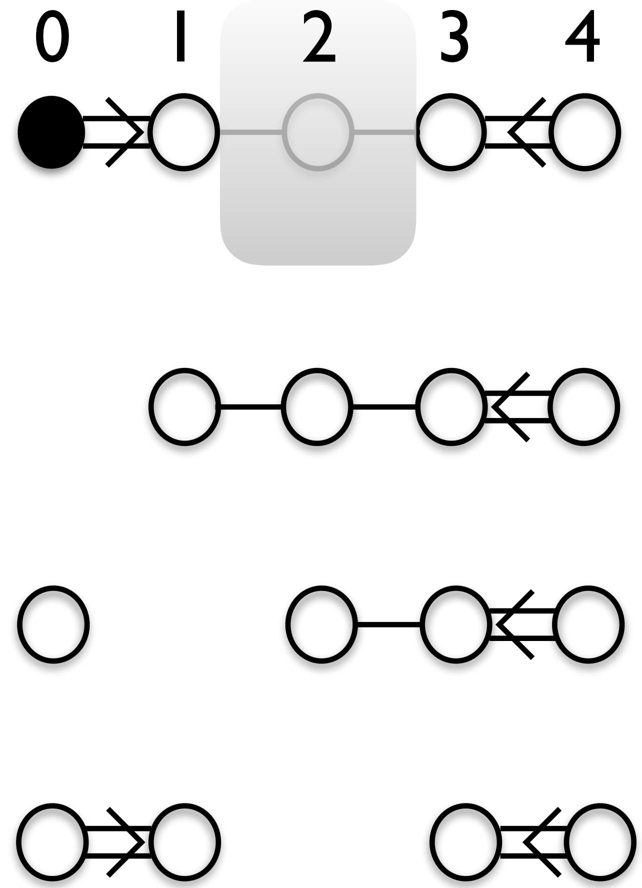
$U_q(C_4)$

$p = 1$



$U_q(C_3) \otimes U_q(C_1)$

Ex: $\hat{g} = C_4^{(1)}$

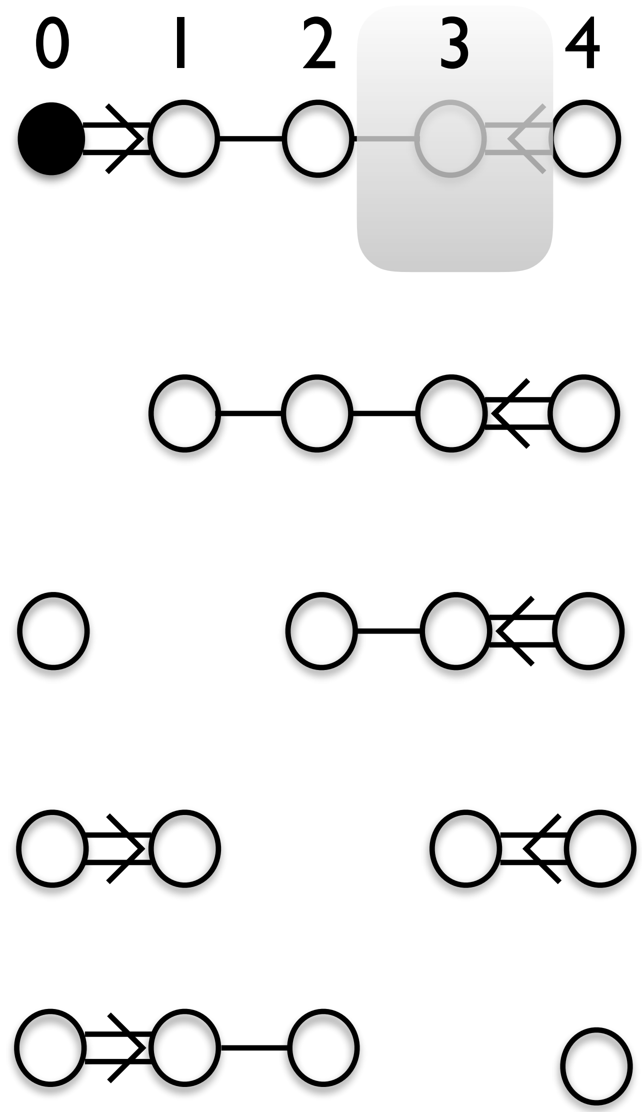


QG symmetry

$p = 0$	$U_q(C_4)$
$p = 1$	$U_q(C_3) \otimes U_q(C_1)$
$p = 2$	$U_q(C_2) \otimes U_q(C_2)$

