Phase structure of *G*-spin chains
– an informal overview –

Peter Vecsernyés
partially joint works with
Volkhard F. Müller and Balázs Megyeri

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Content

G-spin chains and well-known examples

G-spin chain ($\subset$ Hopf spin chain):
quantum chain based on an arbitrary finite group G
(on dual pair of finite dimensional Hopf algebras $H, \hat{H}$,
e.g. $H = \mathbb{C}G$ group algebra, $\hat{H} = \mathbb{C}(G)$ algebra of functions on G)

$G = \mathbb{Z}_2$ – Ising quantum chain

$$H_{Ising}(J, L) := \sum_{i=1}^{L-1} J\sigma_i^z \sigma_{i+1}^z + \sum_{i=1}^{L} \sigma_i^x, \quad J \in \mathbb{R}$$

$G = \mathbb{Z}_2 \times \mathbb{Z}_2$ – spin-$\frac{1}{2}$ XYZ quantum chain

$$H_{XYZ}(\mathbf{J}, L) := \sum_{i=1}^{L-1} J_x\sigma_i^x \sigma_{i+1}^x + J_y\sigma_i^y \sigma_{i+1}^y + J_z\sigma_i^z \sigma_{i+1}^z, \quad \mathbf{J} = (J_x, J_y, J_z) \in \mathbb{R}^3$$

given by Hamiltonians on $\mathcal{H}_L := \otimes_1^L \mathbb{C}^2$

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Difficulties in the traditional formulation

- Hamiltonian formulation of quantum chains is inspired by QM: no inequivalent representations of the quantum observables (CCR in QM), only the spectrum of the Hamiltonian is interesting
- however parameter dependent Hamiltonians are given here to describe the expected 'essentially different' behaviour of ground states in the $L \to \infty$ thermodynamic limit (TL), i.e. the expected existence of phases in the TL in the parameter space (phases: not only unitary, but 'essentially inequivalent' representations of quantum observables)
- difficulties in rigorous derivation and classification of phases due to parallel TL of algebra of quantum observables (Hamiltonians) and (ground) states on them
- circumvention: decompose the TL into two consecutive steps
  1. TL of local algebras of quantum observables, i.e. C^*-inductive limit leading to the $\mathcal{A} \equiv \mathcal{A}_H$ quasilocal observable C^*-algebra of the given quantum chain
  2. examine the pure state space of $\mathcal{A}$ leading to irreps of $\mathcal{A}$ $S(\mathcal{A}) := \{\omega : \mathcal{A} \to \mathbb{C} \,|\, \omega \text{ positive linear normalized pure}\}$

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Relation of Hamiltonians and the algebraic approach

prescription of correspondence: local algebras of a given model generated by monoms in the corresponding local Hamiltonian

$$\mathcal{A}(L) := \langle \text{ monoms in } H(L)\rangle_{C^*}, \qquad \mathcal{A} := \overline{\bigcup_{L\nearrow\mathbb{Z}} \mathcal{A}(L)}^{C^*}$$

- Ising chain on $\frac{1}{2}\mathbb{Z}$

$$\mathcal{A}_i = \langle A_i := \sigma_i^x\rangle_{C^*} \simeq M_1 \oplus M_1 \simeq \mathbb{C}\mathbb{Z}_2, \qquad i \in \mathbb{Z}$$
$$\mathcal{A}_{i+\frac{1}{2}} = \langle A_{i+\frac{1}{2}} := \sigma_i^z\sigma_{i+1}^z\rangle_{C^*} \simeq M_1 \oplus M_1 \simeq \mathbb{C}(\mathbb{Z}_2), \qquad i \in \mathbb{Z}$$

$$A_iA_j = \begin{cases} -A_jA_i, & |i-j| = \frac{1}{2}, \\ A_jA_i, & \text{otherwise.} \end{cases} \Rightarrow \text{crossed products of n.n. algebras}$$

$$\mathcal{A}(L) := \langle A_i, i \in L \subset \tfrac{1}{2}\mathbb{Z}\rangle_{C^*} = \mathcal{A}_j \rtimes \mathcal{A}_{j+\frac{1}{2}} \rtimes \ldots \rtimes \mathcal{A}_k, \ L = [j,k]$$

inductive limit C^*-algebra: two-sided iterated crossed products

$$\mathcal{A}_{Ising} := \ldots \rtimes \mathbb{C}\mathbb{Z}_2 \rtimes \mathbb{C}(\mathbb{Z}_2) \rtimes \mathbb{C}\mathbb{Z}_2 \rtimes \mathbb{C}(\mathbb{Z}_2) \rtimes \mathbb{C}\mathbb{Z}_2 \rtimes \ldots$$

