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PION LASERS

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Analytic solution of a pion-laser model

full n-particle symmetrization,

overlapping wave-packets

Motivation: Thousands of bosons (pions)

to be produced at RHIC accelerator (BNL, 1999)

Open problem:

Relation to Atom Lasers,

Relativistic generalization

[1] D. Kleppner, *"A Beginner's Guide to the Atom Laser"* Physics Today, Aug. 1997 p 11

[2] J. Zimányi and T. Csörgő: *"Multi- Boson Correlations Using Wave-Packets"* http://xxx.lanl.gov/hep-ph/9705432

[3] T. Csörgő and J. Zimányi: *"Analytic Solution of the Pion-Laser Model"* http://xxx.lanl.gov/hep-ph/9705433, Phys. Rev. Lett (1997) to be published

Lasers, Atomic Lasers and Pion Lasers:

Laser = Light Amplification by Stimulated Emission of Radiation

(originally)

x-ray lasers, infra-red lasers and masers = microwave lasers

Atom lasers = Coherent State Atom Amplification by Stimulated Scattering of Atoms "CSAASSA" (Ref.1).

Coherent states or Glauber states: follow classical trajectory with minimalizing the uncertainty of both position and momentum.

They can be defined e.g. as

$$a|\beta\rangle = \beta|\beta\rangle \tag{1}$$

The vacuum is in the coherent state $|0\rangle$ as

$$a|0\rangle = 0 \tag{2}$$

For a single mode harmonic oscillator, the coherent state is

$$|\beta\rangle = \mathrm{e}^{-\frac{|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n}} |n\rangle \tag{3}$$

Lasers and Coherent States:

In a one-mode coherent state,

$$p_n = \langle \beta | \hat{N} | \beta \rangle = \frac{|\beta|^{2n}}{n!} \exp(-|\beta|^2), \tag{4}$$

the mean multiplicity

$$\langle n \rangle = \sum_{n=0}^{\infty} n p_n = |\beta|^2, \tag{5}$$

the multiplicity distribution is Poissonian,

$$p_n = \frac{\langle n \rangle}{n!} \exp(-\langle n \rangle). \tag{6}$$

which yields

$$\langle n^k \rangle = \langle n \rangle^k. \tag{7}$$

Well satisfied in optical and atomic lasers:

$$\langle n^2 \rangle = (1.0 \pm 0.2) \langle n \rangle^2 \tag{8}$$

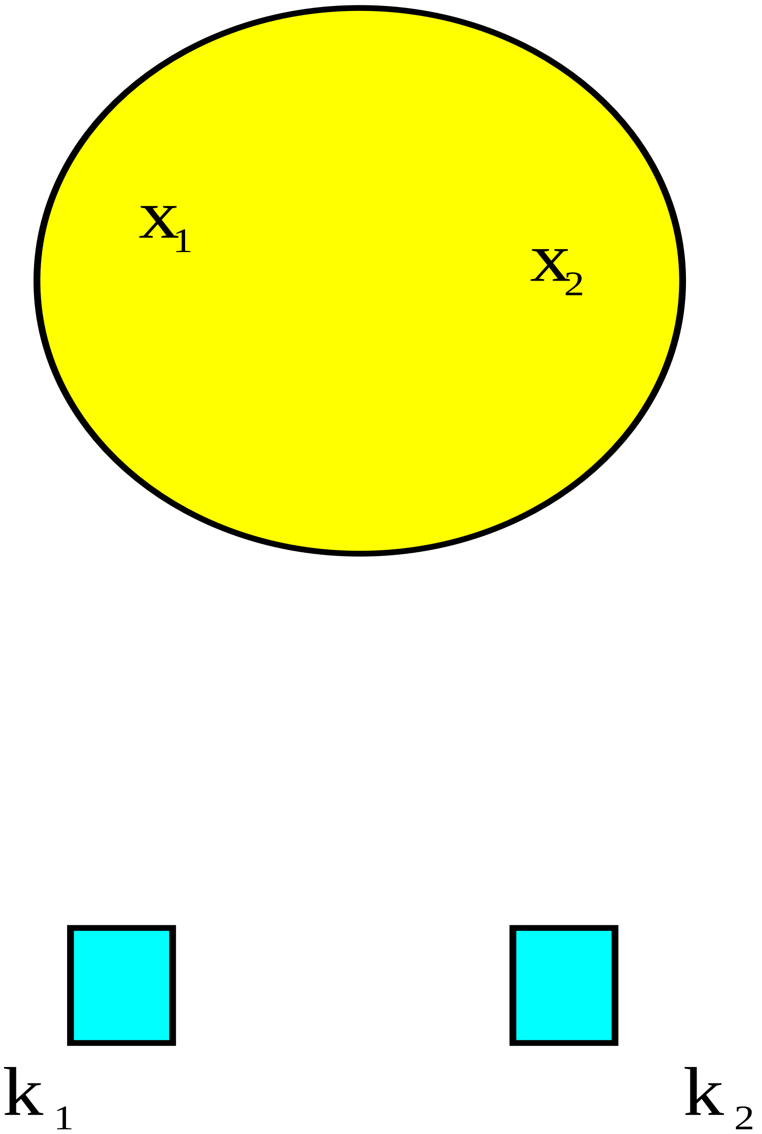
similar result holds for third order coherence, in contrast to thermal source of radiation, where

