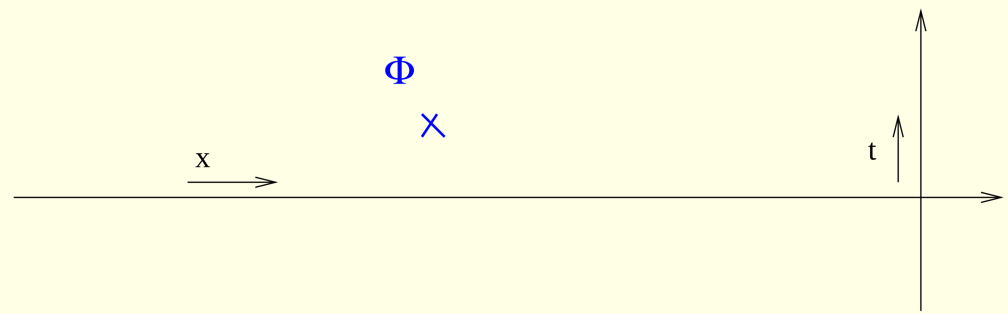
Workshop and Summer School: From Statistical Mechanics to Conformal and Quantum Field Theory
8 January - 8 February, 2007

Bulk and Boundary Form Factors in QFT (1)

Z. Bajnok
Institute for Theoretical Physics, Eötvös University, Budapest

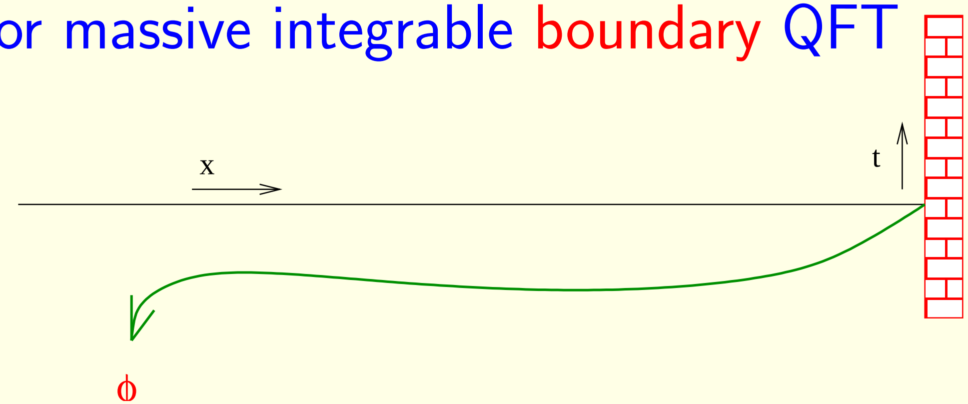
Bulk bootstrap programme for massive integrable QFT in 1+1 D

$$\langle 0|\Phi(x,t)|\theta_1,\theta_2,\ldots,\theta_n\rangle^{in}$$



Boundary bootstrap programme for massive integrable boundary QFT

$$_B\langle 0|\varphi_B(t)|\theta_1,\theta_2,\ldots,\theta_n\rangle_B^{in}$$



Definition of a QFT: Schemes
QFT={ Correlators of local operators}

$$\mathcal{L} = \frac{1}{2}(\partial_t\Phi)^2 - \frac{1}{2}(\partial_x\Phi)^2 - V(\Phi)$$

Schemes based on a Lagrangian

$\swarrow$

free scheme $V = \frac{1}{2}m^2\Phi^2$

operators: $\mathcal{O} =: \partial^k\Phi \dots \partial^l\Phi :$

solved $\langle 0|T(\mathcal{O}_1 \dots \mathcal{O}_N)|0\rangle$

$\downarrow$

perturbative scheme $H = H_0 + H_{pert}$

Correlators: $\langle 0|T(\mathcal{O}_1 \dots \mathcal{O}_N)|0\rangle =$

$$\frac{{}_0\langle 0|T(\mathcal{O}_1 \dots \mathcal{O}_N e^{i\int H_{pert}})|0\rangle_0}{{}_0\langle 0|Te^{i\int H_{pert}}|0\rangle_0}$$

$\searrow$

Path integral scheme $(t = -iy)$

operators: $\mathcal{O} = \partial^k\Phi \dots \partial^l\Phi$

Correlators: $\langle \mathcal{O}_1 \dots \mathcal{O}_N\rangle =$

$$\frac{\int \mathcal{D}\Phi\, \mathcal{O}_1 \dots \mathcal{O}_N e^{-\int d^2x\, \mathcal{L}_E}}{\int \mathcal{D}\Phi\, e^{-\int d^2x\, \mathcal{L}_E}}$$

Schemes based on symmetries

$\swarrow$

CFT scheme $V = 0$

spectrum from the symmetry

$$H = L_0 + \bar{L}_0 = \frac{1}{2}(\partial_x\Phi)^2 + \frac{1}{2}(\partial_x\Phi)^2$$

scaling operators: $\mathcal{O} =: \partial^k\Phi \dots \partial^l\Phi e^{i\beta\Phi} :$

Structure constants: $\langle u|\mathcal{O}|v\rangle$ from consisteny

$$\langle 0|R(\mathcal{O}_1\mathcal{O}_2)|0\rangle = \sum_n \langle 0|\mathcal{O}_1|n\rangle\langle n|\mathcal{O}_2|0\rangle$$

$\searrow$

IFT scheme $V = \cosh\beta\Phi$

spectrum from the symmetry

$$H = \frac{1}{2}(\partial_x\Phi)^2 + \frac{1}{2}(\partial_x\Phi)^2 + V(\Phi)$$

S-matrix bootstrap

local operators: $[\mathcal{O}_1, \mathcal{O}_2]_{spl} = 0$

Form factor bootstrap $\langle u|\mathcal{O}|v\rangle$ satiesfies axioms

$$\langle 0|\mathcal{O}_1\mathcal{O}_2|0\rangle = \sum_n \langle 0|\mathcal{O}_1|n\rangle\langle n|\mathcal{O}_2|0\rangle$$

Free scheme: requirements

Aim: explicit solution $\boxed{\mathcal{L} = \frac{1}{2}(\partial_t \Phi)^2 - \frac{1}{2}(\partial_x \Phi)^2 - \frac{1}{2} m^2 \Phi^2}$ canonical quantization

Requirements: $\Phi(x,t)$ satisfies:

$$(\partial_t^2 - \partial_x^2 + m^2)\Phi = 0 \quad ; \qquad [\Phi(x,t), \partial_t \Phi(x', t)] = i\delta(x - x')$$

Maintain Poincare invariance: (implemented by the conserved charges)

$$\Phi(x,t) = e^{-iHt}\Phi(x,0)e^{iHt} \quad ; \qquad \Phi(x,t) = e^{iPx}\Phi(0,t)e^{-iPx}$$

Conserved energy $H = \int \left[\frac{1}{2}(\partial_t \Phi)^2 + \frac{1}{2}(\partial_x \Phi)^2 + \frac{1}{2} m^2 \Phi^2\right] dx \quad ; \quad \dot{H} = 0$

Conserved momentum $P = \int [\partial_t \Phi \partial_x \Phi]\, dx \quad ; \qquad \dot{P} = i[H, P] = 0$

Energy-momentum tensor $\partial_\mu T^{\mu\nu} = 0$ and $H = \int T^{00} dx \; ; \quad P = \int T^{01} dx$

Free scheme: solution

Solution $\Phi(x,t) = \int \frac{dk}{2\pi\omega(k)} \left[e^{i\omega(k)t - ikx} a^+(k) + e^{-i\omega(k)t + ikx} a(k) \right]$

from equation of motion: $\omega(k) = \sqrt{k^2 + m^2}$ from commutation relation

$$[a(k), a^+(k')] = 2\pi\omega(k)\delta(k-k') \quad ; \qquad [a^+(k), a^+(k')] = 0 = [a(k), a(k')]$$

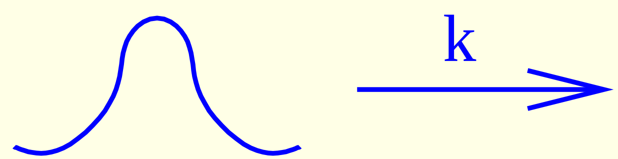
Energy: $H = \int \frac{dk}{2\pi\omega(k)} \omega(k) a^+(k) a(k)$

Momentum: $P = \int \frac{dk}{2\pi\omega(k)} k a^+(k) a(k)$

Hilbert space $\mathcal{H} = \left\{ a^+(k_1) a^+(k_2) \ldots |0\rangle = |k_1, k_2, \ldots\rangle \right\}$ where $a(k)|0\rangle = 0$

state	Energy	Momentum
vacuum $\|0\rangle$	0	0
one particle state $\|k\rangle$	$\omega(k)$	k
multiparticle state $\|k_1, \ldots, k_N\rangle$	$\omega(k_1) + \ldots \omega(k_N)$	$k_1 + \ldots + k_N$

'localized' state $|f(k)\rangle = \int \frac{dk}{2\pi\omega(k)} f(k) a^+(k) |0\rangle$



Free scheme: correlators

Local operators : $\partial_x^{n_1}\partial_t^{m_1}\Phi(x_1,t_1)\ldots\partial_x^{n_N}\partial_t^{m_M}\Phi(x_N,t_N)$:

Normal ordered products (creation operators are on the left)

time ordered two point function

$$\langle 0|T(\Phi(x_1,t_1)\Phi(x_2,t_2))|0\rangle = \int \frac{dk_0 dk_1}{(2\pi)^2}\frac{i}{k_0^2-k_1^2-m^2+i\epsilon}e^{ik_0(t_1-t_2)-ik_1(x_1-x_2)}$$

Matrix elements of Φ

Using Poincare invariance: $\langle 0|\Phi(x,t)|k\rangle = \langle 0|e^{-iHt}\Phi(x,0)e^{iHt}|k\rangle = e^{i\omega(k)t-ikx}$

and so for $t_1 > t_2$

$$\langle 0|\Phi(x_1,t_1)\Phi(x_2,t_2)|0\rangle = \int \frac{dk}{2\pi\omega(k)}\langle 0|\Phi(x_1,t_1)|k\rangle\langle k|\Phi(x_2,t_2)|0\rangle$$

Wick theorem

Perturbative scheme

Use pertrubation theory: $\mathcal{L} = \frac{1}{2}(\partial_t \Phi)^2 - \frac{1}{2}(\partial_x \Phi)^2 - \frac{1}{2} m^2 \Phi^2 - \lambda U(\Phi) = \mathcal{L}_0 - \mathcal{L}_{pert}$

$\Phi(x,t)$ satisfies: $(\partial_t^2 - \partial_x^2 + m^2)\Phi = -\lambda \frac{dU}{d\Phi}$; $[\Phi(x,t), \partial_t \Phi(x', t)] = i\delta(x - x')$

Poincare invariance: $\Phi(x,t) = e^{-iHt}\Phi(x,0)e^{iHt}$; $\Phi(x,t) = e^{iPx}\Phi(0,t)e^{-iPx}$

Energy $H = H_0 + \lambda \int U(\Phi) dx = H_0 + H_{pert}$ Momentum $P = P_0$

Perturbation does not change the operators $\mathcal{O} = : \partial^k \Phi \dots \partial^l \Phi :$
and the spectrum (except mass) [special perturbation]

Correlators: $\langle 0 | T(\mathcal{O}_1 \dots \mathcal{O}_N) | 0 \rangle = \frac{{}_0\langle 0 | T(\mathcal{O}_1 \dots \mathcal{O}_N e^{i \int H_{pert}}) | 0 \rangle_0}{{}_0\langle 0 | T e^{i \int H_{pert}} | 0 \rangle_0}$

$$\langle 0 | T(\mathcal{O}_1 \dots \mathcal{O}_N) | 0 \rangle = \sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{n!} \, {}_0\langle 0 | T(\mathcal{O}_1 \dots \mathcal{O}_N (\textstyle\int U d^2x)^n) | 0 \rangle_0^{conn}$$

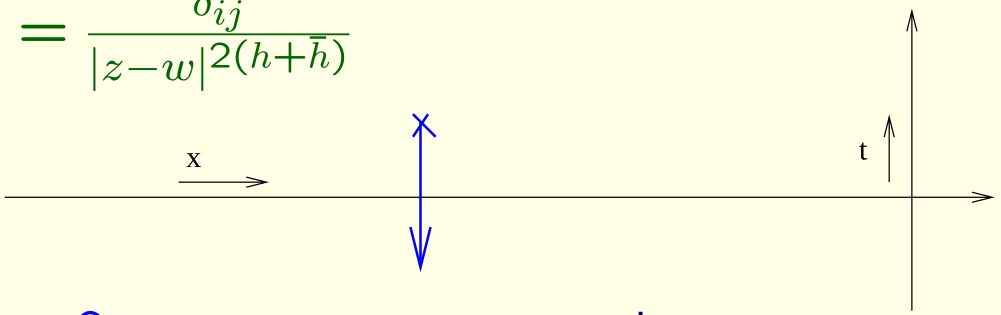
for normal ordered perturbations all terms are finite

CFT scheme

Hilbert space $\mathcal{H} = \sum_{(i,i)} \mathcal{V}_i \otimes \bar{\mathcal{V}}_i$ where $\mathcal{V}_i$ h.w. reps of the left chiral algebra

with eigenvalue of $L_0(\bar{L}_0)$ as h $(\bar{h})$ Energy $H = L_0 + \bar{L}_0$, Momentum $P = L_0 - \bar{L}_0$

h.w. operators: $\langle 0|\Phi_i(z,\bar{z})\Phi_j(w,\bar{w})|0\rangle = \frac{\delta_{ij}}{|z-w|^{2(h+\bar{h})}}$