Ex: $\hat{g} = C_4^{(1)}$

QG symmetry



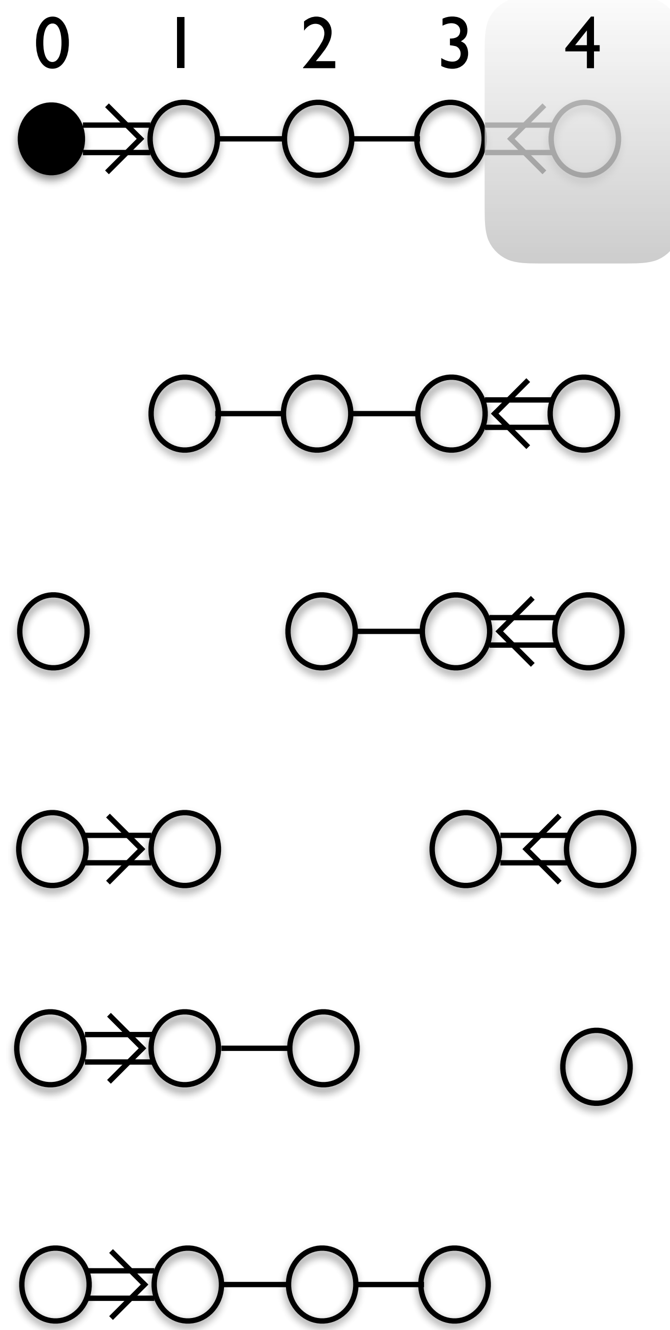
$p = 0$ — $U_q(C_4)$

$p = 1$ — $U_q(C_3) \otimes U_q(C_1)$

$p = 2$ — $U_q(C_2) \otimes U_q(C_2)$

$p = 3$ — $U_q(C_1) \otimes U_q(C_3)$

Ex: $\hat{g} = C_4^{(1)}$



QG symmetry

$p = 0$ — $U_q(C_4)$