$$\langle n^k \rangle = k! \, \langle n \rangle^k. \tag{9}$$

Cornell, Wieman et al, Phys. Rev. Lett. **79** (1997) 337

Kleppner defined lasers in general with the help of eq. (8).

Correlation Basics in High Energy Physics:



In $A + B \to \pi + \pi + X$ reactions, the momentum of the two pions are detected but we do not know which pion is emitted from which part of the source.

Illustration I.

Illustration II.

Correlation Basics in High Energy Physics II:

Simplest quantum mechanical description:

$$A_{1,2} = \frac{1}{\sqrt{2}}\left[\exp(ik_1\cdot x_1 + ik_2\cdot x_2) + \exp(ik_1\cdot x_2 + ik_2\cdot x_1)\right] \quad (10)$$

$$N(k_1,k_2) = \int d^4x_1 \int d^4x_2\, \rho(x_1)\rho(x_2)\, |A_{1,2}|^2 \quad (11)$$

$$C(k_1,k_2) = \frac{N_2(k_1,k_2)}{N_1(k_1)\,N_1(k_2)} = 1 + |\tilde{\rho}(k_1-k_2)|^2, \quad (12)$$

where $\tilde{\rho}(q) = \int d^4x \rho(x) \exp(iq\cdot x)$ = the Fourier-transformed $\rho(x)$.

Information about the space-time structure of the source .

Analogous to the Hanbury - Twiss – Brown effect in astrophysics.

Assumtions:

No spin, isospin effects
Negligible final state Coulomb and strong interaction effects
Incoherent emission of particles (x).
Static Sources $\rightarrow C(k_1,k_2)$ depends on k_1-k_2 only (x).
Negligible multiparticle symmetrization effects (x) .

Model Assumptions:

A model system is prescribed as

$$\hat{\rho} = \sum_{n=0}^{\infty} p_n \, \hat{\rho}_n, \tag{13}$$

the density matrix of the whole system is normalized as

$$\mathrm{Tr}\ (\hat{\rho}) = 1, \tag{14}$$

similarly to the density matrix for events with fixed numer of n:

$$\mathrm{Tr}\ (\hat{\rho}_n) = 1, \tag{15}$$

from which it follows that

$$\sum_{n=0}^{\infty} p_n = 1. \tag{16}$$

The multiplicity distribution is prescribed by $\{p_n\}_{n=0}^{\infty}$.

A system with a fixed number of boson wave packets is

$$\hat{\rho}_n = \int d\alpha_1 ... d\alpha_n \ \rho_n(\alpha_1, ..., \alpha_n) \, |\alpha_1, ..., \alpha_n\rangle\langle\alpha_1, ..., \alpha_n|, \tag{17}$$

where $|\alpha_1, ..., \alpha_n\rangle$ denote properly normalized n-particle wave-packet boson states.

Wave-Packets:

A wave packet creation operator is

$$\alpha_i^\dagger = \int \frac{d^3\mathbf{p}}{(\pi\sigma^2)^{\frac{3}{4}}} \, \mathrm{e}^{-\frac{(\mathbf{p}-\pi_i)^2}{2\sigma_i^2} - i\xi_i(\mathbf{p}-\pi_i) + i\omega(\mathbf{p})(t-t_i)} \, \hat{a}^\dagger(\mathbf{p}). \tag{18}$$

The non-vanishing commutator is

$$\left[\alpha_i, \alpha_j^\dagger\right] = \langle \alpha_i | \alpha_j \rangle \tag{19}$$

Here $\alpha_i = (\xi_i, \pi_i, \sigma_i, t_i)$ refers to: the center in space, in momentum space, the width in momentum space and the production time, respectively.