Operator algebra $[\Phi_i][\Phi_j] = \sum_k C_{ij}^k [\Phi_k]$

Operator - state correspondence
$\{\Phi(x,t)\} \leftrightarrow \mathcal{H}$

Lee-Yang model: the only pr. field $\Phi_{(-\frac{1}{5},-\frac{1}{5})} = \Phi$ and $[\Phi][\Phi] = [1] + C_{\Phi\Phi}^{\Phi}[\Phi]$

Relevant (integrable) perturbations $S = S_{CFT} - \lambda_{bulk} \int dt \int dx\, \Phi(x,t)$

Hilbert space does not change

Assumptions: spectrum, operator algebra smoothly changes $\longrightarrow$

Operator content
Bulk operators $[1]$,$[\Phi(x,t)]$

Perturbative $\to$ IFT scheme

Integrable Langrangian: $\boxed{\mathcal{L} = \frac{1}{2}(\partial_t \Phi)^2 - \frac{1}{2}(\partial_x \Phi)^2 - \frac{m^2}{\lambda}\cosh(\sqrt{\lambda}\Phi)}$

Local operator algebra, spectrum smoothly changes:

Operator content $\mathcal{O} =: \partial^k \Phi \dots \partial^l \Phi :$

Only one selfconjugate particle with mass $\mu(\lambda)$

Energy $H = H_0 + H_{pert}$, Momentum P are conserved ($\dot{E} = \dot{P} = 0$) and generate

$$\Phi(x,t) = e^{iHt}\Phi(x,0)e^{-iHt} \quad ; \qquad \Phi(x,t) = e^{iPx}\Phi(0,t)e^{-iPx}$$

Energy: $E = \omega(k) = \sqrt{\mu^2 + k^2} = \mu \cosh\theta$ Momentum $P = k = \mu \sinh\theta$

notation $|k\rangle = |\theta\rangle$

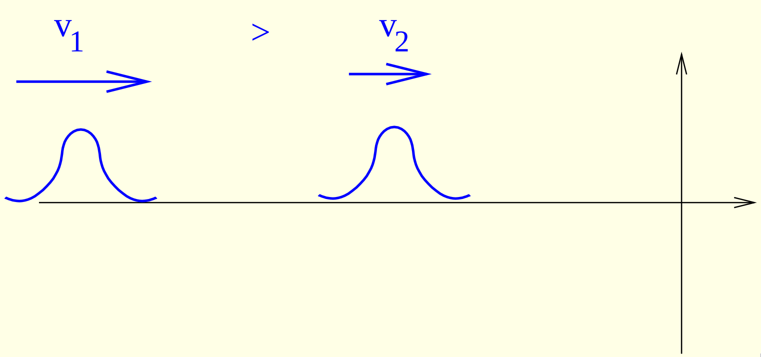
Higher spin conserved charges $E_n = q_n \cosh n\theta$ and $P_n = q_n \sinh n\theta$

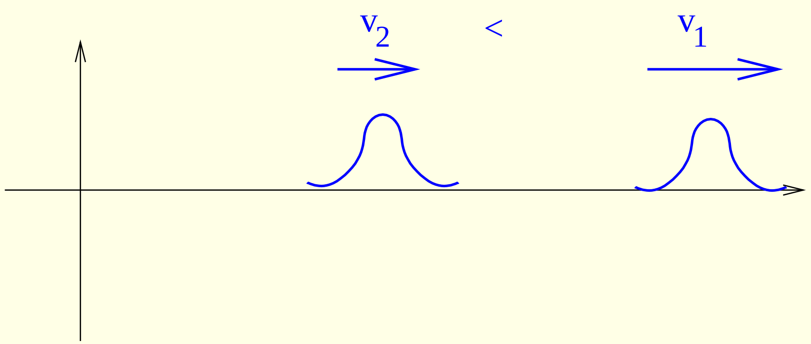
Integrable Field Theory scheme

$v_i = \sinh\theta_i$

Bulk two particle in state: $t \to -\infty$

Bulk two particle out state: $t \to \infty$





Free, noninteracting in particles $\leftarrow$ **S-matrix** $\rightarrow$ **Free, noninteracting out particles**

$$|\theta_1, \theta_2\rangle^{in} = S(|\theta_1 - \theta_2|)|\theta_1, \theta_2\rangle^{out}$$

$\theta_1 > \theta_2$

$$|\theta_1, \theta_2\rangle = S(\theta_1 - \theta_2)|\theta_2, \theta_1\rangle$$

Integrability $\longrightarrow$ factorizability:

$$|\theta_1, \theta_2, \ldots, \theta_n\rangle = \prod_{i<j} S(\theta_i - \theta_j)|\theta_n, \theta_{n-1}, \ldots, \theta_1\rangle$$

Consequences of integrability

Integrability $\longrightarrow$ infinite conserved charges $\longrightarrow$ shift the trajectories

Properties of the S-matrix from the perturbativ scheme

Definition $S(\theta) =_{out} \langle\theta_1,\theta_2|S|\theta_1,\theta_2\rangle_{in} = \langle\theta_2,\theta_1|\theta_1,\theta_2\rangle$ where $\theta = \theta_1 - \theta_2 > 0$

need: analytic structure of S matrix $\leftrightarrow$ reduction formula $\leftrightarrow$ known: analytic structure of correlators

$$< out|\mathcal{O}(x,it)|\theta_1,\theta_2,\ldots,\theta_n > = 2\pi\delta(\theta_1-\theta)\langle out\backslash\theta|\mathcal{O}(x,it)|\theta_2,\ldots,\theta_n >$$

$$-Z^{-1/2}\int_{-\infty}^{\infty} d(it')\, e^{-i(i\omega(\theta_1))(it')}\int_{-\infty}^{\infty} dx'\, e^{-ip(\theta_1)x'}\{-\partial^2_{it'}-\partial^2_{x'}+m^2\}$$

$$< out|T(\Phi(x',it')\mathcal{O}(x,it))|\theta_2,\ldots,\theta_n >$$

Analytic continuation in θ_1:

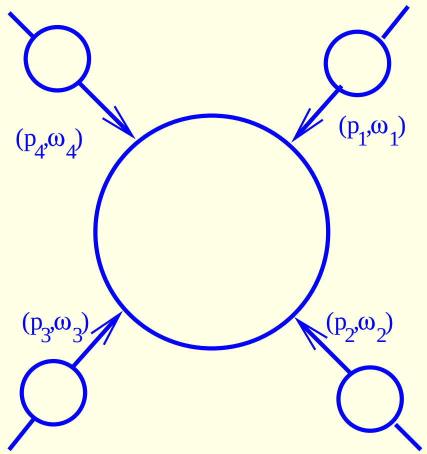
(time reversal) $it \to -it$ continuation: $\theta_1 \to i\pi - \theta_1$ Unitarity $S^{-1}(\theta) = S(-\theta)$

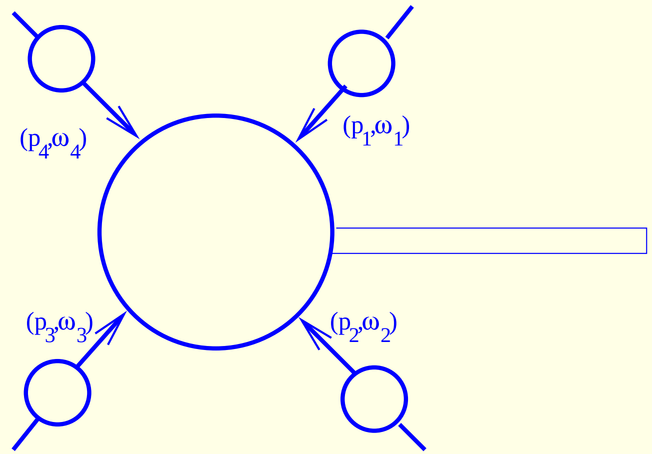
Crossing $(x,it) \to (-iT,X)$ $\theta \to \frac{i\pi}{2} - \theta$ Crossing $S(\theta) = S(i\pi - \theta)$

Singularity properties of the correlators: Landau equations

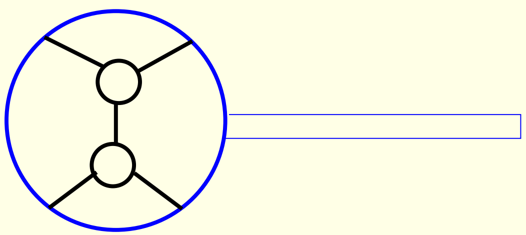
Singularity structure: Landau equations

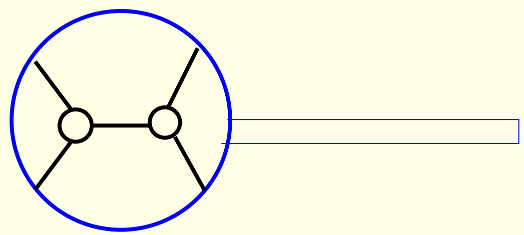
Singularity in correlation function $=$ on mass shell particles

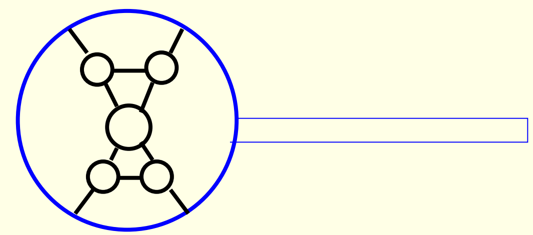




Coleman-Norton interpretation





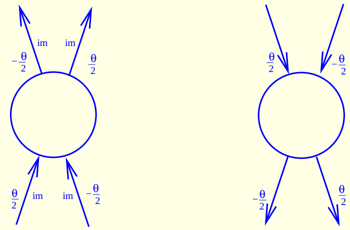


Cutkosky rules: replace $(\omega^2 - p^2 - m^2 + i\epsilon)^{-1}$ with $2\pi\delta^+(\omega^2 - p^2 - m^2)$

$(G^3(u_1))^2$ $\quad$ $(G^3(u_1))^2$ $\quad$ $(G^3(u_2))^4 G^4(u_3)$

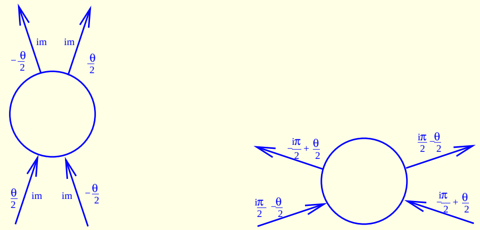
S-matrix bootstrap

Unitarity



$$S(\theta)^{-1} = S(-\theta)$$

Crossing

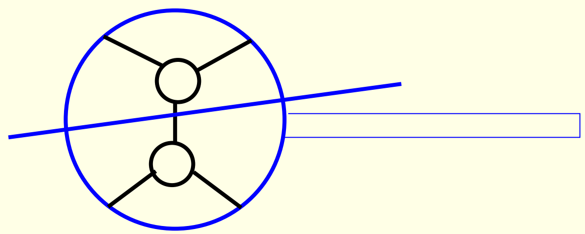


$$S(\theta) = S(i\pi - \theta)$$

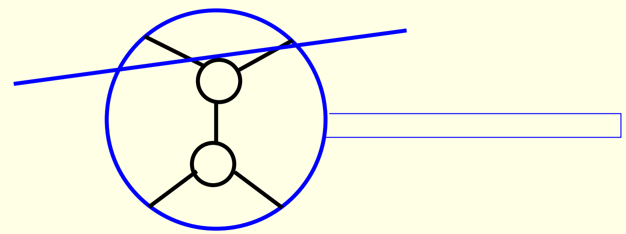
Simplest nontrivial solution $S(\theta) = \frac{\sinh\theta - i\sin\gamma}{\sinh\theta + i\sin\gamma}$

Minimality: all singularity has physical origin: $\gamma > 0$ end of the story (sinh-Gordon):

Bootstrap :



$S_{new}(\theta)$



$S_{old}(\theta + iu)S_{old}(\theta - iu)$

for $\gamma = -\frac{2}{3}$ self-fusion: Lee-Yang,
for generic γ sine-Gordon $B_2, B_3, \ldots, B_n, s, \bar{s}$

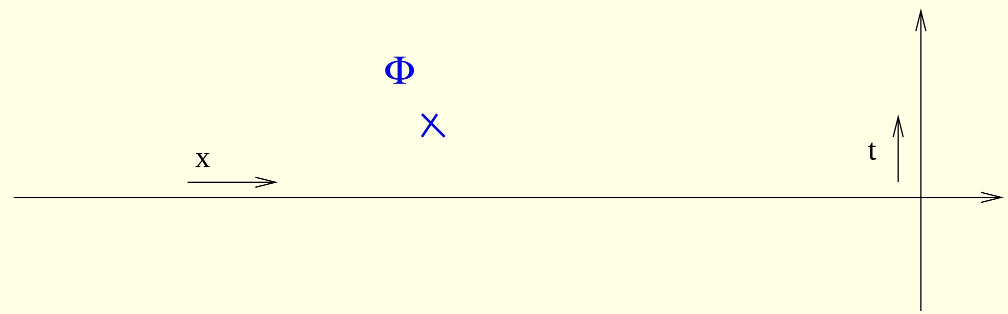
Workshop and Summer School: From Statistical Mechanics to Conformal and Quantum Field Theory
8 January - 8 February, 2007

Bulk and Boundary Form Factors in QFT (2)

