BPS equations for interacting scalar field theories in (1+1) dimensions

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[1]based on work carried out with Luiz Ferreira (Sao Carlos) and others

Introduction

General case of 2 fields in (1+1) dimensions

Examples

BPS theories from representation theory

Numerical support

Summary

Introduction

- Two interesting concepts studied a lot in field theories
  - Integrability
  - BPS properties.
- More has been done on Integrabilities etc.
- But BPS properties are quite interesting too and have important implications.
- Here we will look at the properties and some implications of the BPS conditions.

What are BPS conditions?

- Consider static solutions of a (1+1) field theory with the energy density given by

$$\mathcal{E} = \frac{1}{2}\,\partial_x\phi\,\partial_x\phi + V(\phi);$$

- Rewrite this as (Bogomolny' trick)

$$\mathcal{E} = \frac{1}{2}\left(\partial_x\phi - \sqrt{2V(\phi)}\right)^2 + \frac{1}{\sqrt{2}}\sqrt{V}\partial_x\phi$$

- The last term is clearly $\frac{\partial U}{\partial x}$, for some U

So for the fields which satisfy BPS condition *i.e.*

$$\partial_x \phi = \sqrt{2V(\phi)}$$

- Energy $\int E\, dx \;=\; \int dx \frac{\partial U}{\partial x} \;=\; U(\phi(+\infty)) - U(\phi(-\infty))$.
- So if $U(\phi(\pm\infty))$ are finite
  - the energy is finite
  - this energy is a minimum for the fields which satisfy the BPS condition

Many questions

- How does this generalise to more fields and to higher dimensions?
- Do we just complete the squares, and how do we do this in general?
- What are the properties of such fields (in particular, the condition of finiteness of $U(\phi(\pm\infty)))$?
- What are the implications for the scattering properties of such fields?
- In this talk we will look at the question of more fields in $(1+1)$ dimensions.
- All complications are already visible for 2 fields - so we will really look only at the case of 2 fields.

General case of 2 fields for (1+1) dimensions.

- General discussion (for many fields in (1+1) and higher dimensions) already given in two papers
  - C. Adam, L. A. Ferreira, E. da Hora, A. Wereszczynski and W. J. Zakrzewski, "Some aspects of self-duality and generalised BPS theories," JHEP **1308**, 062 (2013) [arXiv:1305.7239 [hep-th]].
  - C. Adam and F. Santamaria, "The First-Order Euler-Lagrange equations and some of their uses," [arXiv:1609.02154 [hep-th]].
- In this talk we follow the procedure of the first paper.

- Consider a scalar field theory in $(1+1)$ dimensions defined by the Lagrangian $(c=1)$

$$\mathcal{L} = \frac{1}{2}\,\eta_{ab}\,\partial_\mu\varphi_a\,\partial^\mu\varphi_b - V(\varphi_a); \quad a = 1, \ldots N.$$

- The corresponding Euler-Lagrange eqs. are

$$\eta_{ab}\,\partial^2\varphi_b + \frac{\partial V}{\partial \varphi_a} = 0; \quad (\partial^2 \equiv \partial_t^2 - \partial_x^2)$$

- The static energy is given by

$$E = \int_{-\infty}^{\infty} dx \left(\frac{1}{2}\,\eta_{ab}\,\partial_x\varphi_a\,\partial_x\varphi_b + V(\varphi_a)\right).$$

- The static eqs. of motion are given by:

$$\eta_{ab}\,\partial_x^2\varphi_b = \frac{\partial V}{\partial \varphi_a}.$$

Topological Charge

- We need a topological charge.

$$Q = \int_{-\infty}^{\infty} dx\, W_a\, \partial_x \varphi_a,$$

where $W_a = W_a(\varphi_b)$ only.

- For Q to be topological we require that

$$\delta Q = 0 \qquad\qquad \text{for any } \delta\varphi_a,$$

without the use of the eqs. of motion.