For simplicity we assume that $\alpha_i = (\pi_i, \xi_i, \sigma, t_0)$.

The n boson states, normalized to unity, are given as

$$|\ \alpha_1,\ \ldots\ ,\ \alpha_n\rangle = \frac{1}{\sqrt{\sum\limits_{\sigma^{(n)}} \prod\limits_{i=1}^{n} \langle \alpha_i | \alpha_{\sigma_i} \rangle}} \, \alpha_n^\dagger \, \ldots \, \alpha_1^\dagger |0\rangle. \tag{20}$$

Here $\sigma^{(n)}$ denotes the set of all the permutations of the indexes $\{1, 2, ..., n\}$ and the subscript sized $_{\sigma_i}$ denotes the index that replaces the index $_i$ in a given permutation from $\sigma^{(n)}$.

A New Type of Density Matrix

There is one special density matrix, for which one can overcome the difficulty, related to the non-vanishing overlap of many hundreds of wave-packets, even in an explicit analytical manner.

$$\rho_n(\alpha_1, ..., \alpha_n) = \frac{1}{\mathcal{N}(n)} \left(\prod_{i=1}^{n} \rho_1(\alpha_i) \right) \left(\sum_{\sigma^{(n)}} \prod_{k=1}^{n} \langle \alpha_k | \alpha_{\sigma_k} \rangle \right) \qquad (21)$$

Normalization to 1 yields $\mathcal{N}(n)$. The weight factors describe induced emission:

$$\frac{\rho_n(\alpha_1, ..., \alpha_n)}{\prod_{j=1}^{n} \rho_1(\alpha_j)} = \frac{\sum_{\sigma^{(n)}} \prod_{k=1}^{n} \langle \alpha_k | \alpha_{\sigma_k} \rangle}{\mathcal{N}(n)} \qquad \text{(overlap)}. \qquad (22)$$

This is maximal if all are emitted with same α_1:

$$\frac{\rho_n(\alpha_1, ..., \alpha_1)}{[\rho_1(\alpha_1)]^n} = \frac{n!}{\mathcal{N}(n)}, \qquad \text{(full overlap)}, \qquad (23)$$

and minimal if the overlap is negligible,

$$\frac{\rho_n(\alpha_1, ..., \alpha_n)}{\prod_{j=1}^{n} \rho_1(\alpha_j)} = \frac{1}{\mathcal{N}(n)} \qquad \text{(no overlap)}. \qquad (24)$$

Wildly fluctuating weight. E.g. for 800 pions the induced emission weight fluctuates between $[1, 800!] \simeq [1, 10^{2323}]$.

Single-Particle Density Matrix :

Factorizable Gaussian forms:

$$\begin{aligned}
\rho_1(\alpha) &= \rho_x(\xi)\,\rho_p(\pi)\,\delta(t-t_0) \\
\rho_x(\xi) &= \frac{1}{(2\pi R^2)^{\frac{3}{2}}}\exp(-\xi^2/(2R^2)) \\
\rho_p(\pi) &= \frac{1}{(2\pi m T)^{\frac{3}{2}}}\exp(-\pi^2/(2mT))
\end{aligned} \tag{25}$$

Non-relativistic, non-expanding static source at rest in the frame where the calc. performed.

A multiplicity distribution when Bose-Einstein effects are switched off (denoted by $p_n^{(0)}$), is a FREE choice in the model. We assume

$$p_n^{(0)} = \frac{n_0^n}{n!}\exp(-n_0), \tag{26}$$

This completes the specification of the model.

Solution:

1) Reduction of the multi-particle wave-packet problem to an equivalent plane-wave problem with parameters that depend on σ. [2,3]

$n! \propto \exp(n \ln n)$

2) Reduction of the plane-wave problem to a set of recurrence relations.

# of steps $\propto n^2$

[4] S. Pratt, Phys. Lett. **B301** (1993) 159

3) Analytical solution of the recurrence relations. [2,3] n^0.