Z. Bajnok
Institute for Theoretical Physics, Eötvös University, Budapest

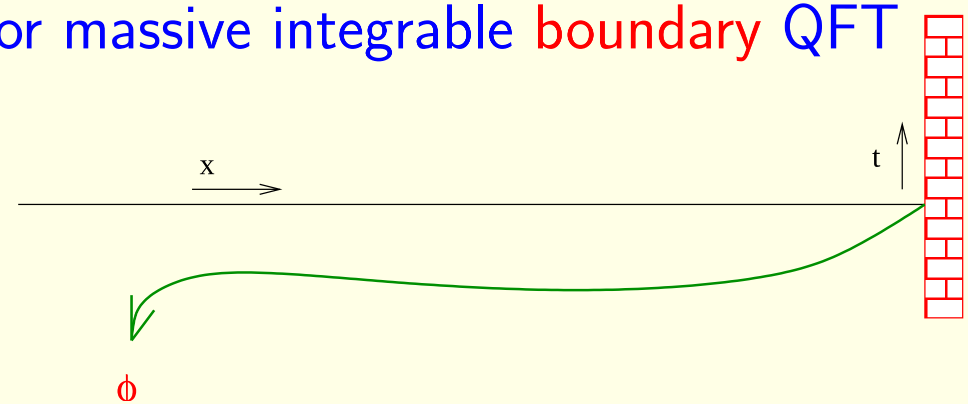
Bulk bootstrap programme for massive integrable QFT in 1+1 D

$\langle 0|\Phi(x,t)|\theta_1,\theta_2,\ldots,\theta_n\rangle^{in}$



Boundary bootstrap programme for massive integrable boundary QFT

${}_B\langle 0|\varphi_B(t)|\theta_1,\theta_2,\ldots,\theta_n\rangle_B^{in}$



Definition of a QFT: Schemes
QFT={ Correlators of local operators}

$$\mathcal{L} = \frac{1}{2}(\partial_t\Phi)^2 - \frac{1}{2}(\partial_x\Phi)^2 - V(\Phi)$$

Schemes based on a Lagrangian

free scheme $V = \frac{1}{2}m^2\Phi^2$

operators: $\mathcal{O} =: \partial^k\Phi \dots \partial^l\Phi :$

solved $\langle 0|T(\mathcal{O}_1 \dots \mathcal{O}_N)|0\rangle$

perturbative scheme $H = H_0 + H_{pert}$

Correlators: $\langle 0|T(\mathcal{O}_1 \dots \mathcal{O}_N)|0\rangle =$

$$\frac{{}_0\langle 0|T(\mathcal{O}_1 \dots \mathcal{O}_N e^{i\int H_{pert}})|0\rangle_0}{{}_0\langle 0|Te^{i\int H_{pert}}|0\rangle_0}$$

Path integral scheme ($t = -iy$)

operators: $\mathcal{O} = \partial^k\Phi \dots \partial^l\Phi$

Correlators: $\langle \mathcal{O}_1 \dots \mathcal{O}_N\rangle =$

$$\frac{\int \mathcal{D}\Phi\, \mathcal{O}_1 \dots \mathcal{O}_N e^{-\int d^2x\, \mathcal{L}_E}}{\int \mathcal{D}\Phi\, e^{-\int d^2x\, \mathcal{L}_E}}$$

Schemes based on symmetries

CFT scheme $V = 0$

spectrum from the symmetry

$$H = L_0 + \bar{L}_0 = \frac{1}{2}(\partial_x\Phi)^2 + \frac{1}{2}(\partial_x\Phi)^2$$

scaling operators: $\mathcal{O} =: \partial^k\Phi \dots \partial^l\Phi e^{i\beta\Phi} :$

Structure constants: $\langle u|\mathcal{O}|v\rangle$ from consisteny

$$\langle 0|R(\mathcal{O}_1\mathcal{O}_2)|0\rangle = \sum_n \langle 0|\mathcal{O}_1|n\rangle\langle n|\mathcal{O}_2|0\rangle$$

IFT scheme $V = \cosh\beta\Phi$

spectrum from the symmetry

$$H = \frac{1}{2}(\partial_x\Phi)^2 + \frac{1}{2}(\partial_x\Phi)^2 + V(\Phi)$$

S-matrix bootstrap

local operators: $[\mathcal{O}_1, \mathcal{O}_2]_{sp} = 0$

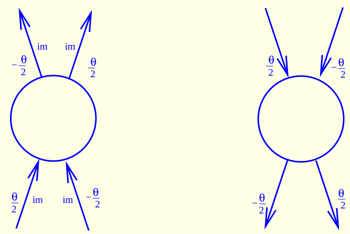
Form factor bootstrap $\langle u|\mathcal{O}|v\rangle$ satiesfies axioms

$$\langle 0|\mathcal{O}_1\mathcal{O}_2|0\rangle = \sum_n \langle 0|\mathcal{O}_1|n\rangle\langle n|\mathcal{O}_2|0\rangle$$

S-matrix bootstrap

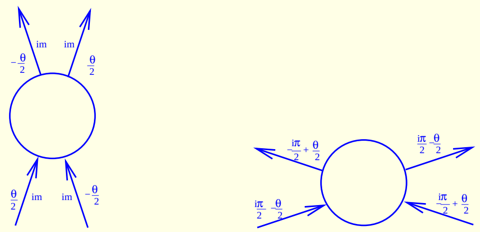
one selfconjugate particle + integrability

Unitarity



$$S(\theta)^{-1} = S(-\theta)$$

Crossing



$$S(\theta) = S(i\pi - \theta)$$

Simplest nontrivial solution $S(\theta) = \frac{\sinh\theta - i\sin\gamma}{\sinh\theta + i\sin\gamma}$

Minimality: all singularity has physical origin: $\gamma > 0$ end of the story (sinh-Gordon):

Bootstrap:

$$S_{new}(\theta) = S_{old}(\theta + iu)S_{old}(\theta - iu)$$

for $\gamma = -\frac{2}{3}$ self-fusion: Lee-Yang

Analytical properties of the bulk form factors

Bulk form factor $\theta_1 > \theta_2 > \ldots > \theta_n$

$$F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n) = \langle 0|\mathcal{O}(0,0)|\theta_1, \ldots, \theta_n\rangle$$

$$F_n^{\mathcal{O}}(\theta_n, \ldots, \theta_1) = \langle 0|\mathcal{O}(0,0)|\theta_n, \ldots, \theta_1\rangle$$

Example $F_2^{\mathcal{O}}(\theta_1, \theta_2) = S(\theta_1 - \theta_2) F_2^{\mathcal{O}}(\theta_2, \theta_1)$

$$F_{mn}^{\mathcal{O}}(\theta_1^{'}, \ldots, \theta_m^{'}|\theta_1, \ldots, \theta_n) = \langle \theta_1^{'}, \ldots, \theta_m^{'}|\mathcal{O}(0,0)|\theta_1, \ldots, \theta_n\rangle$$

Permutation $F_{m2}^{\mathcal{O}}(\theta_1^{'}, \ldots, \theta_m^{'}|\theta_1, \theta_2) = S(\theta_1 - \theta_2) F_{m2}^{\mathcal{O}}(\theta_1^{'}, \ldots, \theta_m^{'}|\theta_2, \theta_1)$

$$< out|\mathcal{O}(x,it)|\theta_1, \theta_2, \ldots, \theta_n > = 2\pi\delta(\theta_1 - \theta)\langle out\backslash\theta|\mathcal{O}(x,it)|\theta_2, \ldots, \theta_n >$$

$$-Z^{-1/2}\int_{-\infty}^{\infty} d(it^{'})\, e^{-i(i\omega(\theta_1))(it^{'})} \int_{-\infty}^{\infty} dx^{'} e^{-ip(\theta_1)x^{'}} \{-\partial_{it^{'}}^2 - \partial_{x^{'}}^2 + m^2\}$$

$$< out|T(\Phi(x^{'}, it^{'})\mathcal{O}(x,it))|\theta_2, \ldots, \theta_n >$$

Crossing from reduction formula

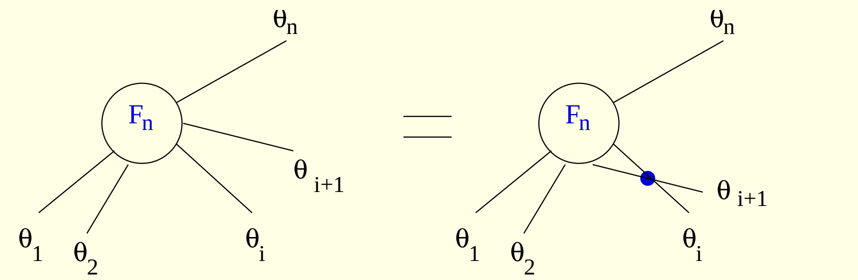
$$F_{1n}^{\mathcal{O}}(\theta|\theta_1, \ldots, \theta_n) = F_{n+1}^{\mathcal{O}}(\theta_1, \ldots, \theta_n, \theta - i\pi) + \delta(\theta - \theta_n) F_{n-1}^{\mathcal{O}}(\theta_1, \ldots, \theta_{n-1})$$

$$F_{1n}^{\mathcal{O}}(\theta|\theta_n, \ldots, \theta_1) = F_{n+1}^{\mathcal{O}}(\theta + i\pi, \theta_n, \ldots, \theta_1) + \delta(\theta - \theta_n) F_{n-1}^{\mathcal{O}}(\theta_{n-1}, \ldots, \theta_1)$$

Bulk form factor axioms

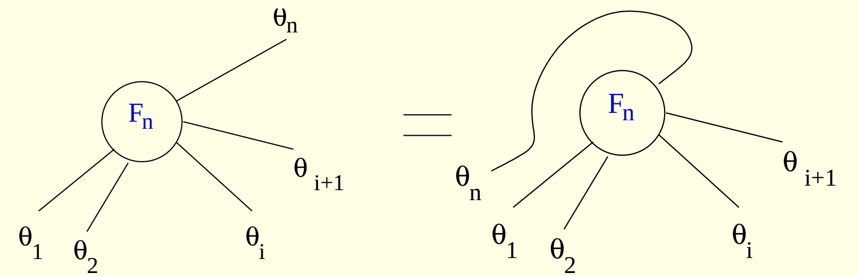
Permutation

$$F_n^{\mathcal{O}}(\theta_1,\ldots\theta_i,\theta_{i+1},\ldots,\theta_n) =$$
$$S(\theta_i-\theta_{i+1})F_n^{\mathcal{O}}(\theta_1,\ldots\theta_{i+1},\theta_i,\ldots,\theta_n)$$



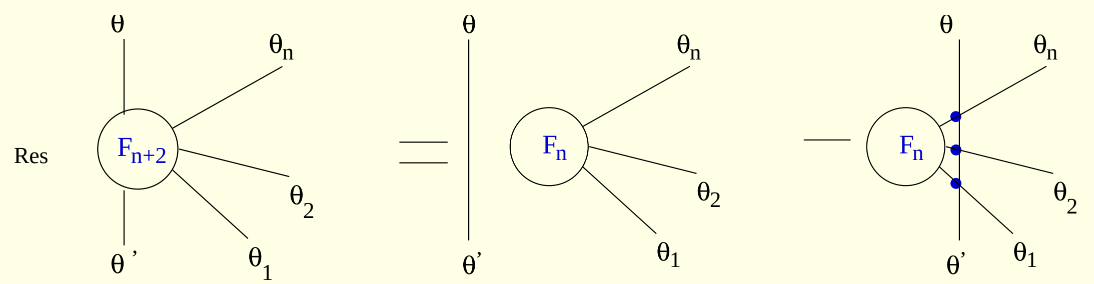
Periodicity

$$F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_{n-1},\theta_n) =$$
$$F_n^{\mathcal{O}}(\theta_n+2\pi i,\theta_1,\ldots,\theta_{n-1})$$



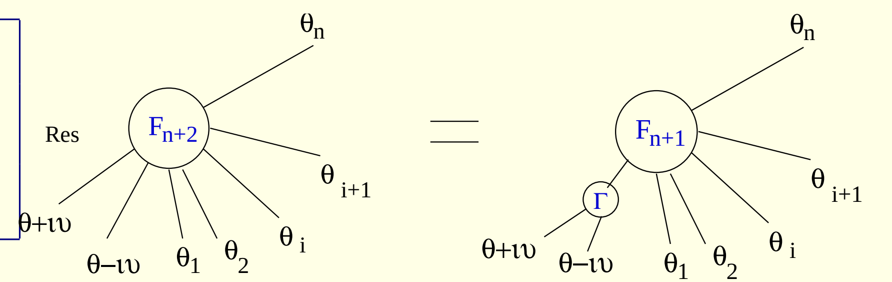
Kinematical singularities

$$-i\mathrm{res}_{\theta=\theta'}F_{n+2}^{\mathcal{O}}(\theta+i\pi,\theta',\theta_1,\ldots,\theta_n) =$$
$$= \left(1-\prod_{i=1}^n S(\theta-\theta_i)\right) F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)$$



Dynamical singularities

$$-i\mathrm{res}_{\theta=\theta'}F_{n+2}^{\mathcal{O}}(\theta+iu,\theta'-iu,\theta_1,\ldots,\theta_n) =$$
$$\Gamma F_{n+1}^{\mathcal{O}}(\theta,\theta_1,\ldots,\theta_n)$$



Solving the bulk form factor equations

Bulk theory with $S(\theta)$: FF axioms + minimality $\longrightarrow$ Solution of the FF equations

Step 1. Solve first the two particle case $F_2(\theta_1 - \theta_2) = f(\theta)$

$$f(\theta) = S(\theta) f(-\theta) \quad ; \quad f(i\pi + \theta) = f(i\pi - \theta)$$

minimality: only dynamical poles, one zero at $\theta = 0$, minimal growth at $\theta \to \infty$

Step 2. Take the Ansatz to satisfy the permutation and the periodicity axioms

$$F_n(\theta_1, \dots, \theta_n) = H_n Q_n(x_1, \dots, x_n) \prod_{i<j} f(\theta_i - \theta_j) \frac{1}{x_i + x_j} \quad ; \quad x = e^{\theta}$$

$Q_n(x_1, \dots, x_n)$: completely symmetric polynomial $\leftrightarrow$ locality