- But

$$\delta Q = \int_{-\infty}^{\infty} dx\, \left[\frac{\partial\, W_a}{\partial\, \varphi_b} - \frac{\partial\, W_b}{\partial\, \varphi_a}\right] \partial_x\varphi_a\, \delta\varphi_b + W_a\, \delta\varphi_a\, \Big|_{-\infty}^{\infty}\, .$$

- If the variations of the fields $\delta\varphi_b$ vanish at infinities, we can make Q topological by assuming that there is a prepotential U such that

$$W_a = \frac{\partial U}{\partial \varphi_a} \qquad \rightarrow \qquad \frac{\partial W_a}{\partial \varphi_b} = \frac{\partial W_b}{\partial \varphi_a}$$

- Then

$$Q = \int_{-\infty}^{\infty} dx \, \frac{\partial U}{\partial \varphi_a} \, \partial_x \varphi_a$$

$$= \int_{-\infty}^{\infty} dx \, \frac{dU}{dx} = U(\varphi_a(\infty)) - U(\varphi_a(-\infty)).$$

- Next we write the density of the topological charge as a sum of products of two terms as

$$Q = \int_{-\infty}^{\infty} dx\, A_a\, \tilde{A}_a.$$

- Then we choose:

$$W_a\, \partial_x \varphi_a = W_b\, \Lambda^{-1}_{bc}\, \Lambda_{ca}\, \partial_x \varphi_a.$$

and

$$A_a = \Lambda_{ab}\, \partial_x \varphi_b, \qquad \tilde{A}_a = W_b\, \Lambda^{-1}_{ba}.$$

- We also need that the static energy becomes

$$E = \int_{-\infty}^{\infty} dx\, \frac{1}{2}\, \left[A_a^2 + \tilde{A}_a^2\right].$$

But Λ is arbitrary

- so we choose: $\Lambda^T \Lambda = \eta$
- and
$$W_a \Lambda^{-1} \Lambda^{-1}{}^T_{ab} W_b = 2\,V = W_a\, \eta^{-1}_{ab}\, W_b.$$
- Here we have used that $(\Lambda^{-1})^T = (\Lambda^T)^{-1}$.

- The BPS equations are given by

$$A_a = \pm \tilde{A}_a$$

- Thus they take the form

$$\partial_x \varphi_a = \pm W_b \Lambda^{-1} {\Lambda^{-1}}^T_{ba} = \pm\, \eta^{-1}_{ab}\, W_b.$$

- It is easy, though a bit tedious, to show that the BPS equations imply the full equations.
- As expected we have the bound on the energy.

$$E = \int_{-\infty}^{\infty} dx\, \frac{1}{2}\, \left[A_a^2 + \tilde{A}_a^2\right] = \int_{-\infty}^{\infty} dx\, \frac{1}{2}\, \left[A_a \pm \tilde{A}_a\right]^2$$

$$\mp \int_{-\infty}^{\infty} dx\, A_a\, \tilde{A}_a.$$

- If the BPS eqs. are satisfied we have $E = \mp Q$, and, if the energy is positive definite, this becomes $E = \mid Q \mid$.

Further comments; finite energy solutions

- If the fields φ_a are real, so are the functions W_a.
- If there are some constant solutions $\varphi_a^{\mathrm{vac.}} = \mathrm{const.}$, of the BPS equations then the quantities W_a have to vanish on these configurations. Hence, the potential also has to vanish on such solutions $V(\varphi_a^{\mathrm{vac.}}) = 0$.
- Moreover, the first derivatives of the potential also have to vanish.
- So,

$$\frac{\partial V}{\partial \varphi_a} \,|_{\varphi_a=\varphi_a^{\mathrm{vac.}}} = 0, \quad \frac{\partial U}{\partial \varphi_a} \,|_{\varphi_a=\varphi_a^{\mathrm{vac.}}} = 0, \quad V \,|_{\varphi_a=\varphi_a^{\mathrm{vac.}}} = 0.$$

- If the real energy E is bounded from below, then the BPS solutions are stable against small perturbations as their energy is the lowest for the given value of Q.