Recurrences:

The probability of finding events with multiplicity n, as well as the single-particle and the two-particle momentum distribution in such events is given as

$$p_n = \frac{\omega_n}{\sum_{k=0}^{\infty} \omega_k}, \tag{27}$$

$$N_1^{(n)}(\mathbf{k}) = \sum_{i=1}^{n} \frac{\omega_{n-i}}{\omega_n} G_i(\mathbf{k}, \mathbf{k}), \tag{28}$$

$$N_2^{(n)}(\mathbf{k}_1, \mathbf{k}_2) = \sum_{l=2}^{n} \sum_{m=1}^{l-1} \frac{\omega_{n-l}}{\omega_n} \left[G_m(\mathbf{k}_1, \mathbf{k}_1) G_{l-m}(\mathbf{k}_2, \mathbf{k}_2) + G_m(\mathbf{k}_1, \mathbf{k}_2) G_{l-m}(\mathbf{k}_2, \mathbf{k}_1) \right], \tag{29}$$

where $\omega_n = p_n/p_0$ and

$$G_n(\mathbf{p}, \mathbf{q}) = n_0^n \, h_n \exp(-a_n(\mathbf{p}^2 + \mathbf{q}^2) + g_n \mathbf{p}\mathbf{q}), \tag{30}$$

$$C_n = \frac{1}{n} \int d^3\mathbf{k} G_n(\mathbf{k}, \mathbf{k}), = h_n \frac{n_0^n}{n} \left(\frac{\pi}{2a_n - g_n} \right)^{\frac{3}{2}}. \tag{31}$$

Using the notation

$$\sigma_T^2 = \sigma^2 + 2mT, \qquad R_{eff}^2 = R^2 + \frac{mT}{\sigma^2 \sigma_T^2}, \tag{32}$$

the plane-wave pion laser model of S. Pratt is reproduced with a replacement $R \to R_{eff}$ and $T \to T_{eff}$.

Solution for the Inclusive Distributions:

Let us introduce the auxiliary quantity

$$G(\mathbf{k}_1, \mathbf{k}_2) = \sum_{i=1}^{\infty} G_i(\mathbf{k}_1, \mathbf{k}_2). \tag{33}$$

(Higher order Bose-Einstein symmetrization effects are negligible, if the first term dominates the above infinite sum, i.e. if $G(\mathbf{k}_1, \mathbf{k}_2) = G_1(\mathbf{k}_1, \mathbf{k}_2)$.)

Now the averaging over the exclusive distributions can be performed,

$$N_1(\mathbf{k}) \;=\; \sum_{n=1}^{\infty} p_n N_1^{(n)}(\mathbf{k}) \;=\; G(\mathbf{k}, \mathbf{k}), \tag{34}$$

and the two-particle inclusive correlation functions can be evaluated as

$$\begin{aligned} C_2(\mathbf{k}_1, \mathbf{k}_2) &= \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1)\, N_1(\mathbf{k}_2)} \\ &= \; 1 + \frac{G(\mathbf{k}_1, \mathbf{k}_2) G(\mathbf{k}_2, \mathbf{k}_1)}{G(\mathbf{k}_1, \mathbf{k}_1) G(\mathbf{k}_2, \mathbf{k}_2)}. \end{aligned} \tag{35}$$

This is an exact result , obtained without the random phase approximation and without assuming that the number of sources is large.

General Analytic Solution:

Auxiliary quantities :

$$\gamma_{\pm} = \frac{1}{2}\left(1 + x \pm \sqrt{1+2x}\right) \qquad x = R_{eff}^2 \sigma_T^2 \tag{36}$$

The *general analytic solution* for $G_n(\mathbf{k}_1, \mathbf{k}_2)$ is given in ref. [2,3].

The C_n will be

$$C_n \ = \ \frac{n_0^n}{n}\left[\gamma_+^{\frac{n}{2}} - \gamma_-^{\frac{n}{2}}\right]^{-3} \tag{37}$$

The generating function for p_n reads [2,3] :

$$G(z) \ = \ \sum_{n=0}^{\infty} p_n z^n \ = \ \exp\left(\sum_{n=1}^{\infty} C_n (z^n - 1)\right), \tag{38}$$

Thus C_n-s are the so called combinants [5,6].

[5] M. Gyulassy and S. K. Kaufmann, Phys. Rev. Lett. **40** (1978) 298

[6] S. Hegyi, Phys. Lett. **B309** (1993) 443-450

Multiplicity Distribution:

The mean and the second factorial moment is

$$\langle n \rangle \;=\; \sum_n n\, p_n \;=\; \sum_{i=1}^{\infty} iC_i, \tag{39}$$

$$\langle n(n-1) \rangle \;=\; \sum_n n\,(n-1)\, p_n \;=\; \langle n \rangle^2 + \sum_{i=2}^{\infty} i\,(i-1)\, C_i. \tag{40}$$

The large n behavior of nC_n depends on $n_0/\gamma_+^{\frac{3}{2}}$.