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Relation of Hamiltonians and quasilocal algebras

- spin-$\frac{1}{2}$ *XYZ* chain on $\mathbb{Z}$

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$\mathcal{A}_i = \langle B_i := \sigma^x_i \sigma^x_{i+1}, A_i := \sigma^y_i \sigma^y_{i+1} \rangle_{C^*} \simeq \oplus_1^4 M_1 \simeq \mathbb{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)$, i odd

$$[A_i, B_j] = 0, \quad A_i A_j / B_i B_j = \begin{cases} -A_j A_i / B_j B_i, & |i-j| = 1, \\ A_j A_i / B_j B_i, & \text{otherwise.} \end{cases}$$

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Quasilocal algebra of Hopf spin chains on $\frac{1}{2}\mathbb{Z}$

General case without prescribed Hamiltonians

• dual pair of finite dimensional Hopf C^*-algebras: $H, \hat{H}$

case of G-spin chains: $H = \mathbb{C}G$, $\hat{H} = \mathbb{C}(G)$

• quasilocal algebra: $\mathcal{A} \equiv \mathcal{A}_H := \ldots H \rtimes \hat{H} \rtimes H \rtimes \hat{H} \rtimes \ldots$

two-sided iterated crossed products wrt left Sweedler actions, $|H|^\infty$-type UHF C^*-algebra

describing models given by any local Hamiltonian in $\mathcal{A}(L)$

Important properties:

• algebraic Haag duality: $\mathcal{A}(L')' \cap \mathcal{A} = \mathcal{A}(L)$, $L \subset \frac{1}{2}\mathbb{Z}$ finite

$L' \equiv [i,j]' := \frac{1}{2}\mathbb{Z} \setminus \{i - \frac{1}{2}, i, \ldots, j, j + \frac{1}{2}\}$ causal complement

$\Leftrightarrow$ local algebras on causally separated regions commute and local algebras saturate the commutants

• integer translation covariance

$\mathbb{Z} \ni n \mapsto \tau_n \in \mathrm{Aut}\,\mathcal{A} : \quad \tau_n(A_i(h)) = A_{i+n}(h), i \in \frac{1}{2}\mathbb{Z}$

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Indication for different phases on $\mathcal{S}(\mathcal{A})$

idea: emergence of different sets of local order/disorder operators in the irreps $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ induced by pure states $\omega \in \mathcal{S}(\mathcal{A})$

having non-zero 'vacuum' expectation values

$\Omega \in \mathcal{H}_\omega, \{B \in \mathcal{B}(\mathcal{H}_\omega)\} : (\Omega, B\Omega) \neq 0$

math: emergence of inequivalent local algebra extensions of $\pi_\omega(\mathcal{A})$ in the irreps $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ induced by pure states $\omega \in \mathcal{S}(\mathcal{A})$

given by the dual observable algebra $\mathcal{A}^d_\omega$

$$\mathcal{A}^d_\omega(L) := \pi_\omega(\mathcal{A}(L'))' \cap \mathcal{B}(\mathcal{H}_\omega) \supseteq \pi_\omega(\mathcal{A}(L'))' \cap \pi_\omega(\mathcal{A}) = \pi_\omega(\mathcal{A}(L)),$$

$$\mathcal{A}^d_\omega := \overline{\bigcup_{L \nearrow \frac{1}{2}\mathbb{Z}} \mathcal{A}^d_\omega(L)}^{C^*} \subseteq \mathcal{B}(\mathcal{H}_\omega)$$

measuring inequivalent violations of Haag duality in the irreps

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Definition of phases on $\mathcal{S}(\mathcal{A})$

phase equivalence of $\omega, \omega' \in \mathcal{S}(\mathcal{A})$:
$\omega \sim \omega'$ if $\exists\, \phi\colon \mathcal{A}^d_\omega \to \mathcal{A}^d_{\omega'}$ $*$-algebra isomorphism such that

$$\phi(\pi_\omega(A)) = \pi_{\omega'}(A),\ A \in \mathcal{A}.$$

phases on $\mathcal{S}(\mathcal{A})$: equivalence classes wrt the equivalence relation $\sim$