$p = 1$ — $U_q(C_3) \otimes U_q(C_1)$

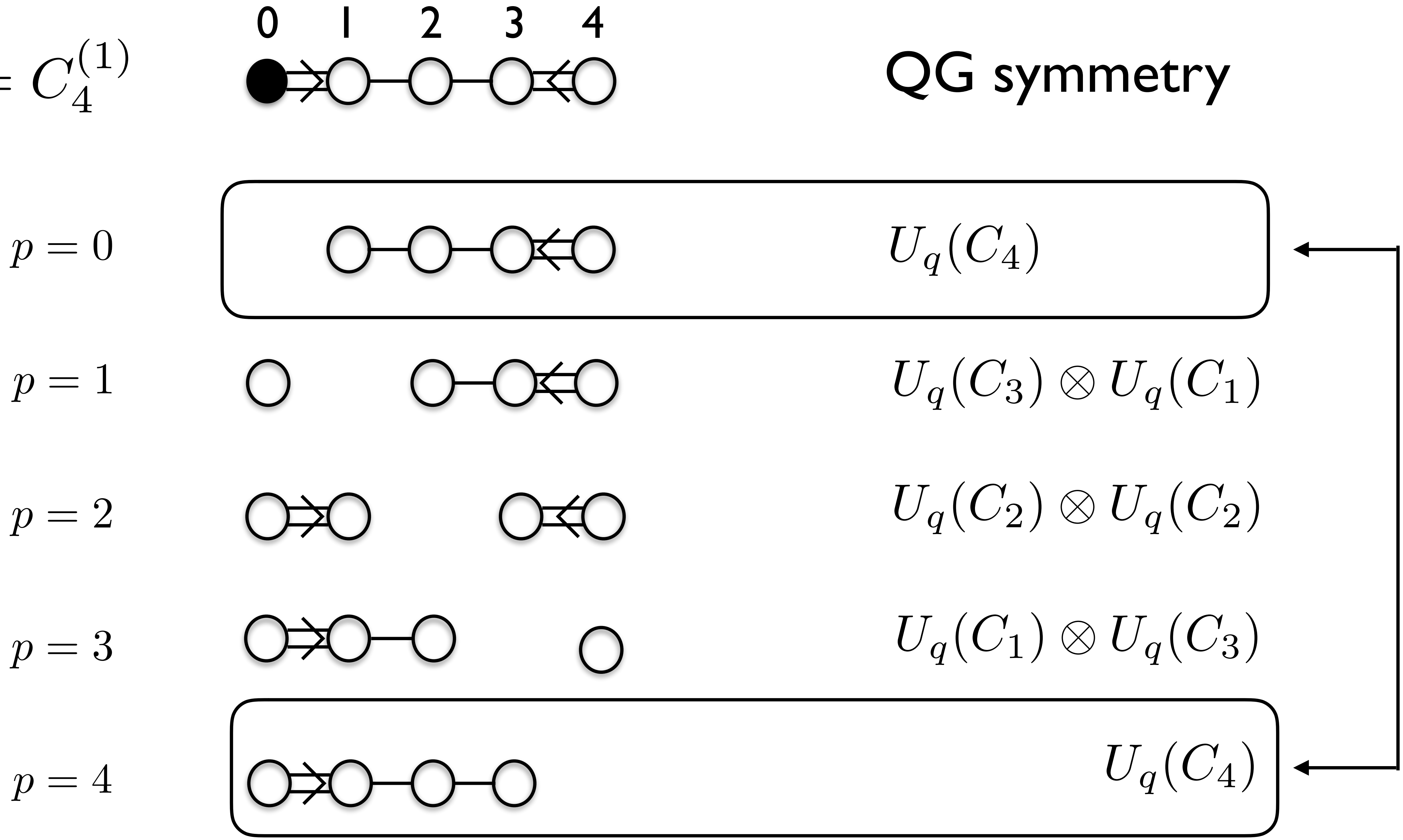
$p = 2$ — $U_q(C_2) \otimes U_q(C_2)$

$p = 3$ — $U_q(C_1) \otimes U_q(C_3)$

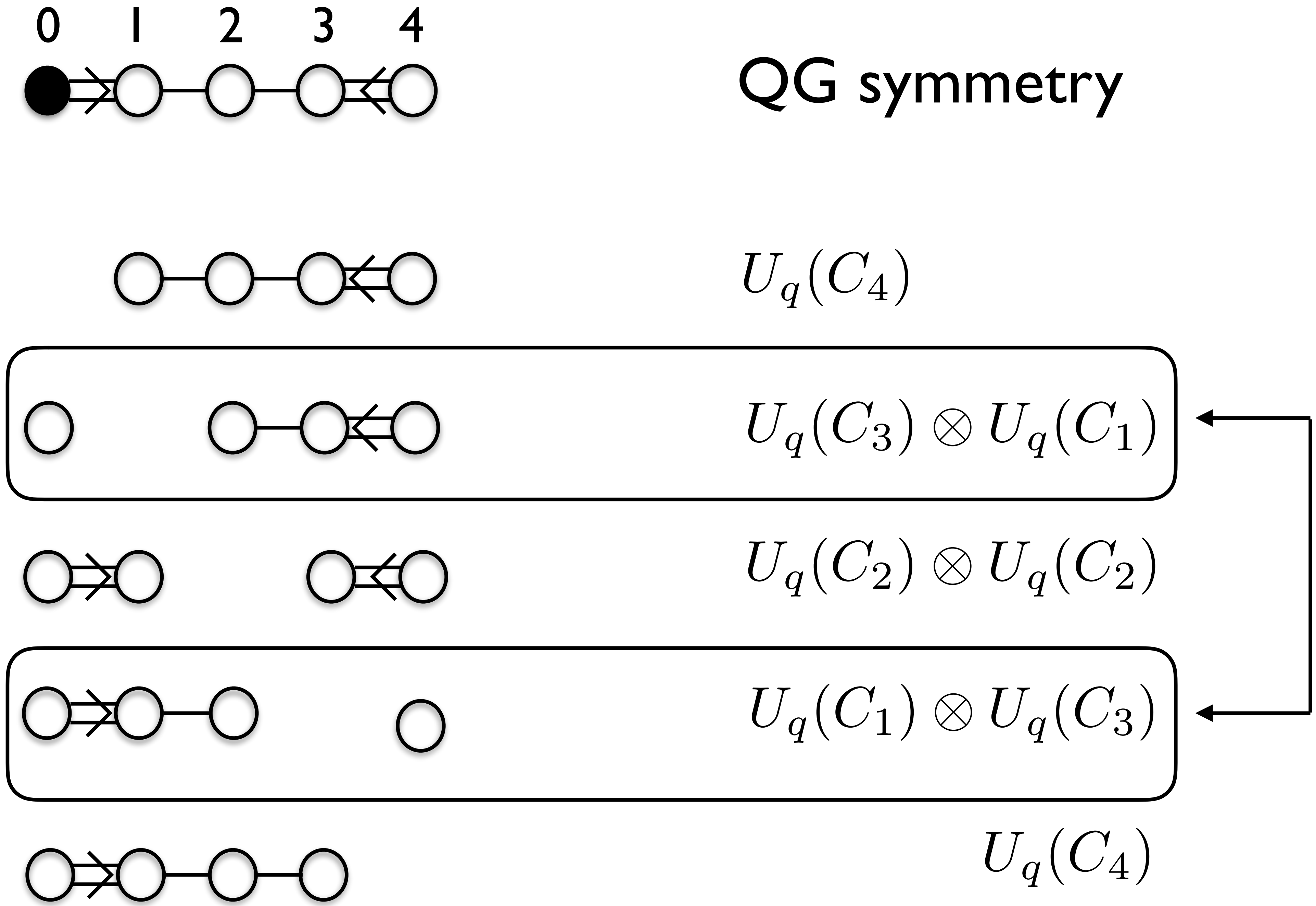
$p = 4$ — $U_q(C_4)$

Ex: $\hat{g} = C_4^{(1)}$

QG symmetry