Step 3. Derive recursion relations from the singularity axioms

kinematical $Q_{n+2}(-x, x, x_1, \dots, x_n) = P_n(x|x_1, \dots, x_n) Q_n(x_1, \dots, x_n)$

dynamical $Q_{n+1}(e^{iu}x, e^{-iu}x, x_1, \dots, x_n) = R_n(x|x_1, \dots, x_n) Q_n(x, x_1, \dots, x_n)$

Step 4. Solve recursion, compute 2ptf

$$\langle 0|\mathcal{O}(x,0)\mathcal{O}(0,0)|0\rangle = \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} \frac{d\theta_1 \dots d\theta_n}{n!(2\pi)^n} |F_n^{\mathcal{O}}(\theta_1, \dots, \theta_n)|^2 e^{-mx \sum_{i=1}^n \cosh\theta_i}$$

Perturbed Lee-Yang model

Bulk theory with $S(\theta)=\frac{\sinh\theta+i\sin\frac{\pi}{3}}{\sinh\theta-i\sin\frac{\pi}{3}}$

Step 1. Solution of the two particle case $F_2(\theta_1-\theta_2)=f(\theta)$

$$f(\theta)=\frac{\cosh\theta-1}{\cosh\theta+\frac{1}{2}}v(i\pi-\theta)v(-i\pi+\theta)\quad;\quad v(\theta)=\exp\left\{2\int_0^\infty\frac{dx}{x}e^{i\theta x/\pi}\frac{\sinh\frac{x}{2}\sinh\frac{x}{3}\sinh\frac{x}{6}}{\sinh^2 x}\right\}$$

minimality: dynamical pole at $\theta=\frac{2i\pi}{3}$, zero at $\theta=0$, growth: $f(\theta)\to 1$ for $\theta\to\infty$

Step 2. The Ansatz satisfying the permutation and periodicity axioms

$$F_n(\theta_1,\dots,\theta_n)=H_nQ_n(x_1,\dots,x_n)\prod_{i<j}\frac{f(\theta_i-\theta_j)}{x_i+x_j}\quad;\qquad H_n=<\mathcal{O}>\left(\frac{3^{1/4}}{2^{1/2}v(0)}\right)^n$$

$Q_n(x_1,\dots,x_n)$: completely symmetric polynomial

Step 3. Recursion relations from the singularity axioms

kinematical $Q_{n+2}(-x,x,x_1,\dots,x_n)=P_n(x|x_1,\dots,x_n)Q_n(x_1,\dots,x_n)$

$$P_n(x|x_1,\dots,x_n)=\frac{(-1)^nx^2}{2(x_+-x_-)}\left(\prod_{i=1}^n(x_i+x_+)(x_i-x_-)-\prod_{i=1}^n(x_i-x_+)(x_i+x_-)\right)\quad x_\pm=e^{\mp i\frac{\pi}{3}}x$$

dynamical $Q_{n+1}(x_+,x_-,x_1,\dots,x_n)=x\prod_{i=1}^n(x+x_i)Q_n(x,x_1,\dots,x_n)$

Perturbed bulk Lee-Yang: minimal form factor solution

Step 4. Solve recursion, compute 2pt function

$$Q_1 = 1; \quad Q_2 = \sigma_1; \quad Q_3 = \sigma_1\sigma_2 \ldots ; \quad Q_n = det(\sigma_{2i-j}[i-j+1])$$

$$\prod_{i=1}^{n}(x + x_i) = \sum_{k=0}^{n} x^k \sigma_{n-k}$$

The solution with minimal degree is unique. Which operator is it? Compute 2pt:

$$\langle 0|\mathcal{O}(x,0)\mathcal{O}(0,0)|0\rangle = \sum_{n=0}^{\infty}(-1)^n \int_0^{\infty} \frac{d\theta_1 \ldots d\theta_n}{n!(2\pi)^n} |F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)|^2 e^{-mx\sum_{i=1}^{n}\cosh\theta_i}$$

Large distance expansion $\leftrightarrow$ Compare to short distance (UV) expansion

$$\mathcal{A} = \mathcal{A}_{(2,5)} + g\int d^2x \Phi_{(-\frac{1}{5},-\frac{1}{5})}$$

$\Phi_{(-\frac{1}{5},-\frac{1}{5})} = \Phi$ field with smallest scaling dimension

$$\langle 0|\Phi(x,0)\Phi(0,0)|0\rangle = x^{\frac{4}{5}} + x^{\frac{2}{5}} C_{\Phi\Phi}^{\Phi}\langle\Phi\rangle + \ldots$$

Perturbed bulk Lee-Yang two point function

Conformal limit compared to form factor expansion

$$\langle 0|\Phi(x,0)\Phi(0,0)|0\rangle$$

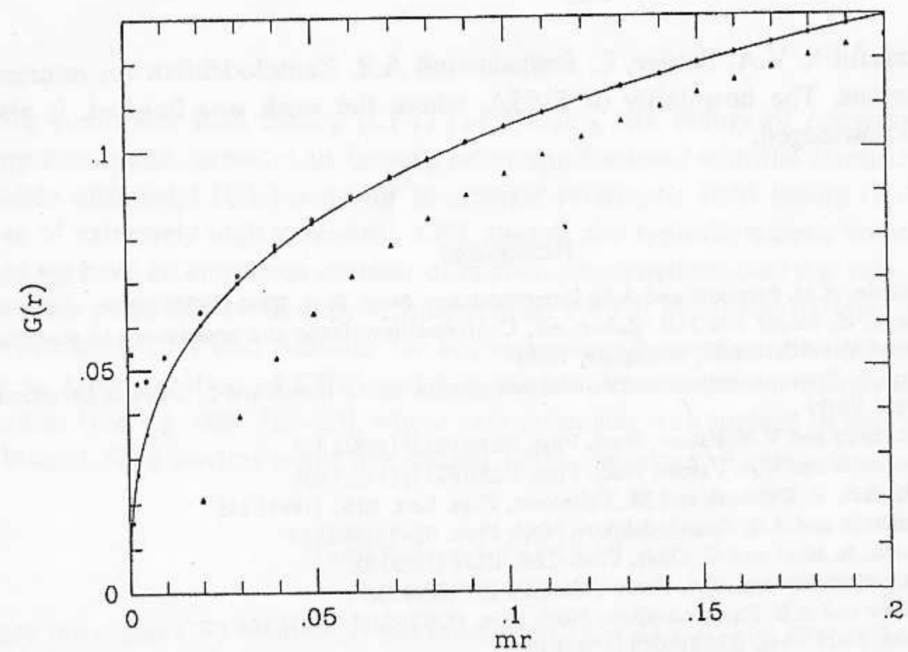


Fig. 3. Convergence of the large-distance expansion for small *mr*. Empty triangles: zero- and one-particle contributions. Empty circles: the same plus two-particle term. Full circles: up to three-particle state contributions. Full curve: the short-distance data.

Conformal limit: $x^{\frac{4}{5}} + x^{\frac{2}{5}} C^{\Phi}_{\Phi\Phi}\langle\Phi\rangle$

FF expansion 0+1+2+3pt: $|F_0^{\Phi}|^2 +$

$$-\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} |F_1^{\Phi}|^2 e^{-mx\cosh\theta}$$

$$+\int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{2(2\pi)^2} |F_2^{\Phi}(\theta_1,\theta_2)|^2 \cdot e^{-mx(\cosh\theta_1+\cosh\theta_2)}$$

$$+\int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2 d\theta_3}{6(2\pi)^3} |F_3^{\Phi}(\theta_1,\theta_2,\theta_3)|^2 \dots$$

Operator classification: visit Aldo's talk

FF solutions + asymptotics (Q is a polynomial) $\longrightarrow$ locality

FF solutions $\leftrightarrow$local operator content

Problem: identification of the operators $\longrightarrow$ compare with other schemes,

Example: derivative operators

$$\langle 0|\Phi(x,t)|\theta_1,\ldots,\theta_N\rangle = e^{i\sum_i \cosh\theta_i t - i\sum_i \sinh\theta_i x} F(\theta_1,\ldots,\theta_N)$$

$\partial_\pm = \partial_t \pm \partial_x$ corresponds to multiplication by σ_1 or $\frac{\sigma_{n-1}}{\sigma_n}$

perturbed free: Energy momentum tensor, spin, conservation laws, equation of motion

pertubed conformal, Δ theorem, clustering, spin

Generalization

Solve the form factor equations for $\gamma > 0$: sinh-Gordon model

solve other models with one selfconjugate particle: Bullough-Dodd

Extend the FF axioms for more particles with diagonal scattering

Extend the FF axioms for more particles with nondiagonal scattering

Use FFs to compute a perturbation of an integrable model

Try to sum up the FF series

Try to describe FFs in finite volume

Generalize the FF program to integrable boundary QFTs

History of form factors

M. Karowski et al Nucl.Phys.B139:455,1978, Phys.Rept.49:229-237,1979, FF program without periodicity, explicite solutions upto two particles, Phys.Rev.D19:2477,1979, summing up the FFs in the Ising case

F. Smirnov et al, Nucl.Phys.B337:156-180,1990. solution of Lee-Yang, restricted sG, Advanced Series in Mathematical physics Vol. 14: the nondiagonal FF program, with solutions to s-G, O(3) σ, $SU(N)$ Thirring, locality is proved.

Mussardo et al Nucl.Phys. B393 (1993) 413-441, Phys.Lett. B307 (1993) 83-90, Phys.Lett. B311 (1993), Phys.Lett. B317 (1993) 573-580, Int. J. Mod. Phys. A9 (1994) 3307-3338, Phys.Lett. B324 (1994) 40-44, FF for sh-G, Bullough-Dodd, staircase, $\mathcal{M}_{3,5} + \Phi_{1,3}$... on FF perturbation theory see Nucl.Phys.B473:469-508,1996, Nucl.Phys.B516:675-703,1998., Nucl.Phys.B737:291-303,2006.

Operator identifications: ask Aldo

Finite volume FF: ask Giuseppe or Aldo

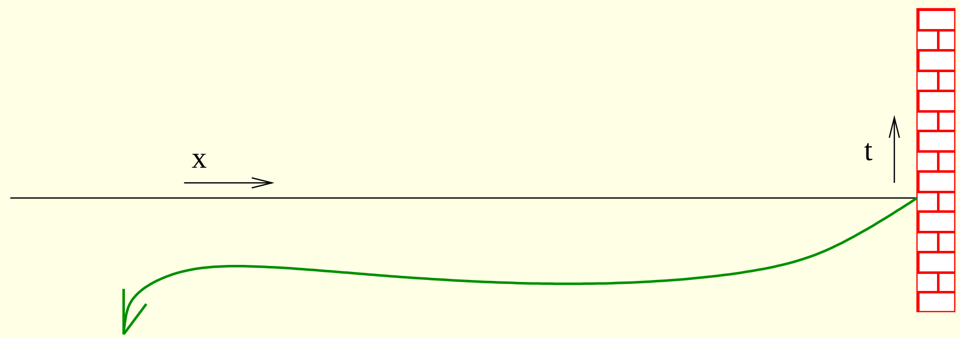
And many other works as well ...

Coming soon: a book about FFs, by O. Castro-Alvaredo, A. Fring, M. Karowski

Bulk and Boundary Form Factors in QFT (3)

Z. Bajnok, L. Palla, and G. Takács
Institute for Theoretical Physics, Eötvös University, Budapest

Massive integrable boundary QFT in 1+1 D (diagonal)



$$_B\langle 0|\varphi_B(t)|\theta_1,\theta_2,\dots,\theta_n\rangle_B^{in}$$

Plan of talk

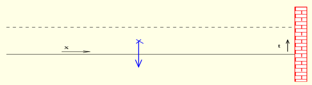
$$ {}_B\langle 0|\varphi_B(t)|\theta_1,\theta_2,\dots,\theta_n\rangle_B^{in} $$

Boundary form factor

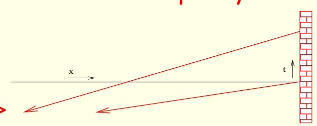
$\swarrow$ $\searrow$

$\varphi_B(t)$: local boundary operator

Bulk $< >$



Bulk form factors$+|B\rangle$

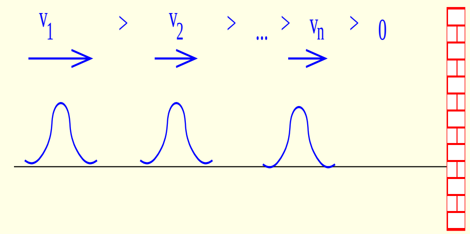


Boundary $< >$

Boundary form factors!

$|\theta_1,\theta_2,\dots,\theta_n\rangle_B^{in}$

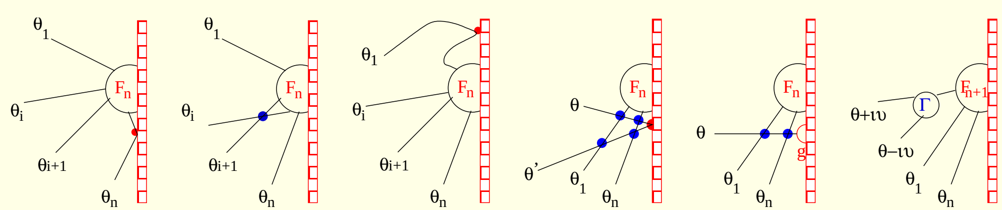
Asymptotic states $>$



Boundary reduction formula $>$

$\downarrow$

Consistency equations !