One may introduce

$$n_c \;=\; \gamma_+^{\frac{3}{2}} = \left[\frac{1+x+\sqrt{1+2x}}{2}\right]^{\frac{3}{2}}. \tag{41}$$

If $n_0 < n_c$, $\langle n \rangle < \infty$.

If $n_0 \geq n_c$, $\langle n \rangle = \infty$.

Hence n_c is a critical value for the parameter n_0.

Infinite convolution of Poisson distributions of particle singlets, pairs, triplets,... as well as infinite convolution of Bose-Einstein distributions with coupled parameters, [2].

Rare gas limiting case:

For large source sizes or large effective temperatures , $x >> 1$:

$$G_n(\mathbf{k}_1,\mathbf{k}_2) \propto \left(\frac{2}{x}\right)^{\frac{3}{2}(n-1)} \exp\left[-\frac{n}{2\sigma_T^2}(\mathbf{k}_1^2+\mathbf{k}_2^2) - \frac{R_{eff}^2}{2n}(\mathbf{k}_1-\mathbf{k}_2)^2\right], \quad (42)$$

$$C_n = \frac{n_0^n}{n^4}\left(\frac{2}{x}\right)^{\frac{3}{2}(1-n)} . \quad (43)$$

Reduced effective temperatures and reduced effective radii .

The multiplicity distribution is shifted:

$$p_n = \frac{n_0^n}{n!}\exp(-n_0)\left[1+\frac{n(n-1)-n_0^2}{2(2x)^{\frac{3}{2}}}\right]. \quad (44)$$

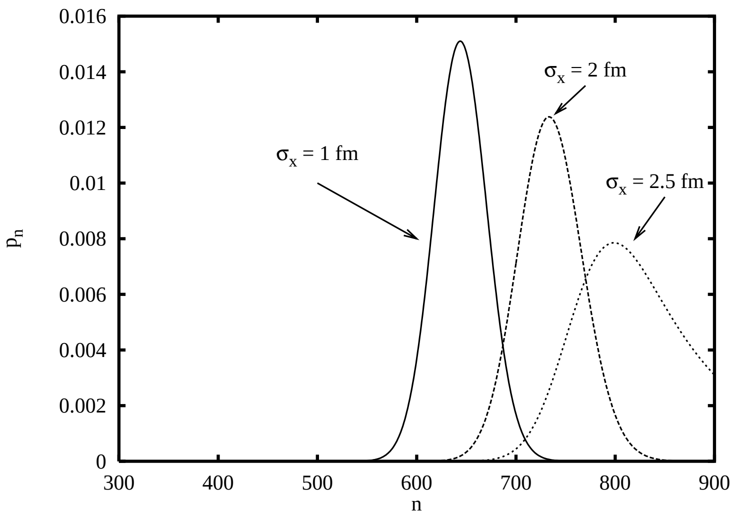
A small expansion parameter is $\epsilon = \left(\frac{2}{x}\right)^{\frac{3}{2}}$.

$$N_1(\mathbf{k}) = \frac{n_0}{(\pi\sigma_T^2)^{\frac{3}{2}}}\exp\left(-\frac{\mathbf{k}^2}{\sigma_T^2}\right) + \frac{n_0^2}{(\pi\sigma_T^2 x)^{\frac{3}{2}}}\exp\left(-\frac{2}{\sigma_t^2}\mathbf{k}^2\right) \quad (45)$$