Ex: $\hat{g} = C_4^{(1)}$ — QG symmetry



$p = 0$ — $U_q(C_4)$

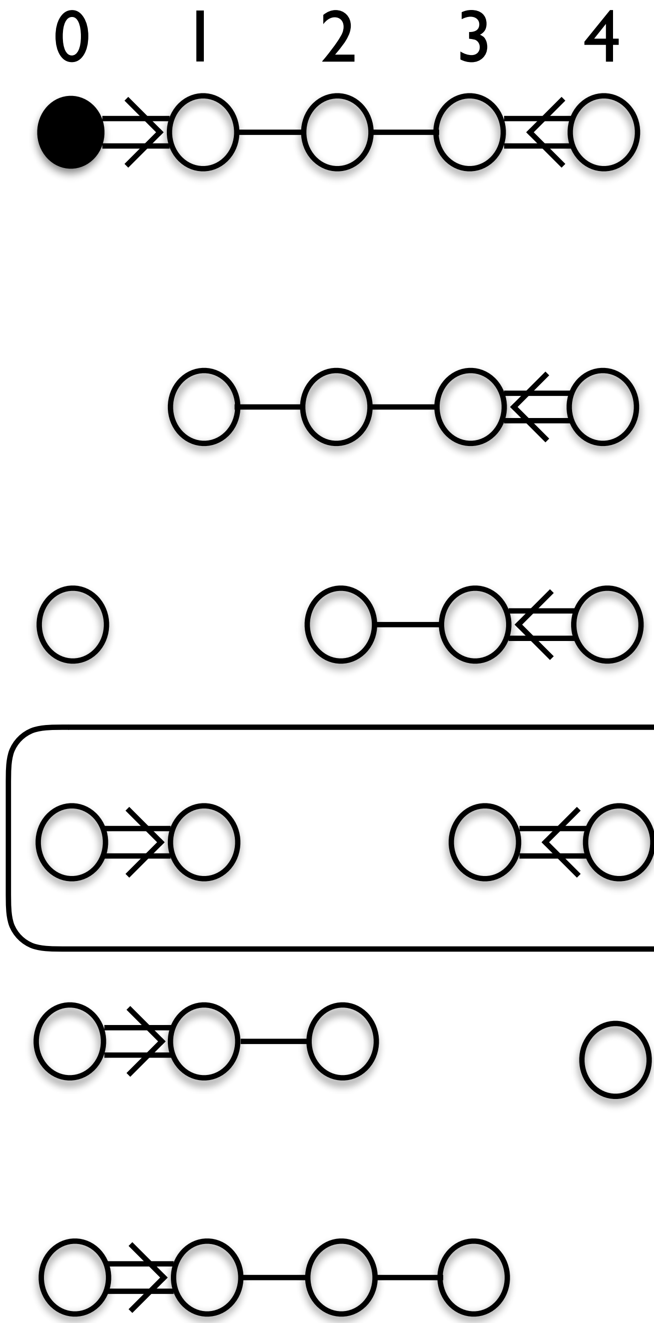
$p = 1$ — $U_q(C_3) \otimes U_q(C_1)$

$p = 2$ — $U_q(C_2) \otimes U_q(C_2)$

$p = 3$ — $U_q(C_1) \otimes U_q(C_3)$

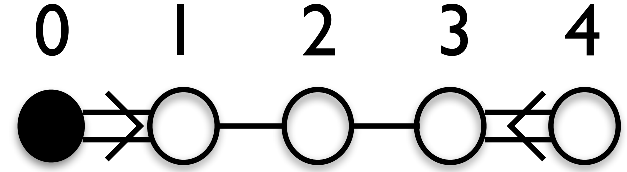
$p = 4$ — $U_q(C_4)$

Ex: $\hat{g} = C_4^{(1)}$

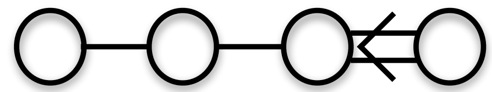


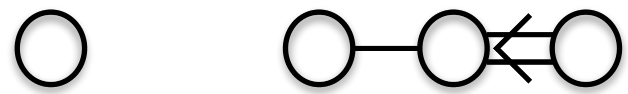
	QG symmetry
$p = 0$	$U_q(C_4)$
$p = 1$	$U_q(C_3) \otimes U_q(C_1)$
$p = 2$	$U_q(C_2) \otimes U_q(C_2)$
$p = 3$	$U_q(C_1) \otimes U_q(C_3)$
$p = 4$	$U_q(C_4)$