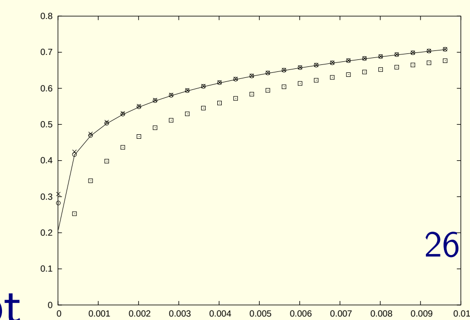


$\longrightarrow$ Axioms $>$

Minimal solution $>$

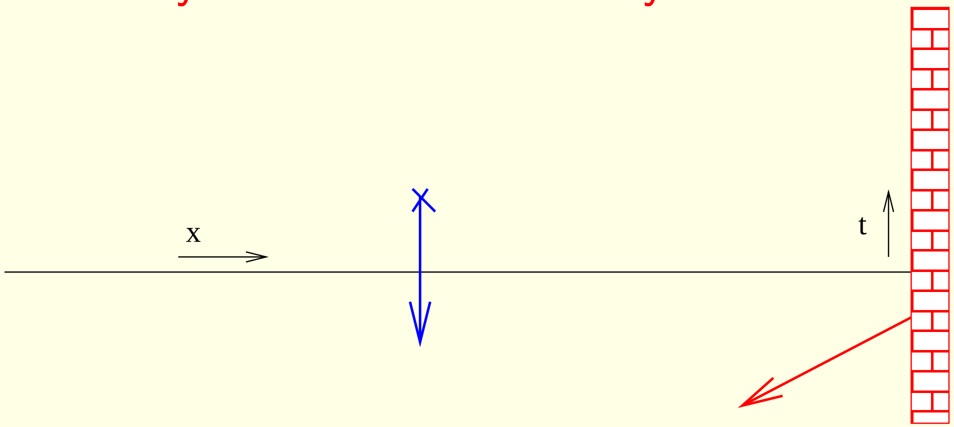
Perturbed boundary Lee-Yang

2pt



Integrable massive BQFT as perturbed BCFT

Boundary conformal field theory



Operator - state correspondence

$$\{\varphi_B(t)\} \leftrightarrow \mathcal{H}_B$$

Bulk operators $\Phi(x,t)$ $\varphi_B(t)$ Boundary operators Bulk - Boundary OPE

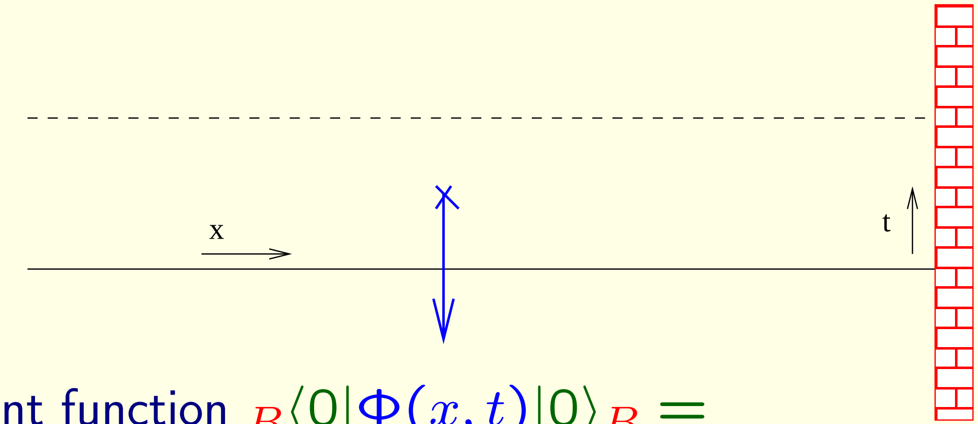
Integrable perturbations

$$S = S_{BCFT} - \lambda_{bulk} \int dt \int_{-\infty}^{0} dx\, \Phi(x,t) - \lambda_{bdry} \int dt\, \varphi_B(t)$$

Assumptions: Hilbert space does not change; spectrum, operator algebra smoothly changes $\longrightarrow$

Operator content
Bulk operators $\Phi(x,t)$
Boundary operators $\varphi_B(t)$

Correlation functions I: bulk operators



One point function ${}_B\langle 0|\Phi(x,t)|0\rangle_B =$

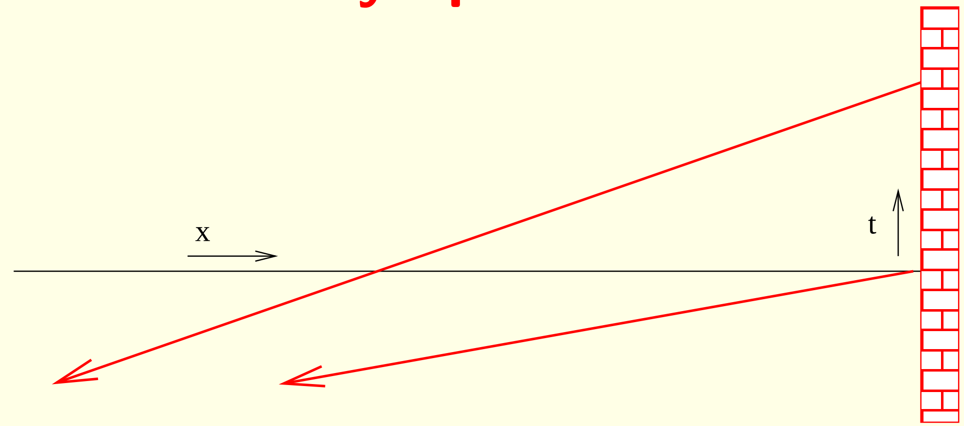
$$\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{d\theta_1\dots d\theta_n}{n!(2\pi)^n}\langle 0|\Phi(x,t)|\theta_1,\dots,\theta_n\rangle\langle\theta_1,\dots,\theta_n|B\rangle e^{-mx\sum_{i=1}^{n}\cosh\theta_i}$$

$\updownarrow$ Bulk form factor $\qquad$ $\updownarrow$ boundary state

Expansion for large x using bulk form factors and the boundary state

Boundary operators $x = 0$ $\longrightarrow$ new (boundary) technic is needed

Correlation functions II: boundary operators



Boundary two point function $\quad {}_B\langle 0|\varphi_B(t)\varphi_B(0)|0\rangle_B =$

$$\sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\theta_1\dots d\theta_n}{n!(2\pi)^n}\,{}_B\langle 0|\varphi_B(t)|\theta_1,\dots,\theta_n\rangle_B\,{}_B\langle\theta_1,\dots,\theta_n|\varphi_B(0)|0\rangle_B$$

time dependence:

$$\sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\theta_1\dots d\theta_n}{n!(2\pi)^n}|F_n^{\varphi_B}(\theta_1,\dots,\theta_n)|^2 e^{-mt\sum_{i=1}^{n}\cosh\theta_i}$$

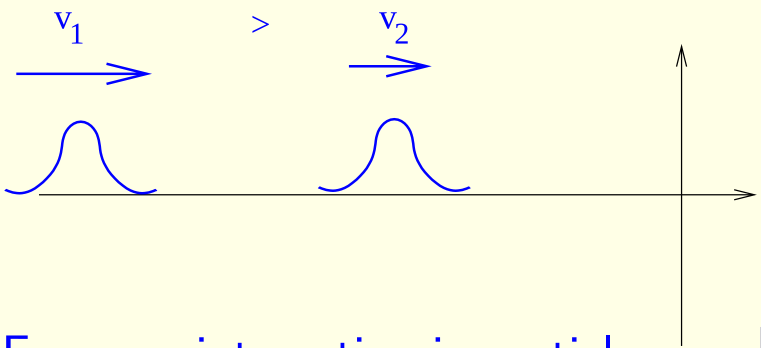
boundary form factor

$$\boxed{F_n^{\varphi_B}(\theta_1,\dots,\theta_n) = {}_B\langle 0|\varphi_B(t)|\theta_1,\dots,\theta_n\rangle_B^{in}}$$

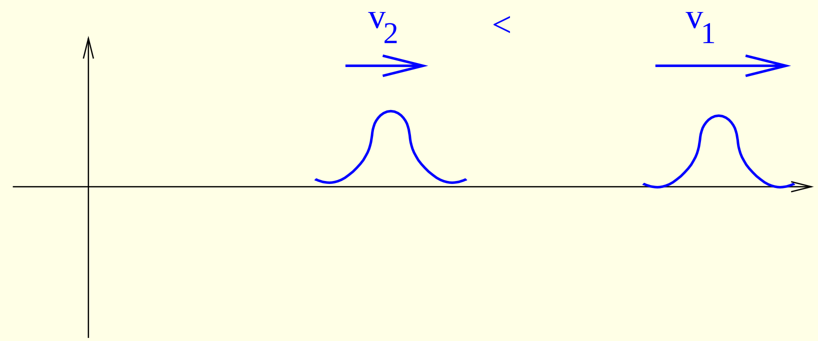
Boundary correlation functions, large t expansion$\longrightarrow$ boundary form factors $|\!>$

Asymptotic states: $|\theta_1, \theta_2, \ldots, \theta_n\rangle^{in}$ $\boxed{v_i = \sinh \theta_i}$

Bulk two particle in state: $t \to -\infty$



Bulk two particle out state: $t \to \infty$



Free, noninteracting in particles $\leftarrow$ **S-matrix** $\rightarrow$ **Free, noninteracting out particles**

$$|\theta_1, \theta_2\rangle^{in} = S(|\theta_1 - \theta_2|)|\theta_1, \theta_2\rangle^{out}$$

$$\boxed{\theta_1 > \theta_2}$$

$$|\theta_1, \theta_2\rangle = S(\theta_1 - \theta_2)|\theta_2, \theta_1\rangle$$

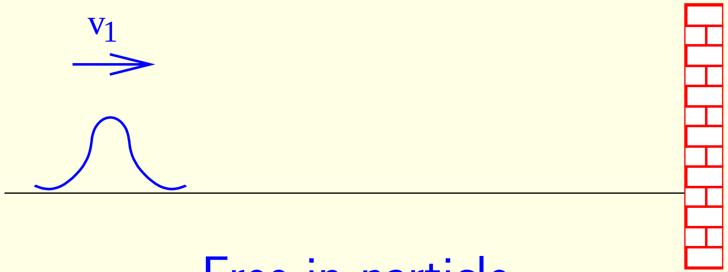
Unitarity $\quad S^*(\theta_1 - \theta_2) = S^{-1}(\theta_1 - \theta_2) = S(\theta_2 - \theta_1)$

Integrability $\longrightarrow$ factorizability:

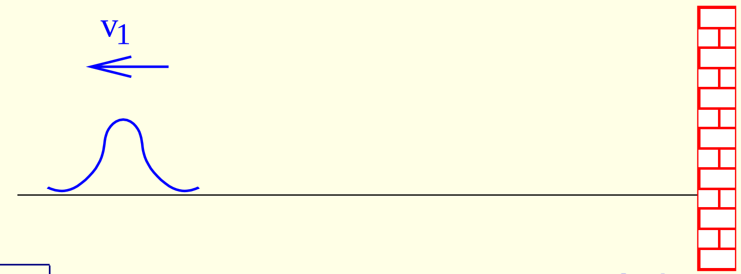
$$|\theta_1, \theta_2, \ldots, \theta_n\rangle = \prod_{i<j} S(\theta_i - \theta_j)|\theta_n, \theta_{n-1}, \ldots, \theta_1\rangle$$

Boundary asymptotic states: $|\theta_1, \theta_2, \ldots, \theta_n\rangle_B^{in}$

Boundary one particle in state: $t \to -\infty$



Boundary one pt out state: $t \to \infty$



Free in particle $\longleftarrow$ **R-matrix** $\longrightarrow$ **Free out particle**

$$|\theta_1\rangle_B^{in} = R(|\theta_1|)|\theta_1\rangle_B^{out}$$

$\theta_1 > 0$

$$|\theta_1\rangle_B = R(\theta_1)|-\theta_1\rangle_B$$

Unitarity: $R^*(\theta) = R^{-1}(\theta) = R(-\theta)$

Integrability $\longrightarrow$ factorizability:

$$|\theta_1, \theta_2, \ldots, \theta_n\rangle_B = \prod_{i<j} S(\theta_i - \theta_j) \prod_i R(\theta_i)|-\theta_1, -\theta_2, \ldots, -\theta_n\rangle_B$$

!>

Analytic structure: reduction formula

Unitarity $S^*(\theta_1-\theta_2) = S^{-1}(\theta_1-\theta_2) = S(\theta_2-\theta_1)$ not restrictive enough

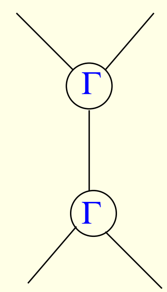
need: analytic structure of S matrix	$\leftrightarrow$	reduction formula	$\leftrightarrow$	known: analytic structure of correlators

$$< \theta_1, \theta_2, \ldots, \theta_n | \mathcal{O}(x,t) | in > = \delta(\theta_1 - \theta)\langle \theta_2, \ldots, \theta_n | \mathcal{O}(x,t) | in \setminus \theta >$$

$$iZ^{-1/2} \int_{-\infty}^{\infty} dt^{'}\, e^{-i\omega(\theta_1)t^{'}} \int_{-\infty}^{\infty} dx^{'}\, e^{ip(\theta_1)x^{'}} \{\partial_{t^{'}}^2 - \partial_{x^{'}}^2 + m^2\}$$

$$< \theta_2, \ldots, \theta_n | T(\Phi(x^{'}, t^{'})\mathcal{O}(x,t)) | in >$$

Crossing $S(\theta_1-\theta_2) = S(i\pi - \theta_1 + \theta_2)$



perturbed Lee Yang model $S_{LY}(\theta) = \frac{\sinh\theta + i\sin\frac{\pi}{3}}{\sinh\theta - i\sin\frac{\pi}{3}}$