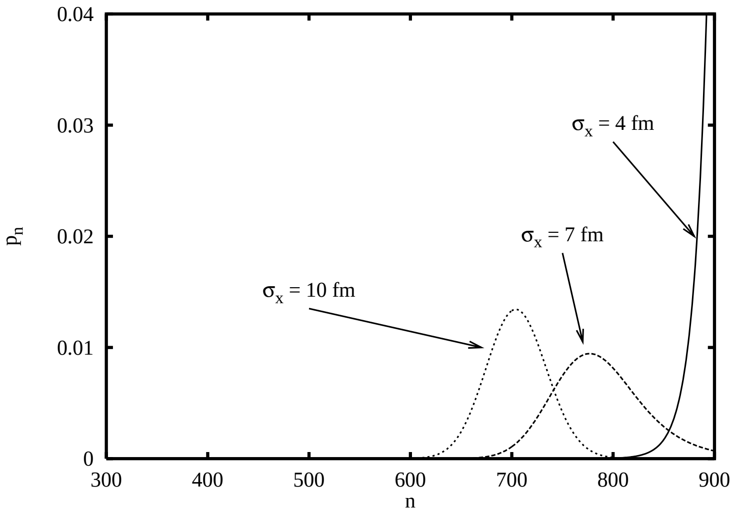
$$N_1^{(n)}(\mathbf{k}) = \frac{n}{(\pi\sigma_T^2)^{\frac{3}{2}}}\exp\left(-\frac{\mathbf{k}^2}{\sigma_T^2}\right)\left\{1+\frac{(n-1)}{(2x)^{\frac{3}{2}}}\left[2^{\frac{3}{2}}\exp\left(-\frac{\mathbf{k}^2}{\sigma_t^2}\right)-1\right]\right\}, \quad (46)$$

properly normalized . The single-particle inclusive and exclusive momentum distributions are enhanced at low momentum.

Modification of pion number distribution:



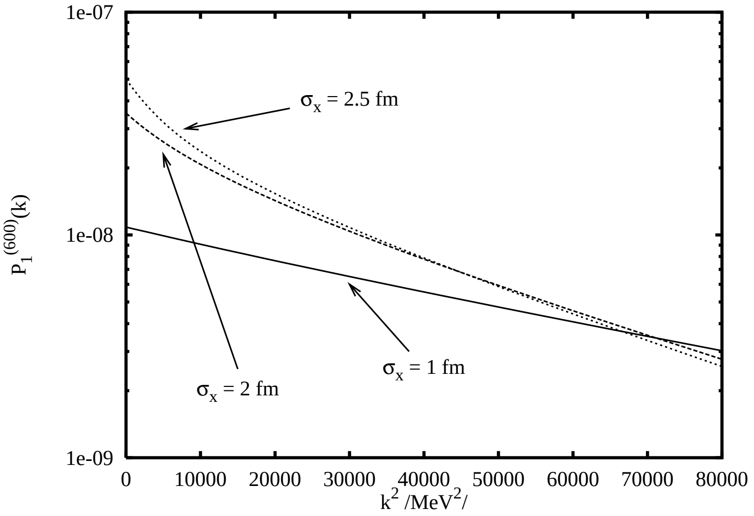
As the size of the wave packets is changed, the overlap changes and stimulated emission results in larger multiplicities.



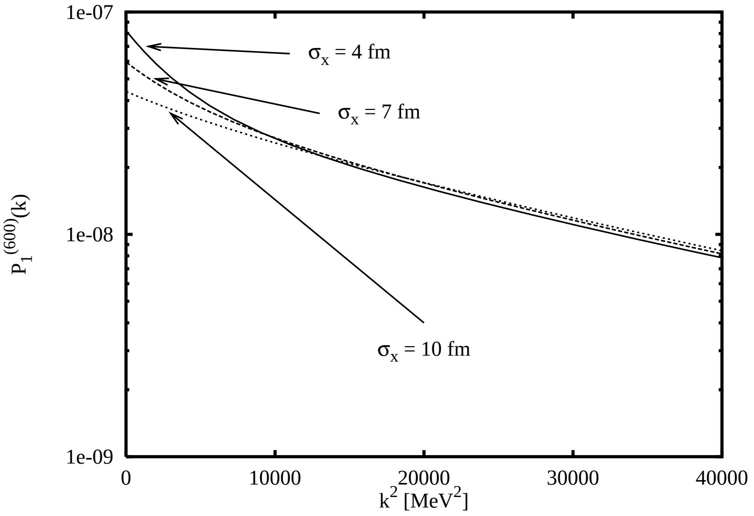
As the size of wave packets approaches 4 fm, the induced emission over-compensates , and the pion laser is set on .

If the wave-packet size is increased further, the overlap decreases and lasing disappears.

Modification of momentum distribution:



Stimulated emission results in enhancement of pions in the low momentum modes.



The induced emission results in a maximal enhancement of pions at low momentum, if pion laser is set on . If the overlap decreases and enhancement at low momentum disappears.