Ex: $\hat{g} = C_4^{(1)}$

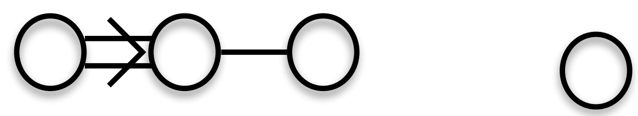


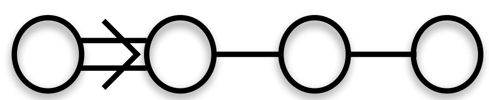
QG symmetry

$p = 0$  $U_q(C_4)$

$p = 1$  $U_q(C_3) \otimes U_q(C_1)$

$p = 2$  $U_q(C_2) \otimes U_q(C_2)$

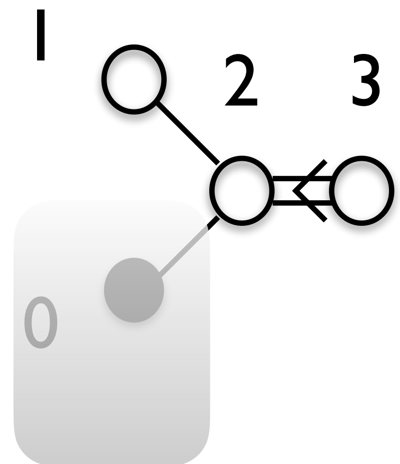
$p = 3$  $U_q(C_1) \otimes U_q(C_3)$

$p = 4$  $U_q(C_4)$

$\hat{g} = C_n^{(1)}$ **duality symmetry** $p \leftrightarrow n - p$

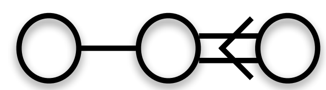
self-duality $p = \dfrac{n}{2}$ $n = \text{even}$