Analytic structure: boundary reduction formula

Unitarity $\quad R^*(\theta_1) = R^{-1}(\theta_1) = R(-\theta_1)$ not restrictive enough

need: analytic structure of R matrix	$\leftrightarrow$	boundary reduction formula	$\leftrightarrow$	known: analytic structure of boundary correlators

$$
{}_B\langle \theta_1,\theta_2,\dots,\theta_n|\mathcal{O}(t)|in\rangle_B = \delta(\theta_1-\theta)\,{}_B\langle \theta_2,\dots,\theta_n|\mathcal{O}(t)|in\setminus\theta\rangle_B
$$

$$
i2Z^{-1/2}\int_{-\infty}^{\infty} dt'\, e^{-i\omega(\theta_1)t'}\int_{-\infty}^{0} dx'\cos(p(\theta_1)x')\{\partial_{t'}^2-\partial_{x'}^2+m^2+\delta(x')\partial_{x'}\}
$$

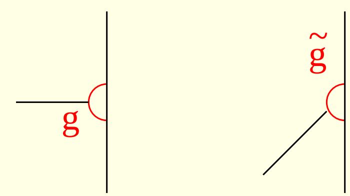
$$
{}_B\langle \theta_2,\dots,\theta_n|T(\Phi(x',t')\mathcal{O}(t))|in\rangle_B
$$

Boundary crossing $\quad R(\theta_1) = S(2\theta_1)R(i\pi-\theta_1)$

perturbed boundary Lee Yang

$$
S(\theta) = -\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = -\left[\frac{1}{3}\right] \quad , \quad [x] = (x)(1-x) \quad ; \quad (x) = \frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi x}{2}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi x}{2}\right)}
$$

$$
R(\theta) = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(-\frac{2}{3}\right)\left[\frac{b+1}{6}\right]\left[\frac{b-1}{6}\right]
$$



Analytical properties of the bulk form factors

Bulk form factor $\theta_1 > \theta_2 > \ldots > \theta_n$

$$F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n) = \langle 0 | \mathcal{O}(0,0) | \theta_1, \ldots, \theta_n \rangle$$

$$F_n^{\mathcal{O}}(\theta_n, \ldots, \theta_1) = \langle 0 | \mathcal{O}(0,0) | \theta_n, \ldots, \theta_1 \rangle$$

Example $F_2^{\mathcal{O}}(\theta_1, \theta_2) = S(\theta_1 - \theta_2) F_2^{\mathcal{O}}(\theta_2, \theta_1)$

$$F_{mn}^{\mathcal{O}}(\theta_1', \ldots, \theta_m' | \theta_1, \ldots, \theta_n) = \langle \theta_1', \ldots, \theta_m' | \mathcal{O}(0,0) | \theta_1, \ldots, \theta_n \rangle$$

Permutation $F_{m2}^{\mathcal{O}}(\theta_1', \ldots, \theta_m' | \theta_1, \theta_2) = S(\theta_1 - \theta_2) F_{m2}^{\mathcal{O}}(\theta_1', \ldots, \theta_m' | \theta_2, \theta_1)$

Crossing from reduction formula

$$F_{1n}^{\mathcal{O}}(\theta | \theta_1, \ldots, \theta_n) = F_{n+1}^{\mathcal{O}}(\theta_1, \ldots, \theta_n, \theta - i\pi) + \delta(\theta - \theta_n) F_{n-1}^{\mathcal{O}}(\theta_1, \ldots, \theta_{n-1})$$

$$F_{1n}^{\mathcal{O}}(\theta | \theta_n, \ldots, \theta_1) = F_{n+1}^{\mathcal{O}}(\theta + i\pi, \theta_n, \ldots, \theta_1) + \delta(\theta - \theta_n) F_{n-1}^{\mathcal{O}}(\theta_{n-1}, \ldots, \theta_1)$$

!>

Analytical properties of the boundary form factors

Boundary form factor

$$\theta_1 > \theta_2 > \ldots > \theta_n > 0$$

$$F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n) =_B \langle 0|\mathcal{O}(0)|\theta_1,\ldots,\theta_n\rangle_B$$

$$F_n^{\mathcal{O}}(-\theta_1,\ldots,-\theta_n) =_B \langle 0|\mathcal{O}(0)|-\theta_1,\ldots,-\theta_n\rangle_B$$

Example

$$F_1^{\mathcal{O}}(\theta_1) = R(\theta_1)F_1^{\mathcal{O}}(-\theta_1)$$

$$F_{mn}^{\mathcal{O}}(\theta_1',\ldots,\theta_m'|\theta_1,\ldots,\theta_n) = {}_B\langle\theta_1',\ldots,\theta_m'|\mathcal{O}(0)|\theta_1,\ldots,\theta_n\rangle_B$$

Reflection

$$F_{m1}^{\mathcal{O}}(\theta_1',\ldots,\theta_m'|\theta_1) = R(\theta_1)F_{m1}^{\mathcal{O}}(\theta_1',\ldots,\theta_m'|-\theta_1) + \frac{g}{2}\delta(\theta-\frac{i\pi}{2})F_{m0}^{\mathcal{O}}(\theta_1',\ldots,\theta_m')$$

$$F_{1n}^{\mathcal{O}}(\theta_1'|\theta_1,\ldots,\theta_n) = R(-\theta_1')F_{1n}^{\mathcal{O}}(-\theta_1'|\theta_1,\ldots,\theta_n) + \frac{g}{2}\delta(\theta'+\frac{i\pi}{2})F_{0n}^{\mathcal{O}}(\theta_1,\ldots,\theta_n)$$

Crossing from reduction formula

$$F_{1n}^{\mathcal{O}}(\theta'|\theta_1,\ldots,\theta_n) = F_{n+1}^{\mathcal{O}}(\theta'+i\pi,\theta_1,\ldots,\theta_n) + \delta(\theta-\theta_1)F_{n-1}^{\mathcal{O}}(\theta_2,\ldots,\theta_n)$$

$$F_{1n}^{\mathcal{O}}(-\theta'|-\theta_1,\ldots,-\theta_n) = F_{n+1}^{\mathcal{O}}(-\theta'+i\pi,-\theta_1,\ldots,-\theta_n) + \delta(\theta'-\theta_1)F_{n-1}^{\mathcal{O}}(-\theta_2,\ldots,-\theta_n)$$

!>

Comparison to other approaches

Consistency eqs.: derived

$$F^{\mathcal{O}}_{m1}(\theta'_1,\ldots,\theta'_m|\theta_1) = R(\theta_1)F^{\mathcal{O}}_{m1}(\theta'_1,\ldots,\theta'_m|-\theta_1) + \frac{g}{2}\delta(\theta-\frac{i\pi}{2})F^{\mathcal{O}}_{m0}(\theta'_1,\ldots,\theta'_m)$$

$$F^{\mathcal{O}}_{1n}(\theta'_1|\theta_1,\ldots,\theta_n) = R(-\theta'_1)F^{\mathcal{O}}_{1n}(-\theta'_1|\theta_1,\ldots,\theta_n) + \frac{g}{2}\delta(\theta'+\frac{i\pi}{2})F^{\mathcal{O}}_{0n}(\theta_1,\ldots,\theta_n)$$

$$F^{\mathcal{O}}_{1n}(\theta'|\theta_1,\ldots,\theta_n) = F^{\mathcal{O}}_{n+1}(\theta'+i\pi,\theta_1,\ldots,\theta_n) + \delta(\theta-\theta_1)F^{\mathcal{O}}_{n-1}(\theta_2,\ldots,\theta_n)$$

$$F^{\mathcal{O}}_{1n}(-\theta'|-\theta_1,\ldots,-\theta_n) = F^{\mathcal{O}}_{n+1}(-\theta'+i\pi,-\theta_1,\ldots,-\theta_n) + \delta(\theta'-\theta_1)F^{\mathcal{O}}_{n-1}(-\theta_2,\ldots,-\theta_n)$$

Similar equations in spin models BUT WITHOUT $\frac{g}{2}\delta(\theta-\frac{i\pi}{2})\ldots$:

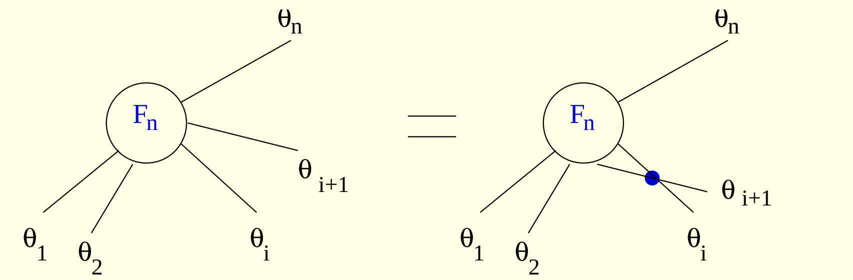
XXZ,XYZ	M. Jimbo, R. Kedem, H. Konno T. Miwa, R. Weston	Nucl.Phys. B448 (1995) 429-456
Higher rank XXZ	Y.-H. Quano	Int.J.Mod.Phys. A15 (2000) 3699-3716
Belavin's Z_n-symmetric	Y.-H. Quano	J.Phys. A33 (2000) 8275
$A^{(1)}_{n-1}$ face model	Y.-H. Quano	J.Phys. A34 (2001) 8445-8464

Equations are DERIVED $\longrightarrow$ valid for any QFT for form factors of local operators!

Bulk form factor axioms

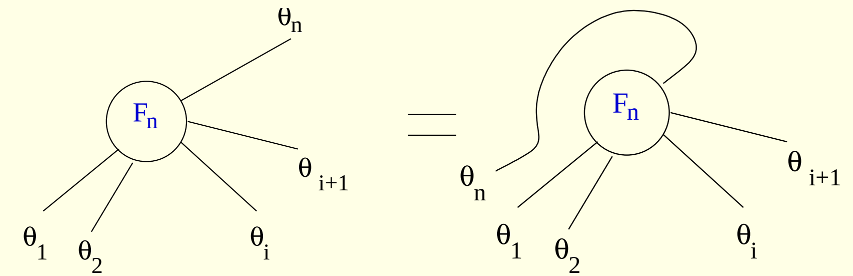
Permutation

$$F_n^{\mathcal{O}}(\theta_1,\dots\theta_i,\theta_{i+1},\dots,\theta_n) =$$
$$S(\theta_i-\theta_{i+1})F_n^{\mathcal{O}}(\theta_1,\dots\theta_{i+1},\theta_i,\dots,\theta_n)$$



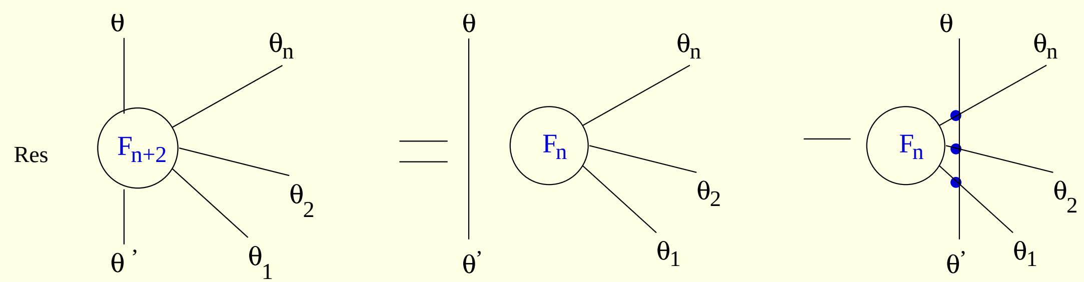
Periodicity

$$F_n^{\mathcal{O}}(\theta_1,\dots,\theta_{n-1},\theta_n) =$$
$$F_n^{\mathcal{O}}(\theta_n+2\pi i,\theta_1,\dots,\theta_{n-1})$$



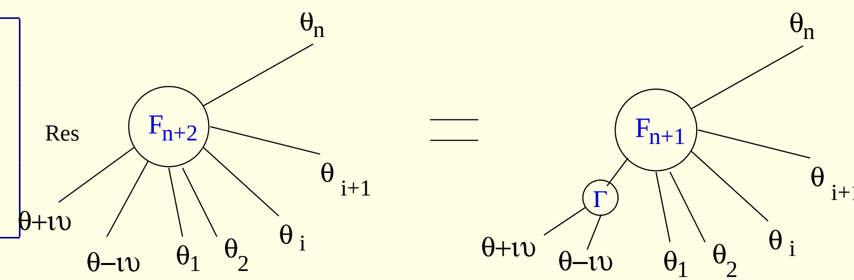
Kinematical singularities

$$-i\mathrm{res}_{\theta=\theta'}F_{n+2}^{\mathcal{O}}(\theta+i\pi,\theta',\theta_1,\dots,\theta_n) =$$
$$= \left(1-\prod_{i=1}^n S(\theta-\theta_i)\right)F_n^{\mathcal{O}}(\theta_1,\dots,\theta_n)$$



Dynamical singularities

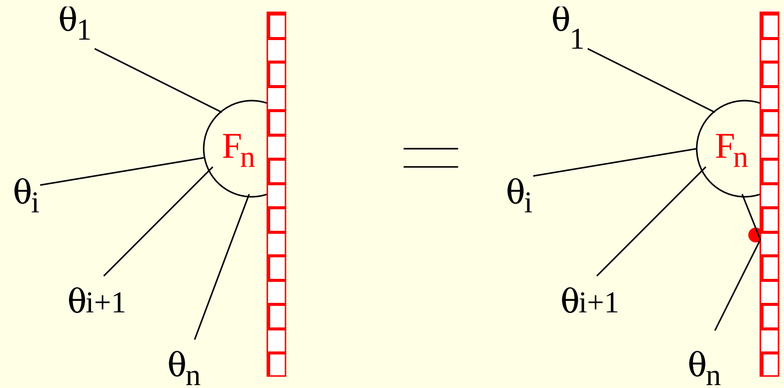
$$-i\mathrm{res}_{\theta=\theta'}F_{n+2}^{\mathcal{O}}(\theta+iu,\theta'-iu,\theta_1,\dots,\theta_n) =$$
$$\Gamma F_{n+1}^{\mathcal{O}}(\theta,\theta_1,\dots,\theta_n)$$