- Note that the BPS eq. imply that the topological charge Q can be rewritten as

$$Q = \int_{-\infty}^{\infty} dx \, \frac{\partial U}{\partial \varphi_a} \partial_x \varphi_a = \int_{-\infty}^{\infty} dx \, \frac{d U}{d x} = \pm 2 \int_{-\infty}^{\infty} dx \, V .$$

- Moreover,

$$\eta^{-1}_{ab} \frac{\partial U}{\partial \varphi_a} \frac{\partial U}{\partial \varphi_b} = \mp \frac{\partial U}{\partial \varphi_a} \partial_x \varphi_a = \mp \frac{\partial U}{\partial x} = 2 \, V(\varphi_a).$$

  This is hard to solve as $V = V(\varphi_a(x))$

- So, on any BPS solution, we have that

$$U(x) = \pm 2 \int_{-\infty}^{x} dx' \, V + U(x = -\infty).$$

- Hence, for any given lagrangian, the problem reduces to having to find a prepotential U for the potential V in this lagrangian.

Two Fields.

- Consider

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu\varphi_1)^2 + (\partial_\mu\varphi_2)^2 - \lambda\partial_\mu\varphi_1\,\partial^\mu\varphi_2\right) - V(\varphi_1,\varphi_2),$$

- where $V(\varphi_1,\varphi_2)$ is given by:

$$V = \frac{2}{4-\lambda^2}\left[M^2 + N^2 + \lambda MN\right].$$

- Let us choose $M = M(\varphi_1)$, $N = N(\varphi_2)$.
- Note that when $\lambda = 0$ we have two real independent real fields, and when $\lambda \neq 0$ this Lagrangian corresponds to η_{ab}

$$\eta = \begin{pmatrix} 1 & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} & 1 \end{pmatrix}$$

- Moreover, the potential V

$$V = 2\eta^{-1}_{ab}G_aG_b, \quad \text{where} \quad G_1 = M, \quad G_2 = N.$$

- Recall that the matrix Λ was essentially arbitrary. We have chosen it in terms of our matrix η with a parameter λ. This choice of Λ led to the coupling of the two fields.
- Note that the appearance of η^{-1} in V leads to $\frac{\partial U}{\partial \varphi_i} = G_i(\varphi_i)$ and so the topological charge is a sum of two terms

$$U(\varphi_1, \varphi_2) = \int d\varphi_1 M(\varphi_1) + \int d\varphi_2 N(\varphi_2) + \text{constant}$$

- The selfduality equations now take the form

$$\partial_x \varphi_i = F_i$$

- where F_i are given by

$$F_1 = \frac{2}{4-\lambda^2}\left(\lambda \frac{\partial U}{\partial \varphi_2} + 2\frac{\partial U}{\partial \varphi_1}\right),$$

$$F_2 = \frac{2}{4-\lambda^2}\left(\lambda \frac{\partial U}{\partial \varphi_1} + 2\frac{\partial U}{\partial \varphi_2}\right).$$

- Later we will discuss the choices of $N(\varphi_2)$ and $M(\varphi_1)$ and, in particular, will discuss the case when each of them are proportional to $\sin(\varphi_i)$. Have also looked at other expressions for M and N but the results were pretty much as expected.
- As the relation between U and V is complicated it is not easy to find U for a given V. Even here this was not easy (here we were helped by the fact that U was a sum of two terms (one dependent on φ_1, the other on φ_2)). In general, this is not the case.
- Next we will construct various models by starting with various expressions of the topological charge and then checking what potential they correspond to.

BPS theories from representation theory.

- We have realised that we can construct BPS theories involving fields which are components of a field which takes values in a representation of a Lie algebra $\mathcal{G}$.
- This involves a lot of group theory (have to think of weight vectors, the corresponding weight lattices and co-weight latticesetc)
- We then construct the prepotential U through a given representation R of G as

$$U = \sum_{i=1}^{\dim R} c_i \, e^{i\,\vec{\mu}_i \cdot \vec{\varphi}}$$

- Here $\vec{\mu}_i$, $i = 1, 2, \ldots \dim R$, are the weights of R.
- For the matrix η we can take the λ-modified Cartan matrix

$$K_{ab} = \frac{2\,\vec{\alpha}_a \cdot \vec{\alpha}_b}{\vec{\alpha}_b^2}.$$

$SU(3)$ **case.**

- In this case we can take η in the form

$$\eta = \begin{pmatrix} 2 & -\lambda \\ -\lambda & 2 \end{pmatrix}, \qquad \eta^{-1} = \frac{1}{(4-\lambda^2)} \begin{pmatrix} 2 & \lambda \\ \lambda & 2 \end{pmatrix}$$