Correlation functions:

A lasing behavior in the very dense $x << 1$ Bose-gas limit, an optically coherent behavior [2], characterized by

$$C(\mathbf{k}_1, \mathbf{k}_2) = 1.$$

In the rare gas limit,

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = 1 + \exp\left(-R^2_{eff} \mathbf{\Delta k}^2\right) \tag{47}$$

Exclusive Correlation Functions:

In a dense gas limiting case, $x << 1$, the exclusive and inclusive correlation functions coincide , $C(\mathbf{k}_1, \mathbf{k}_2) = 1$.

In the rare gas limit, $x >> 1$, the exclusive and inclusive correlations are different .

$$C_2^{(n)}(\mathbf{k}_1, \mathbf{k}_2) = \frac{n^2}{n(n-1)} \frac{N_2^{(n)}(\mathbf{k}_1, \mathbf{k}_2)}{N_1^{(n)}(\mathbf{k}_1)\, N_1^{(n)}(\mathbf{k}_2)}$$
$$= 1 + \lambda_{\mathbf{K}} \exp\left(-R^2_{\mathbf{K},s} \mathbf{\Delta k}_s^2 - R^2_{\mathbf{K},o} \mathbf{\Delta k}_o^2\right), \qquad (48)$$

where $\mathbf{K} = 0.5(\mathbf{k}_1 + \mathbf{k}_2)$, $\mathbf{\Delta k}_s = \mathbf{\Delta k} - \mathbf{K}(\mathbf{\Delta k} \cdot \mathbf{K})/(\mathbf{K} \cdot \mathbf{K})$ and $\mathbf{\Delta k}_o = \mathbf{K}(\mathbf{\Delta k} \cdot \mathbf{K})/(\mathbf{K} \cdot \mathbf{K})$,

The momentum dependent parameters are

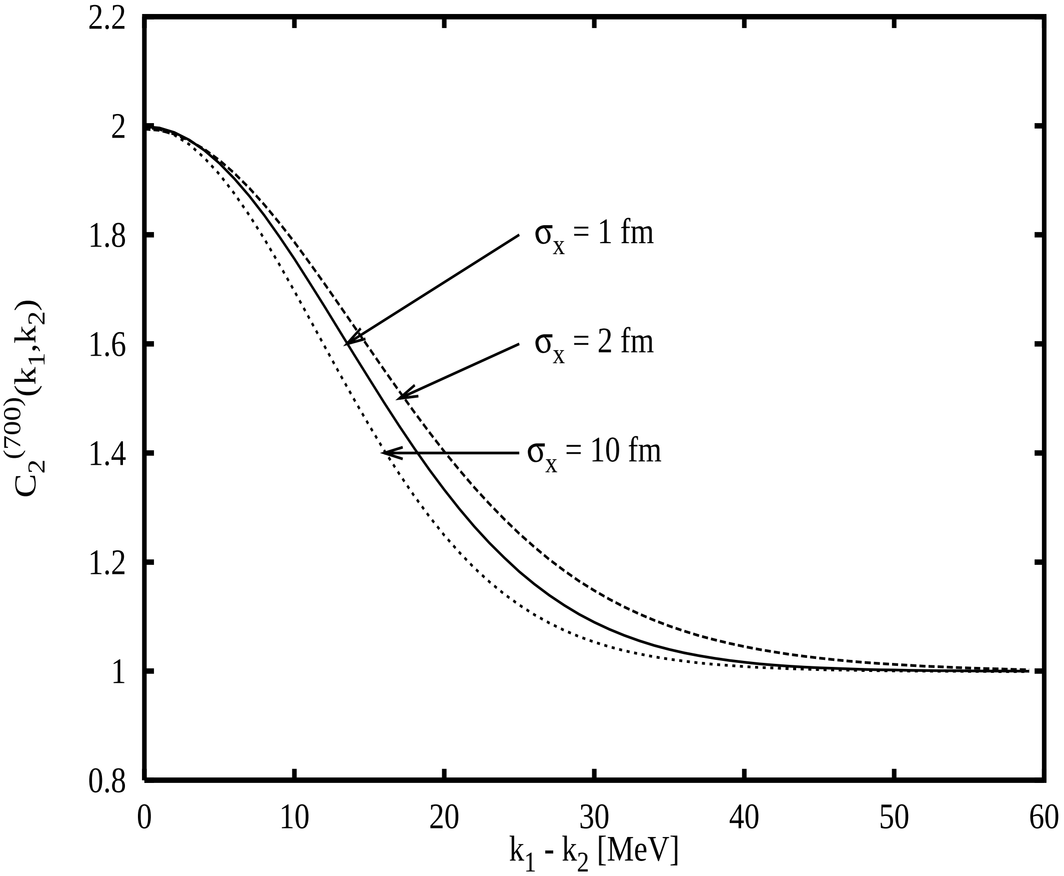
$$\lambda_{\mathbf{K}} = 1 + \frac{2}{(2x)^{\frac{3}{2}}} \left[1 - 2^{(5/2)} \exp\left(-\frac{\mathbf{K}^2}{\sigma_T^2}\right)\right] \qquad (49)$$