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Definition of phases on $\mathcal{S}(\mathcal{A})$

phase equivalence of $\omega, \omega' \in \mathcal{S}(\mathcal{A})$:
$\omega \sim \omega'$ if $\exists\, \phi \colon \mathcal{A}^d_\omega \to \mathcal{A}^d_{\omega'}$ $*$-algebra isomorphism such that

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Algebraic structures emerging in phase classification

Combination of two algebraic structures generalized from

finite groups $\to$ group algebras $\to$ Hopf algebras:

1. subgroup lattice of a finite group G
$\to$ special sublattice of subalgebras in $\mathbb{C}G$:
subgroup algebra lattice of $\mathbb{C}G$
$\Leftrightarrow$ lattice of left (= right) coideal subalgebras of $\mathbb{C}G$
$\to$ lattice of left (right) coideal subalgebras of a Hopf algebra H

2. projective representation $U_\alpha : G \to \mathrm{Aut}\, V$ of a finite group G
$U(g)U(h) = \alpha(g,h)U(gh), \alpha(g,h) \in \mathbb{C}_1$ satisfying 2-cocycle condition,
cohomology class of 2-cocycles: $\alpha \sim \alpha'$ if $\exists \beta \in \mathbb{C}_1(G)$ such that
$\alpha'(g,h) = \beta(g)\beta(h)/\beta(gh)\alpha(g,h)$
$\to$ (representation of the) one-sided deformed group algebra $\mathbb{C}G_\alpha$
by a coproduct cocycle α of $\mathbb{C}(G)$
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Classification of phases

steps of classification:

1. characterization of the extensions $\mathcal{A}^d_\omega(H) \supseteq \pi_\omega(\mathcal{A}(H))$
2. determine the phase equivalent characterizations
3. determine the translation invariant phases

Result:

1.a) $\mathcal{A}^d_\omega = \mathcal{B}_{i+\frac{1}{2},i} \rtimes \pi_\omega(\mathcal{A})$

where $\mathcal{B}_{i+\frac{1}{2},i} := \pi_\omega(\mathcal{A}(-\infty, i-\frac{1}{2}))' \cap \pi_\omega(\mathcal{A}(i+1,\infty))' \cap \mathcal{B}(\mathcal{H}_\omega)$
is a finite dimensional right $\mathcal{D}$-module von Neumann algebra.
$\mathcal{D} \equiv \mathcal{D}(H)$ is the Drinfeld double of H, if $\mathcal{A}_i \simeq H$.

1.b $\mathcal{B}_{i+\frac{1}{2},i}$ is isomorphic as a right $\mathcal{D}$-module *-algebra
to a one-sided deformed left coideal subalgebra $\mathcal{K}$ of $\hat{\mathcal{D}}$
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Classification of translation invariant phases

3.a $\mathcal{A}^d_\omega$ is bosonic, i.e. $[\mathcal{B}_{i+\frac{1}{2},i}, \mathcal{B}_{j+\frac{1}{2},j}] = 0$ if $|i-j| \geq 1$
$\Leftrightarrow$ the IRC (α, k) is bosonic, i.e. $R\alpha = \alpha$.
(R is the universal R-matrix of the quasitringular $\mathcal{D}(H)$.)

3.b $\mathcal{A}^d_\omega$ is bosonic if ω is translation invariant.

$\Rightarrow$ translation invariant phases are characterized by cohomology classes of bosonic IRC-s.