- Clearly we have to restrict the values λ to $|\lambda| < 2$.
- Next we look at the triplet and anti-triplet representations of $SU(3)$
- The pre-potential associated to the triplet is then

$$U^{(3)} = \gamma_1\, e^{i\,\varphi_1} + \gamma_3\, e^{i\,(\varphi_2-\varphi_1)} + \gamma_2\, e^{-i\,\varphi_2}.$$

- The pre-potential associated to the anti-triplet is given by

$$U^{(\bar{3})} = \bar{\gamma}_2\, e^{i\,\varphi_2} + \bar{\gamma}_3\, e^{i\,(\varphi_1-\varphi_2)} + \bar{\gamma}_1\, e^{-i\,\varphi_1}.$$

- To have a real pre-potential U we take $\gamma_i = \bar{\gamma}_i$ and also take

$$U = \frac{1}{2}\left(U^{(3)} + U^{(\bar{3})}\right) = \gamma_1 \cos\varphi_1 + \gamma_2 \cos\varphi_2 + \gamma_3 \cos\left(\varphi_1 - \varphi_2\right)$$

- Hence the static energy is

$$E = \int_{-\infty}^{\infty} dx \, \left[(\partial_x\varphi_1)^2 + (\partial_x\varphi_2)^2 - \lambda\,\partial_x\varphi_1\,\partial_x\varphi_2 + V(\varphi_a)\right]$$

where

$$V = \frac{1}{\lambda^2 - 4}\left[-\gamma_1^2\sin^2(\varphi_1) + \gamma_1\sin(\varphi_1)(\gamma_3(\lambda-2)\sin(\varphi_1-\varphi_2)\right.$$
$$-\gamma_2\lambda\sin(\varphi_2)) + \gamma_3^2(\lambda-2)\sin^2(\varphi_1-\varphi_2)$$
$$\left.-\gamma_2^2\sin^2(\varphi_2) - \gamma_2\gamma_3(\lambda-2)\sin(\varphi_2)\sin(\varphi_1-\varphi_2)\right].$$

- Note that the self-duality equations become

$$\partial_x\varphi_1 = \pm\frac{2\gamma_1\sin(\varphi_1)+\gamma_2\lambda\sin(\varphi_2)-\gamma_3(\lambda-2)\sin(\varphi_1-\varphi_2)}{\lambda^2-4},$$

$$\partial_x\varphi_2 = \pm\frac{\gamma_1\lambda\sin(\varphi_1)+2\gamma_2\sin(\varphi_2)+\gamma_3(\lambda-2)\sin(\varphi_1-\varphi_2)}{\lambda^2-4}.$$

- The vacua are determined by the equations

$$\frac{\partial U}{\partial\varphi_1} = -\gamma_1\sin(\varphi_1)-\gamma_3\sin(\varphi_1-\varphi_2)=0,$$

$$\frac{\partial U}{\partial\varphi_2} = \gamma_3\sin(\varphi_1-\varphi_2)-\gamma_2\sin(\varphi_2)=0.$$

- Note that these conditions imply

$$\gamma_1\sin(\varphi_1) = -\gamma_3\sin(\varphi_1-\varphi_2) = -\gamma_2\sin(\varphi_2),$$

- and that in the case

$$\lambda = 0, \qquad \gamma_3 = 0.$$

the theory decouples into two sine-Gordons fields.

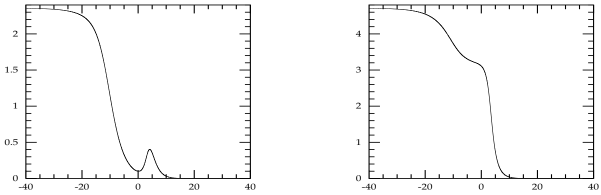
Numerical support.