Ex: $\hat{g} = A_5^{(2)}$



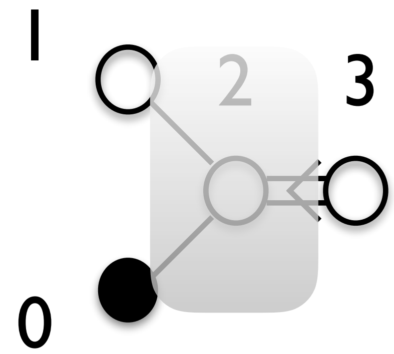
QG symmetry

$p = 0$



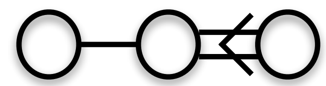
$U_q(C_3)$

Ex: $\hat{g} = A_5^{(2)}$



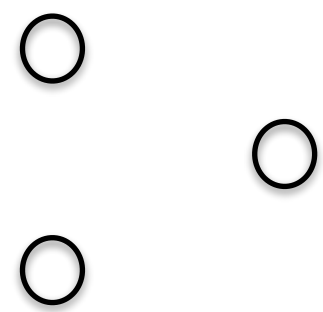
QG symmetry

$p = 0$



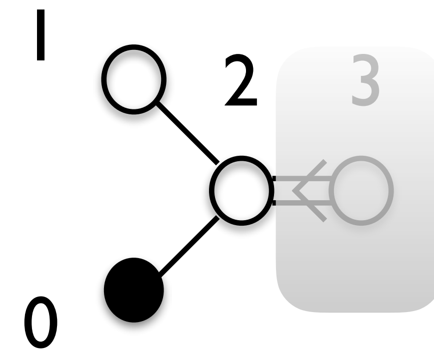
$U_q(C_3)$

$p = 2$



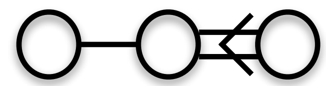
$U_q(C_1) \otimes U_q(D_2)$

Ex: $\hat{g} = A_5^{(2)}$



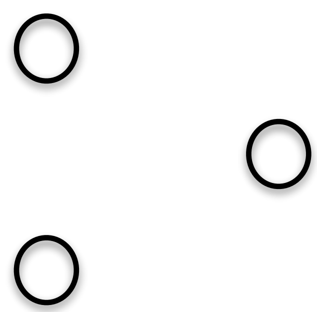
QG symmetry

$p = 0$



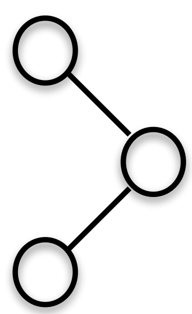
$U_q(C_3)$

$p = 2$



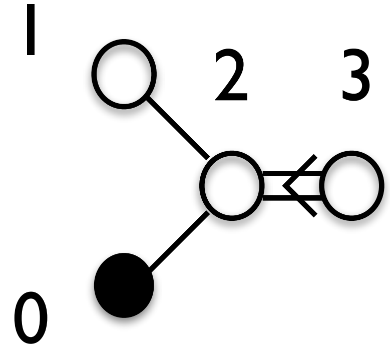
$U_q(C_1) \otimes U_q(D_2)$

$p = 3$



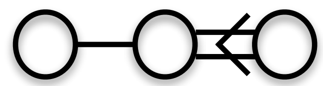
$U_q(D_3)$

Ex: $\hat{g} = A_5^{(2)}$



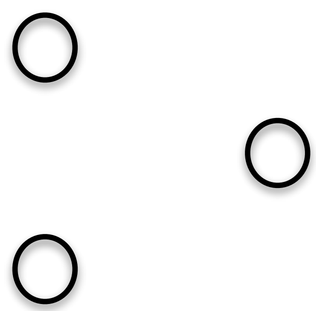
QG symmetry

$p = 0$



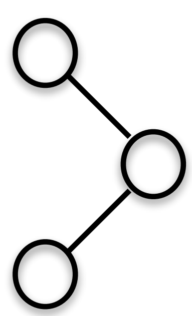
$U_q(C_3)$

$p = 2$



$U_q(C_1) \otimes U_q(D_2)$

$p = 3$



$U_q(D_3)$

$\hat{g} = A_{2n-1}^{(2)}$

Z_2 symmetry $\quad \mathbf{R} \leftrightarrow \bar{\mathbf{R}}$

$U_q(C_{n-p}) \otimes \boxed{U_q(D_p)}$

These symmetries explain the degeneracies!

Ex: $\hat{g} = A_{10}^{(2)}$ with $p = 3, N = 2$

By explicit diagonalization, find degeneracies:

$$\{1, 1, 10, 14, 14, 21, 30, 30\}$$

QG symmetry $U_q(B_2) \otimes U_q(C_3)$

The "spin" at each site is in the representation $(\mathbf{5}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{6})$

LieART $\Rightarrow$

$$((\mathbf{5}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{6}))^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus 2(\mathbf{5}, \mathbf{6}) \oplus (\mathbf{10}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{14}) \oplus (\mathbf{14}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{21})$$

Exactly matches with observed degeneracies!

4. Conclusions

- *Anisotropic* open integrable spin chains constructed with

$$\begin{cases} \text{trigonometric R-matrices} & \hat{g} \\ \text{diagonal K-matrices, parameter p} & \end{cases}$$

exhibit rich patterns of degeneracies

- Due to rich symmetries:

QG: removing p^{th} node of Dynkin diagram of $\hat{g}$

duality: $p \leftrightarrow n-p$

Z_2

- For $N \to \infty$ expect infinite-dimensional symmetry

$$U_q(\hat{g})$$

(regardless of boundary conditions) [Jimbo & Miwa1994, ...]

For finite N, the symmetry algebra is a finite-dimensional sub-algebra of $U_q(\hat{g})$

The maximal sub-algebras can be obtained by removing one node from Dynkin diagram

We have described the boundary conditions & integrable open spin chains with these maximal symmetries, for all non-exceptional $\hat{g}$

Happy 70th Birthday!

& many more birthdays & papers!