Boundary form factor axioms I

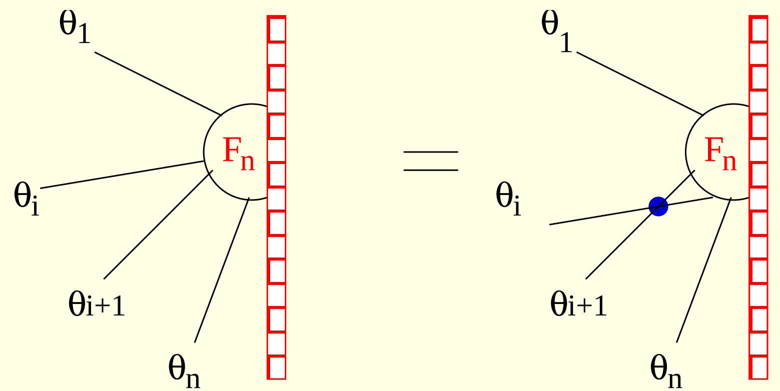
Reflection

$$F_n^{\mathcal{O}}(\theta_1, \dots, \theta_{n-1}, \theta_n) = R(\theta_n) F_n^{\mathcal{O}}(\theta_1, \dots, \theta_{n-1}, -\theta_n)$$



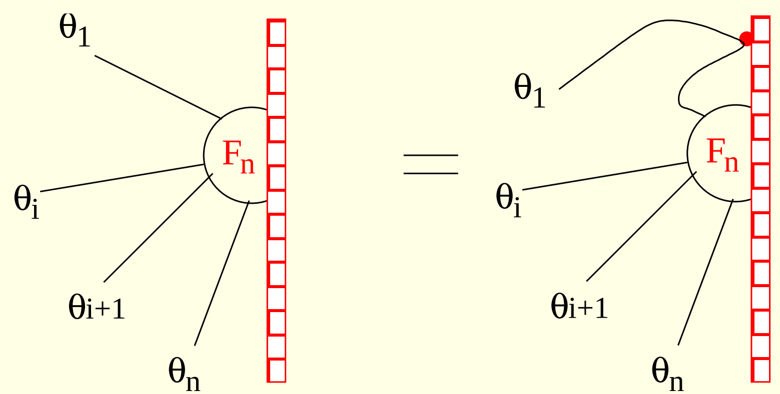
Permutation

$$F_n^{\mathcal{O}}(\theta_1, \dots \theta_i, \theta_{i+1}, \dots, \theta_n) = S(\theta_i - \theta_{i+1}) F_n^{\mathcal{O}}(\theta_1, \dots \theta_{i+1}, \theta_i, \dots, \theta_n)$$



Boundary periodicity

$$F_n^{\mathcal{O}}(\theta_1, \theta_2, \dots, \theta_n) = R(i\pi - \theta_1) F_n^{\mathcal{O}}(2i\pi - \theta_1, \theta_2, \dots, \theta_n)$$

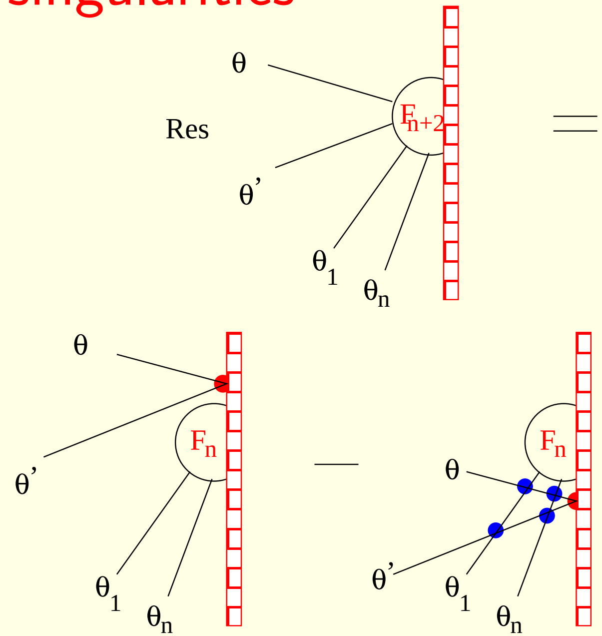


Boundary form factor axioms II: kinematical singularities

Bulk kinematical singularities

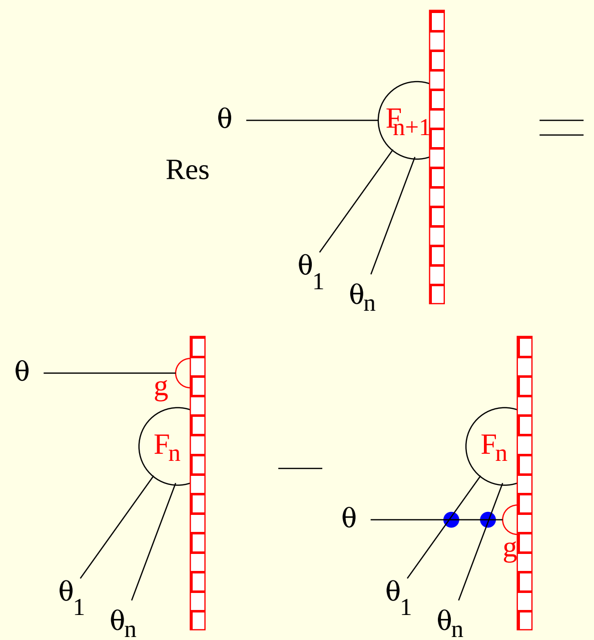
$$-i\,\mathrm{res}_{\theta=\theta'} F^{\mathcal{O}}_{n+2}(\theta+i\pi,\theta',\theta_1,\dots,\theta_n) =$$
$$\left(1-\prod_{i=1}^{n} S(\theta-\theta_i)S(\theta+\theta_i)\right) F^{\mathcal{O}}_{n}(\theta_1,\dots,\theta_n)$$

$$-i\,\mathrm{res}_{\theta=\theta'} F^{\mathcal{O}}_{n+2}(-\theta+i\pi,\theta',\theta_1,\dots,\theta_n) =$$
$$\left(R(\theta)-\prod_{i=1}^{n} S(\theta-\theta_i)R(\theta)S(\theta+\theta_i)\right) F^{\mathcal{O}}_{n}(\theta_1,\dots,\theta_n)$$



Boundary kinematical singularities

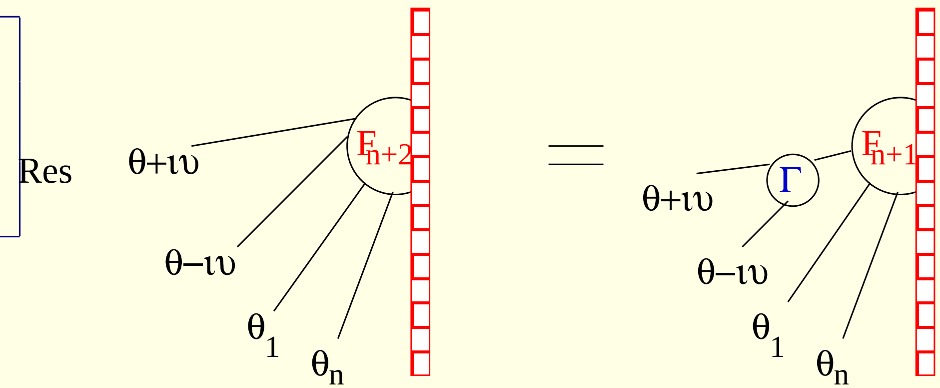
$$-i\,\mathrm{res}_{\theta=\frac{i\pi}{2}} F^{\mathcal{O}}_{n+1}(\theta,\theta_1,\dots,\theta_n) =$$
$$\frac{g}{2}\left(1-\prod_{i=1}^{n} S\left(\frac{i\pi}{2}-\theta_i\right)\right) F^{\mathcal{O}}_{n}(\theta_1,\dots,\theta_n)$$



Boundary form factor axioms III: dynamical singularities

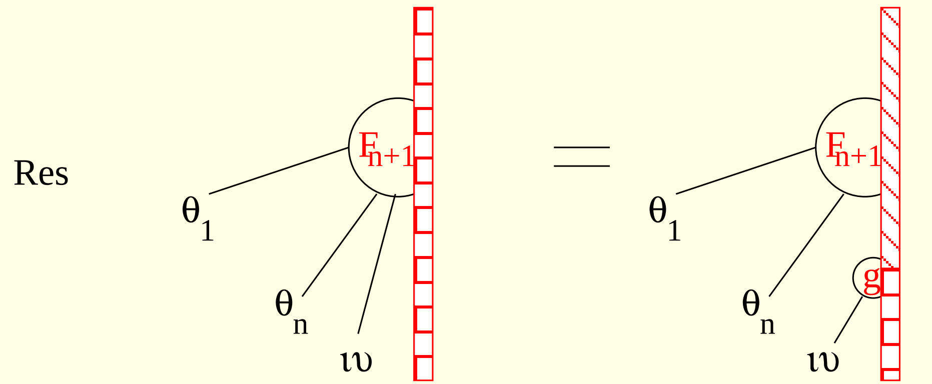
Bulk dynamical singularities

$$-i\mathrm{res}_{\theta=\theta'} F^{\mathcal{O}}_{n+2}(\theta+iu,\theta'-iu,\theta_1,\dots,\theta_n) = \Gamma F^{\mathcal{O}}_{n+1}(\theta,\theta_1,\dots,\theta_n)$$



Boundary dynamical singularities

$$-i\mathrm{res}_{\theta=iu} F^{\mathcal{O}}_{n+1}(\theta_1,\dots,\theta_n,\theta) = \tilde{g}\tilde{F}^{\mathcal{O}}(\theta_1,\dots,\theta_n)$$



!>

Solving the bulk form factor equations

Bulk theory with $S(\theta)$: FF axioms + minimality $\rightarrow$ Solution of the FF equations

Step 1. Solve first the two particle case $F_2(\theta_1 - \theta_2) = f(\theta)$

$$f(\theta) = S(\theta) f(-\theta) \quad ; \quad f(i\pi + \theta) = f(i\pi - \theta)$$

minimality: only dynamical poles, one zero at $\theta = 0$, minimal growth at $\theta \rightarrow \infty$

Step 2. Take the Ansatz to satisfy the permutation and the periodicity axioms

$$F_n(\theta_1, \ldots, \theta_n) = H_n Q_n(x_1, \ldots, x_n) \prod_{i<j} f(\theta_i - \theta_j) \frac{1}{x_i + x_j} \quad ; \quad x = e^{\theta}$$

$Q_n(x_1, \ldots, x_n)$: completely symmetric polynomial

Step 3. Derive recursion relations from the singularity axioms

kinematical $Q_{n+2}(-x, x, x_1, \ldots, x_n) = P_n(x|x_1, \ldots, x_n) Q_n(x_1, \ldots, x_n)$

dynamical $Q_{n+1}(e^{iu}x, e^{-iu}x, x_1, \ldots, x_n) = R_n(x|x_1, \ldots, x_n) Q_n(x, x_1, \ldots, x_n)$

Step 4. Solve recursion, classify the solutions: operator content

Perturbed Lee-Yang model

Bulk theory with $S(\theta)=\frac{\sinh\theta+i\sin\frac{\pi}{3}}{\sinh\theta-i\sin\frac{\pi}{3}}$

Step 1. Solution of the two particle case $F_2(\theta_1-\theta_2)=f(\theta)$

$$f(\theta)=\frac{\cosh\theta-1}{\cosh\theta+\frac{1}{2}}v(i\pi-\theta)v(-i\pi+\theta)\quad;\quad v(\theta)=\exp\left\{2\int_0^\infty\frac{dx}{x}e^{i\theta x/\pi}\frac{\sinh\frac{x}{2}\sinh\frac{x}{3}\sinh\frac{x}{6}}{\sinh^2 x}\right\}$$

minimality: dynamical pole at $\theta=\frac{2i\pi}{3}$, zero at $\theta=0$, growth: $f(\theta)\to 1$ for $\theta\to\infty$

Step 2. The Ansatz satisfying the permutation and periodicity axioms

$$F_n(\theta_1,\dots,\theta_n)=H_nQ_n(x_1,\dots,x_n)\prod_{i<j}\frac{f(\theta_i-\theta_j)}{x_i+x_j}\quad;\qquad H_n=<\mathcal{O}>\left(\frac{3^{1/4}}{2^{1/2}v(0)}\right)^n$$

$Q_n(x_1,\dots,x_n)$: completely symmetric polynomial

Step 3. Recursion relations from the singularity axioms

kinematical $Q_{n+2}(-x,x,x_1,\dots,x_n)=P_n(x|x_1,\dots,x_n)Q_n(x_1,\dots,x_n)$

$$P_n(x|x_1,\dots,x_n)=\frac{(-1)^nx^2}{2(x_+-x_-)}\left(\prod_{i=1}^n(x_i+x_+)(x_i-x_-)-\prod_{i=1}^n(x_i-x_+)(x_i+x_-)\right)\quad x_\pm=e^{\mp i\frac{\pi}{3}}x$$

dynamical $Q_{n+1}(x_+,x_-,x_1,\dots,x_n)=x\prod_{i=1}^n(x+x_i)Q_n(x,x_1,\dots,x_n)$

Perturbed bulk Lee-Yang: minimal form factor solution

Step 4. Solve recursion and classify them: 2pt function and operator content

$$Q_1 = 1; \quad Q_2 = \sigma_1; \quad Q_3 = \sigma_1 \sigma_2 \ldots \quad , \qquad \prod_{i=1}^n (x + x_i) = \sum_{k=0}^n x^k \sigma_{n-k}$$