- We have performed several simulations to solve the BPS equations for both cases and to check their stability.
- First we solved the BPS equations. As these equations are first order, their solutions depend on the values of φ_1 and φ_2 fields at one specific value of x, which in our simulations was always $x = 0$.
- We solved them by first moving forward in x and then going backwards in x and then joining them together. We performed many such simulations, varying both the simulation step dx and how far we were moving in x. For small values of dx ($dx < 0.00001$) the results were essentially the same.
- In the plots, that we include here, we present the results obtained for $dx = 0.000002$.
- Then we used the obtained BPS fields as initial conditions for longer simulations of the full equations of motion corresponding to our cases. The time evolution was simulated by the 4th order Runge Kutta method.

- Our simulations were performed with absorbing boundary conditions but, in fact, the fields at the boundaries were so small so the absorption was infinitesimal.
- Of course, as our BPS fields were static solutions, we did not expect any evolution and ideed this was confirmed by our simulations. In fact, we did not see any significant changes of the fields.
- Thus our simulations confirmed that the BPS solutions were indeed the static solutions of the full field equations and that these solutions were stable.

Concrete results for $SU(3)$ case.

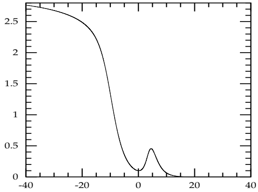
- Here we present the BPS solutions obtained in 5 simulations (the first four for $\lambda = 0.5$)
- Case I) $\gamma_1 = 1$, $\gamma_3 = 1$ and $\gamma_2 = \sqrt{2}$



(a) (b)

$SU(3)$ fields - case I; a) φ_1 and b) φ_2.

- Case II $\gamma_1 = 1$, $\gamma_3 = 1$ and $\gamma_2 = 0.5$



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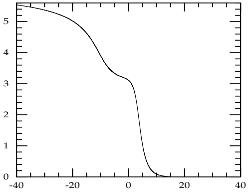
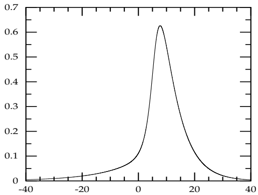


(d)

$SU(3)$ fields - case II; a) φ_1 and b) φ_2.

- Case III) $\gamma_1 = 0.5$, $\gamma_3 = 0.1$ and $\gamma_2 = 0.1$



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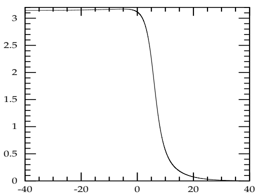
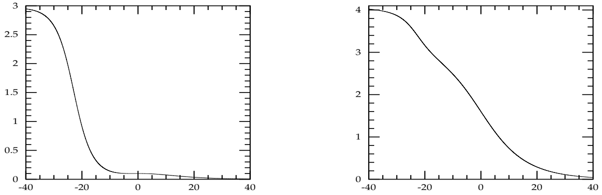


(f)

$SU(3)$ fields - case III; a) fields φ_1 and φ_2.

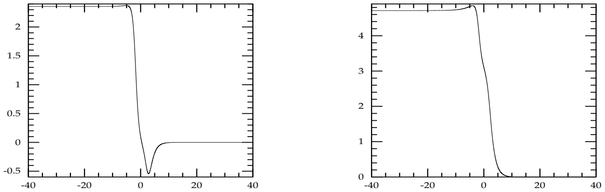
- Case IV) $\gamma_1 = 0.5$, $\gamma_3 = 0.1$ and $\gamma_2 = 0.1$. The values of γ_i are the same as in case III but the starting values (*i.e.* $\varphi_1(0)$ and $\varphi_2(0)$ are different.



(g) (h)

$SU(3)$ fields - case IV; a) fields φ_1 and φ_2.

- Case V. The values of γ_i are the same as in case I but this time $\lambda = 1.8$.



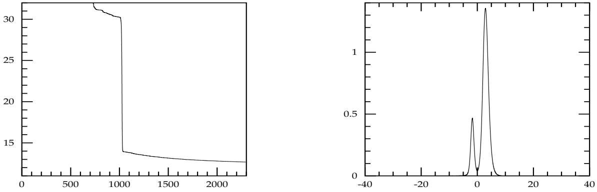
(i) (j)

Figure: $SU(3)$ fields - case V; a) fields φ_1 and φ_2. This case is similar to case I but $\lambda = 1.8$

General comments.