Example 1: $G = \mathbb{Z}_N$ – N-state Potts quantum chains for N prime

$\bullet \mathcal{D}(\mathbb{Z}_N) \simeq \mathbb{C} \cdot \mathbb{Z}_N \times \mathbb{Z}_N \simeq \hat{\mathcal{D}}(\mathbb{Z}_N) \Rightarrow$
• extensions $\mathcal{B}_{i+\frac{1}{2},i} \simeq (k \rightharpoonup \hat{\mathcal{D}})_\alpha$ are given by subgroups and 2-cocycle cohomology classes (2CC) on them:
– N different 2CC on $\mathbb{Z}_N \times \mathbb{Z}_N$
– single 2CC on any of the $N+1$ $\mathbb{Z}_N$ subgroups
– single 2CC on the trivial subgroup (no extension)

• $\hat{\mathcal{D}} \ni (m, n)$ is an anyon with statistics phase $\exp(2\pi imn/N)$
$\Rightarrow$ only three bosonic/translation invariant phase

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Translation invariant phases – examples

Example 2: $G = \mathbb{Z}_4$ – 4-state Potts quantum chain

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• $\hat{\mathcal{D}} \ni (m, n)$ is an anyon with statistics phase $\exp(2\pi imn/4)$

$\Rightarrow$ bosonic/translation invariant phases with maximal subgroups:

– $\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2$ (two 2CC), $\mathbb{Z}_4$

– single 2CC on any of the three $\mathbb{Z}_2$ subgroups

– single 2CC on the trivial subgroup (no extension)

Example 3: $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ – spin-$\frac{1}{2}$ XYZ- chain

• $\mathcal{D}(\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq \mathbb{C} \cdot \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \hat{\mathcal{D}}(\mathbb{Z}_2 \times \mathbb{Z}_2) \Rightarrow$

• bosonic/translation invariant phases with maximal subgroups:

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Ferromagnetic states from ferromagnetic elements

- define ferromagnetic elements in a Hopf algebra H as intersection of positive and positive definite elements: $F(H) := H_+ \cap H^+$
- 'verification': $B(g,h) := \varphi(g^{-1}h), \varphi \in F(\mathbb{C}(G))$ leads to left G-invariant ferromagnetic Boltzmann weight on $G \times G$
$0 \leq B(g,h) = B(fg,fh) = \varphi(g^{-1}h) \leq \varphi(e) = B(g,g)$
- Fourier transformation maps $F(H)$ onto $F(\hat{H})$
$H \ni a \mapsto \mathcal{F}(a) := a \rightharpoonup \chi \in \hat{H}$, where $\chi \in \hat{H}$ is the Haar integral
- $F(H)$ is a convex cone, normalization leads to a convex set
- $\mathcal{A}$ is UHF $\Rightarrow$ unique normalized trace on $\mathcal{A}$: use (power of) ferromagnetic transfer 'matrices' as density metrices

$$T_{[i,j]}(f) := \sqrt{V_{[i,j]}(f)}\, W_{[i,j]}(\hat{f}) \sqrt{V_{[i,j]}(f)}, \quad f \in F(H), \hat{f} = \mathcal{F}(f)$$

$$V_{[i,j]}(f) = \prod_{k \in [i,j] \cap \mathbb{Z}} A_k(f), \quad W_{[i,j]}(\hat{f}) = \prod_{k \in [i,j] \cap \mathbb{Z} + \frac{1}{2}} A_k(\hat{f})$$

$$P_{[i,j]}(f) := \lim_m T_{[i,j]}(f)^m / \mathrm{Tr}\left(T_{[i,j]}(f)^m\right)$$

$\omega_f(-) := \lim_n^{w^*} \mathrm{Tr}\left(P_{[-n,n]}(f) -\right)$ (set of states on $\mathcal{A}$ is w*-compact $\Rightarrow$ convergent subsequences, unique? pure? translation invariant?)

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Ferromagnetic projections and maximal bosonic extensions

- projections $p \in F_0(H) := \{f \in F(H) \,|\, 1 = \langle \hat{\mathbf{1}}, f \rangle\}$ are extremal points
- their Fourier transformed $\hat{p} := \mathcal{F}(p)/\langle \chi, p \rangle \in F_0(\hat{H})$ are projections
- their embeddings in the chain commute: $[A_i(p), A_{i\pm\frac{1}{2}}(\hat{p})] = 0$

$\Rightarrow T_{[-n,n]}(f), n \in \mathbb{N}$ is a sequence of decreasing projections