The solution with minimal degree is unique. Which operator is it? Compute 2pt:

$$\langle 0 | \mathcal{O}(x,0) \mathcal{O}(0,0) | 0 \rangle = \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} \frac{d\theta_1 \ldots d\theta_n}{n!(2\pi)^n} |F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n)|^2 e^{-mx \sum_{i=1}^n \cosh \theta_i}$$

Large distance expansion $\leftrightarrow$ Compare to short distance (UV) expansion

$$\mathcal{A} = \mathcal{A}_{(2,5)} + g \int d^2 x \Phi_{(\frac{1}{5},\frac{1}{5})}$$

$\Phi_{(\frac{1}{5},\frac{1}{5})} = \Phi$ field with smallest scaling dimension

$$\langle 0 | \Phi(x,0) \Phi(0,0) | 0 \rangle = x^{\frac{4}{5}} + x^{\frac{2}{5}} C_{\Phi\Phi}^{\Phi} \langle \Phi \rangle + \ldots$$

Perturbed bulk Lee-Yang two point function

Conformal limit compared to form factor expansion

$$\langle 0|\Phi(x,0)\Phi(0,0)|0\rangle$$

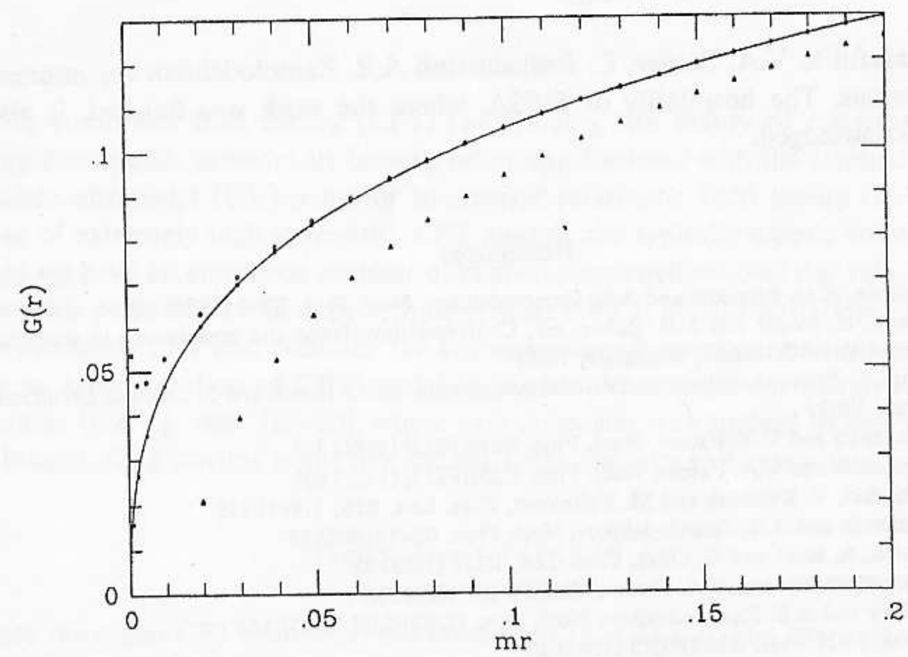


Fig. 3. Convergence of the large-distance expansion for small mr. Empty triangles: zero- and one-particle contributions. Empty circles: the same plus two-particle term. Full circles: up to three-particle state contributions. Full curve: the short-distance data.

Conformal limit: $x^{\frac{4}{5}} + x^{\frac{2}{5}} C^{\Phi}_{\Phi\Phi}\langle\Phi\rangle$

FF expansion 0+1+2+3pt: $|F_0^{\Phi}|^2 +$

$$-\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}|F_1^{\Phi}|^2 e^{-mx\cosh\theta}$$

$$+\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2}{2(2\pi)^2}|F_2^{\Phi}(\theta_1,\theta_2)|^2 \cdot e^{-mx(\cosh\theta_1+\cosh\theta_2)}$$

$$+\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2 d\theta_3}{6(2\pi)^3}|F_3^{\Phi}(\theta_1,\theta_2,\theta_3)|^2 \ldots$$

Al. B. Zamolodchikov NPB348 (1991) 619.

Solving the boundary form factor equations

Boundary theory with $S(\theta)$,$R(\theta)$: BFF axioms + minimality $\rightarrow$ Solution of the BFF eqs

Step 1. Solve first the one particle case $F_1(\theta) = r(\theta)$

$$r(\theta) = R(\theta) r(-\theta) \quad ; \quad r(i\pi - \theta) = R(\theta) r(i\pi + \theta)$$

minimality: only dynamical poles, one zero at $\theta = 0$, minimal growth at $\theta \rightarrow \infty$

Step 2. Take the Ansatz to satisfy the reflection, permutation and the bperiodicity axioms

$$F_n(\theta_1, \dots, \theta_n) = H_n Q_n(y_1, \dots, y_n) \prod_i r(\theta_i) \frac{1}{y_i} \prod_{i<j} f(\theta_i + \theta_j) f(\theta_i - \theta_j) \frac{1}{y_i + y_j}$$

$Q_n(y_1, \dots, y_n)$: completely symmetric polynomial of $y = e^{\theta} + e^{-\theta}$

Step 3. Derive recursion relations from the singularity axioms

bulk kinematical $Q_{n+2}(-y, y, y_1, \dots, y_n) = P_n(y|y_1, \dots, y_n) Q_n(y_1, \dots, y_n)$

boundary kinematical $Q_{n+1}(0, y_1, \dots, y_n) = \tilde{P}_n(y_1, \dots, y_n) Q_n(y_1, \dots, y_n)$

dynamical $Q_{n+1}(y_+, y_-, y_1, \dots, y_n) = R_n(y|y_1, \dots, y_n) Q_n(y, y_1, \dots, y_n)$

Step 4. Solve recursion, classify the solutions: operator content

Perturbed boundary Lee-Yang model

Boundary theory with $S(\theta) = -\left[\frac{1}{3}\right]$, $R(\theta) = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(-\frac{2}{3}\right)\left[\frac{b+1}{6}\right]\left[\frac{b-1}{6}\right]$

Step 1. Solution of the one particle case $F_1(\theta) = r(\theta)$

$$r(\theta) = \frac{i \sinh(\theta)}{(\sinh\theta + i\sin\frac{\pi}{6}(b+1))(\sinh\theta - i\sin\frac{\pi}{6}(b+1))} u(\theta)$$

$$u(\theta) = \exp\left\{ \int_0^\infty \frac{dt}{t} \left[\frac{1}{\sinh\frac{t}{2}} - 2\cosh\frac{t}{2}\cos\left(\frac{i\pi}{2} - \theta\right)\frac{t}{\pi}\frac{\sinh\frac{5t}{6} + \sinh\frac{t}{2} - \sinh\frac{t}{3}}{\sinh^2 t} \right] \right\}$$

minimality: dynamical poles at $\theta = \frac{i\pi(b\pm1)}{6}$, zero at $\theta = 0$, growth: $r(\theta) \to 1$

Step 2. The Ansatz satisfying the permutation and periodicity axioms

$$F_n(\theta_1,\dots,\theta_n) = H_n Q_n(y_1,\dots,y_n) \prod_i \frac{r(\theta_i)}{y_i} \prod_{i<j} \frac{f(\theta_i-\theta_j)f(\theta_i+\theta_j)}{y_i+y_j} \; ; \quad H_n = <\mathcal{O}> \left(\frac{3^{1/4}}{2^{1/2}v(0)}\right)^n$$

Step 3. Recursion relations from the singularity axioms

bulk kinematical $Q_{n+2} = P_n Q_n$ with $\beta_k(b) = 2\cos(\frac{\pi}{6}(b+k))$ and $y_\pm = 2\cosh(\theta \pm \frac{i\pi}{3})$

$$P_n = \frac{(y^2-\beta_1^2(b))(y^2-\beta_{-1}^2(b))}{2(y_+-y_-)} \left(\prod_{i=1}^n (y_i+y_+)(y_i-y_-) - \prod_{i=1}^n (y_i-y_+)(y_i+y_-) \right)$$

dynamical $Q_{n+1}(y_+, y_-, y_1, \dots, y_n) = (y^2 + \beta_{-3}^2(b))\, y \prod_{i=1}^n (y+y_i)\, Q_n(y, y_1, \dots, y_n)$

Perturbed boundary Lee-Yang: minimal form factor solution

Step 4. Solve recursion and classify them: 2pt function and operator content

$$Q_1 = \sigma_1; \quad Q_2 = \sigma_1(\sigma_2 + 3 - \beta^2_{-3}(b));$$

$$Q_3 = \sigma_1(\sigma_1(\sigma_2 + \beta_1^2(b))(\sigma_2 + \beta^2_{-1}(b)) - (\sigma_2 + 3)(\sigma_1\sigma_2 - \sigma_3))\dots$$

The solution with minimal degree is unique. Which operator is it? Compute 2pt:

$$\langle 0|\mathcal{O}(t)\mathcal{O}(0)|0\rangle = \sum_{n=0}^{\infty}(-)^n \int_0^{\infty} \frac{d\theta_1 \dots d\theta_n}{n!(2\pi)^n} |F_n^{\mathcal{O}}(\theta_1,\dots,\theta_n)|^2 e^{-mt\sum_{i=1}^{n}\cosh\theta_i}$$

Large distance expansion. $\leftrightarrow$ Compare to short distance (UV) expansion

$$\mathcal{A} = \mathcal{A}_{BCFT(2,5)} + g\int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx\, \Phi_{(\frac{1}{5},\frac{1}{5})}(x,t) + h\int_{-\infty}^{\infty} dt\, \varphi_{\frac{1}{5}}(t)$$

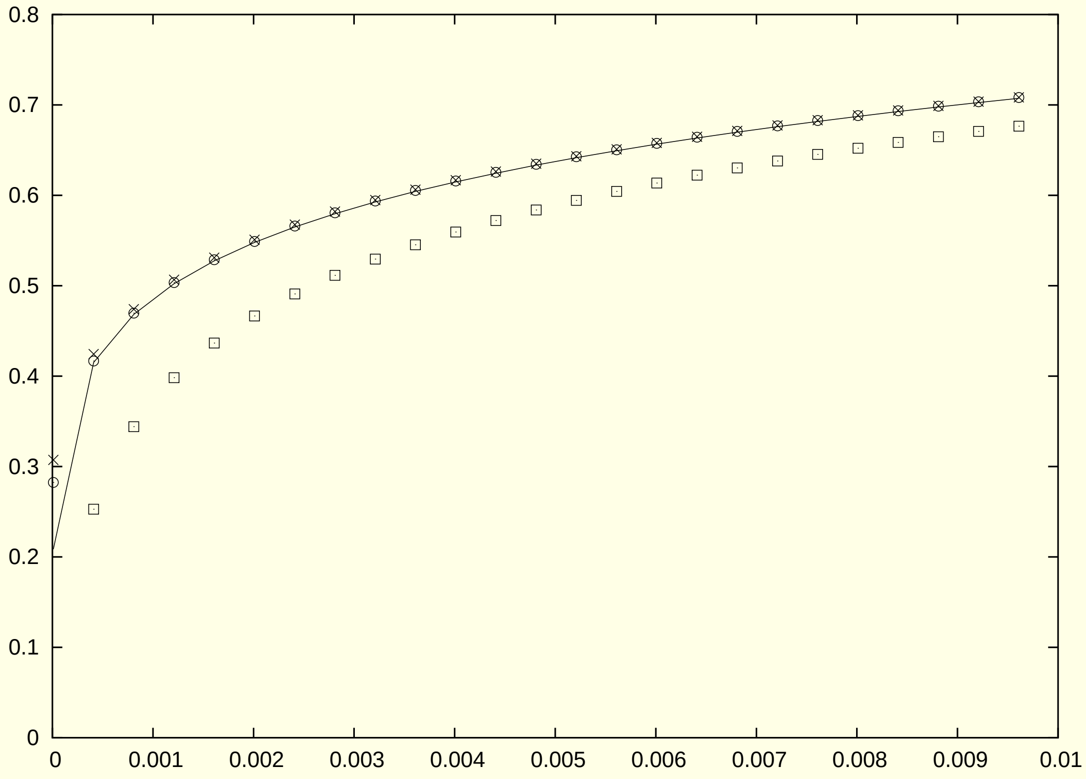
$\varphi_{\frac{1}{5}} = \varphi$ boundary field with smallest scaling dimension

$$\langle 0|\varphi(t)\varphi(0)|0\rangle = -t^{\frac{2}{5}} + t^{\frac{1}{5}} C_{\varphi\varphi}^{\varphi}\langle\varphi\rangle + \dots$$

Perturbed boundar Lee-Yang two point function

Conformal limit compared to form factor expansion

$$\langle 0|\varphi(t)\varphi(0)|0\rangle$$



Conformal limit: $-t^{\frac{2}{5}}+t^{\frac{1}{5}}C_{\varphi\varphi}^{\varphi}\langle\varphi\rangle$

BFF expansion 0+1+2+3pt: $|F_0^{\varphi}|^2+$

$$-\int_0^\infty \frac{d\theta}{2\pi}|F_1^{\varphi}|^2 e^{-mt\cosh\theta}$$

$$+\int_0^\infty \frac{d\theta_1 d\theta_2}{2(2\pi)^2}|F_2^{\varphi}(\theta_1,\theta_2)|^2 \cdot e^{-mt(\cosh\theta_1+\cosh\theta_2)}$$

$$+\int_0^\infty \frac{d\theta_1 d\theta_2 d\theta_3}{6(2\pi)^3}|F_3^{\varphi}(\theta_1,\theta_2,\theta_3)|^2\dots$$

$>$