- All the plots show that the fields always go from one vacuum to another one.
- In all the cases, at the vacuum at $x = \infty$, the fields go to 0. As x gets smaller and smaller the fields go to various vacua.
- In cases I, II and V $\gamma_1 = \gamma_3 = 1$ and so we know that $\varphi_2(-\infty) = 2\varphi_1(-\infty)$.
- The exact values of $\varphi_1(-\infty)$ in cases I and II are different and they depend on the value of γ_2.
- The cases of I and V differ by the value of λ and their plots are slightly different but the values of the energies are the same.
- The case III is slightly different. The values of the parameters λ and γ_i are the same as in the case IV the only difference lies in the values of initial values of $\varphi_i(0)$ (in the case III we took $\varphi_1(0) = 0.1$ and $\varphi_2(0) = 3.1$ while in the case IV we took the same value of $\varphi_1(0)$ but for $\varphi_2(0)$ we took 1.6.)

- Of course, the BPS equations do not 'know about the topology' they are just responsible for the evolution to the 'nearest' vacuum. Hence in the case III the field φ_1 evolved in both directions of x to the same value of the vacuum, namely 0, while the field φ_2 went to π and 0.
- We have looked also at the energy densities of the fields.



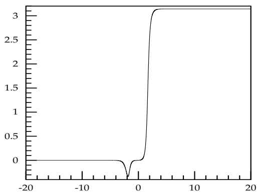
(a) (b)

Energy densities (and topological charge densities) of field configurations seen in cases I (a) and V (b).

- We note that in both cases we have two peaks of the energy. As the total value is the same in both cases the whole effect of λ corresponds to the change of the relative heights of the two peaks and their positions. When λ is larger the peaks are also a little closer. For smaller values of λ these effects are less visible.

Concrete results for 2 Sine Gordon fields.

- We have also looked at the solitons of the model involving 2 Sine Gordon fields.
- Even in this case had to solve the self-duality equations numerically.
- The plots show the fields (for $\lambda = 0.8$) (left φ_1, right φ_2)



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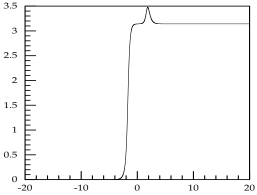
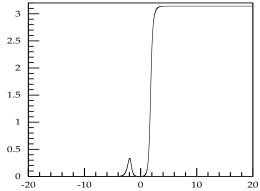


(d)

- Here the plots show the fields (for $\lambda = -0.8$) (left φ_1, right φ_2)



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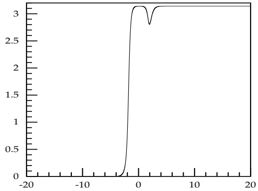


(f)

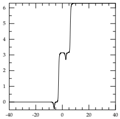
- We note the reversal of the secondary ‘bumps’.
- The height of the bumps depends on λ (vanishes for $\lambda = 0$).

Interactions:

- Have also looked at various interactions of such solitons.
- Take the solutions for one value of λ and use as initial condition for the full simulation with a different value.
- Construct two two-solitons systems by sewing together two self-duality fields.
- Construct similar 'breather-like' systems.
- Interesting results.
- In the first case energy is larger and the soliton gets rid of it radiating and sometimes also moving. Other cases were more interesting.

Other interactions

- Have also looked at simulations involving two solitons in each field interacting with similar solitons (or antisolitons) in the other field.
- As solitons repel (and soliton-antisoliton attract) we have to place solitons close together.
- For two solitons we took one soliton ($0 \le \varphi \le \pi$) in each field and put it for $x \le 0$; The second soliton in each field was taken as $\varphi + \pi$ and placed at $x \ge 0$.
- Here are plots for $\lambda = 1.0$ case.



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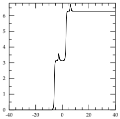
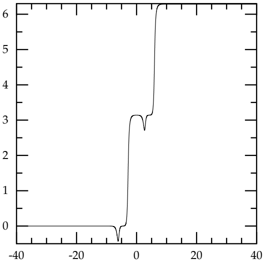


(h)

- At different values of λ plots were slightly different.