$\Rightarrow$ the ferromagnetic state ω_p is translation invariant pure

- $K := \mathcal{D}(p)\mathcal{D}(\hat{p}) \in \mathcal{D}(H)$ are bosonic IRC projections $(K \otimes K, K)$ with $R(K \otimes K) = K \otimes K$ to the (undeformed) maximal bosonic left coideal subalgebras $K \rightharpoonup \hat{\mathcal{D}}(H) \subset \hat{\mathcal{D}}(H)$
- ω_p is contained in the phase characterized by the extension $\mathcal{B}_{i+\frac{1}{2},i} \simeq K \rightharpoonup \hat{\mathcal{D}}(H)$

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PV; Asymptotic commutants and the absence of non-local translation invariant phases

(Bosonic) IRCs

Characterization of phases of $\mathcal{A} = \mathcal{A}(H)$

- phases in $\mathcal{S}(\mathcal{A})$ (in $S_0(\mathcal{A})$) $\simeq$ cohomology classes of (bosonic) intermediate right Δ-cocycles (IRC) in the quasitriangular Drinfeld double $\mathcal{D} \equiv (\mathcal{D}(H), R)$

(bosonic) IRC in $\mathcal{D}$

- the pair (w, k) of a partial isometry $w \in \mathcal{D} \otimes \mathcal{D}$ and a projection $k \in \mathcal{D}$ is an IRC if

$$w^* w = k \otimes k, \qquad \Delta(k)w = w$$
$$S(q^*)q = k, \quad q := S(w_1)w_2$$
$$(\varepsilon \otimes id)(w) = k = (id \otimes \varepsilon)(w)$$
$$(id \otimes \Delta)(w)(\mathbf{1} \otimes w) = (\Delta \otimes id)(w)(w \otimes \mathbf{1}).$$

- an IRC (w, k) is bosonic if $Rw = w$
- (w, k) and (w', k') are cohomologous if $\exists x \in \mathcal{D}$ unitary with

$$\varepsilon(x) = 1, \quad w' = \Delta(x^*)w(x \otimes x).$$

Connection between bosonic IRCs and asymptotic commutants $\mathcal{A}^{ac}$

Lemma If ω is a pure α-invariant state on a norm asymtotically abelian (NAA) algebra $(\mathcal{A},\alpha)$ then $(\mathcal{A}^{ac}_\omega, \alpha_\omega := Ad\, U)$ is SAA (WAA).

- $\mathcal{A}^d_\omega \subseteq \mathcal{A}^{ac}_\omega$ if $\omega \in S_0$
- transportability if $\omega \in S$
  $\exists \tau^{-1}$: $\mathcal{A}^d_\omega(i+\frac{1}{2}, i) \to \mathcal{A}^d_\omega(i-\frac{1}{2}, i-1)$ right $\mathcal{D}$-module *-algebra isomorphism:
  $\tau^{-1}(B) = \sum_\beta \pi_\omega(A_\beta)(B\cdot X_\beta), \quad A_\beta \in \mathcal{A}(i-\frac{1}{2}, i), X_\beta \in \mathcal{D}$
- $\alpha_\omega \circ \tau^{-1}$ right $\mathcal{D}$-module *-algebra automorphism of $\mathcal{A}^d_\omega(i+\frac{1}{2}, i)$
  $\Rightarrow \exists x \in \mathcal{D}$ unitary $\Delta(x)w = w(x\otimes x)$
  $\Rightarrow Spec\, x^2 \subseteq (Spec\, x)^2 = Spec\, x \Rightarrow \exists n \in \mathbb{N} : x^n = \mathbf{1}$
- generalized antiferromagnets ($x \neq \mathbf{1}$) but no quasicrystals ($x^m \neq \mathbf{1}, m \in \mathbb{N}$) if $\omega \in S_0$
- $\mathcal{A}^d_\omega$ is local (Haag dual) if $\omega \in S_0$
  $(\alpha_\omega^{-n}\otimes id)(F^*_{13})F^*_{12}(\alpha_\omega^{-n}\otimes id)(F_{13})F_{12} = \mathbf{1}_\mathcal{B}\otimes w^* R w = \mathbf{1}_\mathcal{B}\otimes k \otimes k$
  $F_{12} = \Phi(k \rightharpoonup \xi_\beta)\otimes X_\beta \in \mathcal{A}^d_\omega(i+\frac{1}{2}, i)\otimes \mathcal{D}$ partial isometry