- We run various simulations for many values of λ.
- Results were very interesting.
- For $\lambda = 0$ nothing much happened - solitons were too far away...
- For $\lambda < 0$ we obtained some slow motion of solitons (sometimes all moved apart ($\lambda = -0.5$), sometimes two outside ones moved away and the two middle ones were trapped at first and then moved away slowly ($\lambda = -1.5$))
- For $\lambda > 0$. For large λ - either all moved away and we ended up with the vacuum or, for smaller λ, we ended up with one soliton (eg $\lambda = 1.0$).

- Fields at $t = 0$ are the previously shown fields - here shown once again



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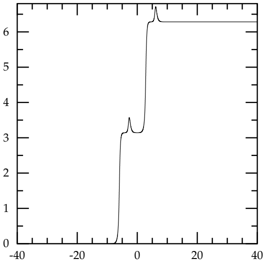
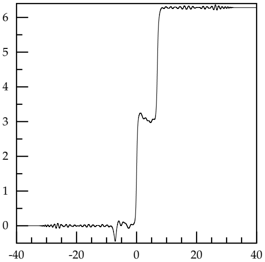


(j)

- Fields at $t = 720$.



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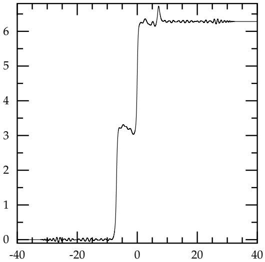
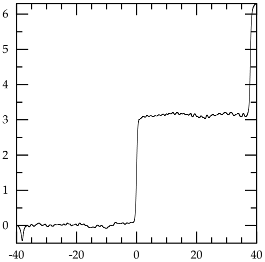
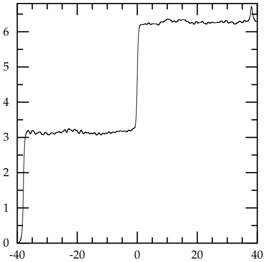


(l)

- Fields at $t = 1000$.

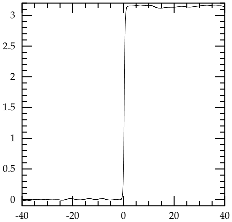


(m)

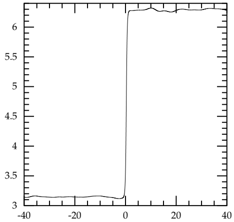


(n)

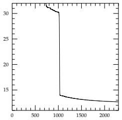
- Fields much later at $t = 6000$ are shown here and the total energy.



(o)



(p)



An explanation

- The final state has a soliton (in φ_1) going from 0 to π and a soliton in φ_2 from π to 2π. Both at $x = 0$.
- Note that then $\sin(\varphi_2) = -\sin(\varphi_1)$.
- Hence the energy of such a field is

$$E = \frac{1}{2}\left[(\partial_x\varphi_1)^2(2-\lambda)\right] + \frac{2}{4-\lambda^2}\sin^2(\varphi_1)(2-\lambda),$$

- This is like a field of one soliton which can satisfy the BPS condition

$$\partial_x\varphi = -\frac{2}{\sqrt{4-\lambda^2}}\sin(\varphi)$$

- Its energy energy is much too large so the field is emitting its excess of energy.

- Interesting results have also been obtained for solitons and antisolitons.
- Overall, for small values of $|\lambda|$ the scattering behaviour resembles the behaviour seen in a model involving only one Sine Gordon field for larger values we have some more interesting behaviour

Summary and some Conclusions

- Have made some progress in understanding the BPS conditions.
- In simple cases, nothing very spectacular and we get stable soliton solutions.
- When one considers more complicated fields consisting of 'relatively' isolated solitons - the system behaves like a system of Sine Gordon fields.
- However, the interactions depend on λ and for larger values of this parameters - behaviour changes (but not that much)
- Most of our work is still somewhat preliminary and so far only in (1+1) dimensions.
- A lot still remains to be done

Thank you for listening.

Many happy returns, Laci.

Sok boldog visszatérés, Laci.