$$R^2_{\mathbf{K},s} = R^2_{eff} + \frac{1}{(2x)^{\frac{3}{2}}} \left[R^2_{eff} - \sqrt{2} \exp\left(-\frac{\mathbf{K}^2}{\sigma_T^2}\right) \left((n+2) R^2_{eff} + \frac{2}{\sigma_T^2}\right)\right], \qquad (50)$$

$$R^2_{\mathbf{K},o} = R^2_{\mathbf{K},s} + \frac{n}{x^{\frac{3}{2}}} \frac{\mathbf{K}^2}{\sigma_T^4} \exp\left(-\frac{\mathbf{K}^2}{\sigma_T^2}\right) \qquad (51)$$

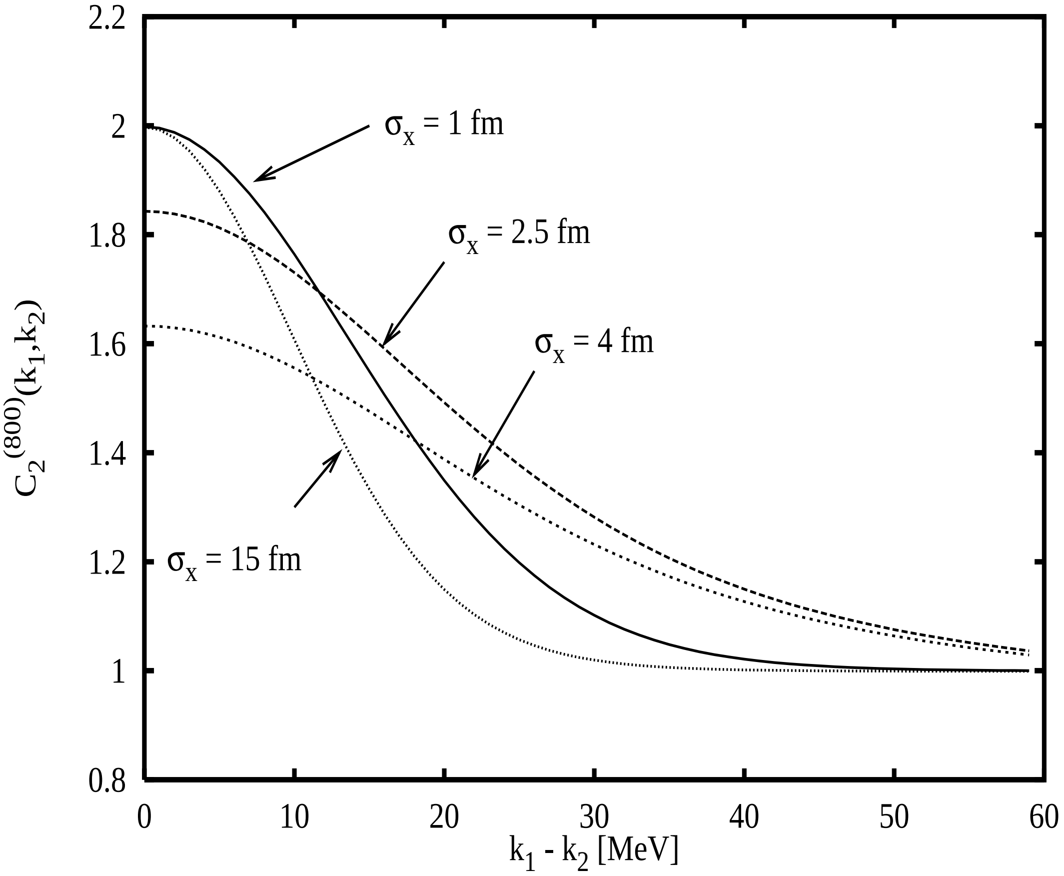
as illustrated on Figure 3.

Dependence on Wave-Packet Width:



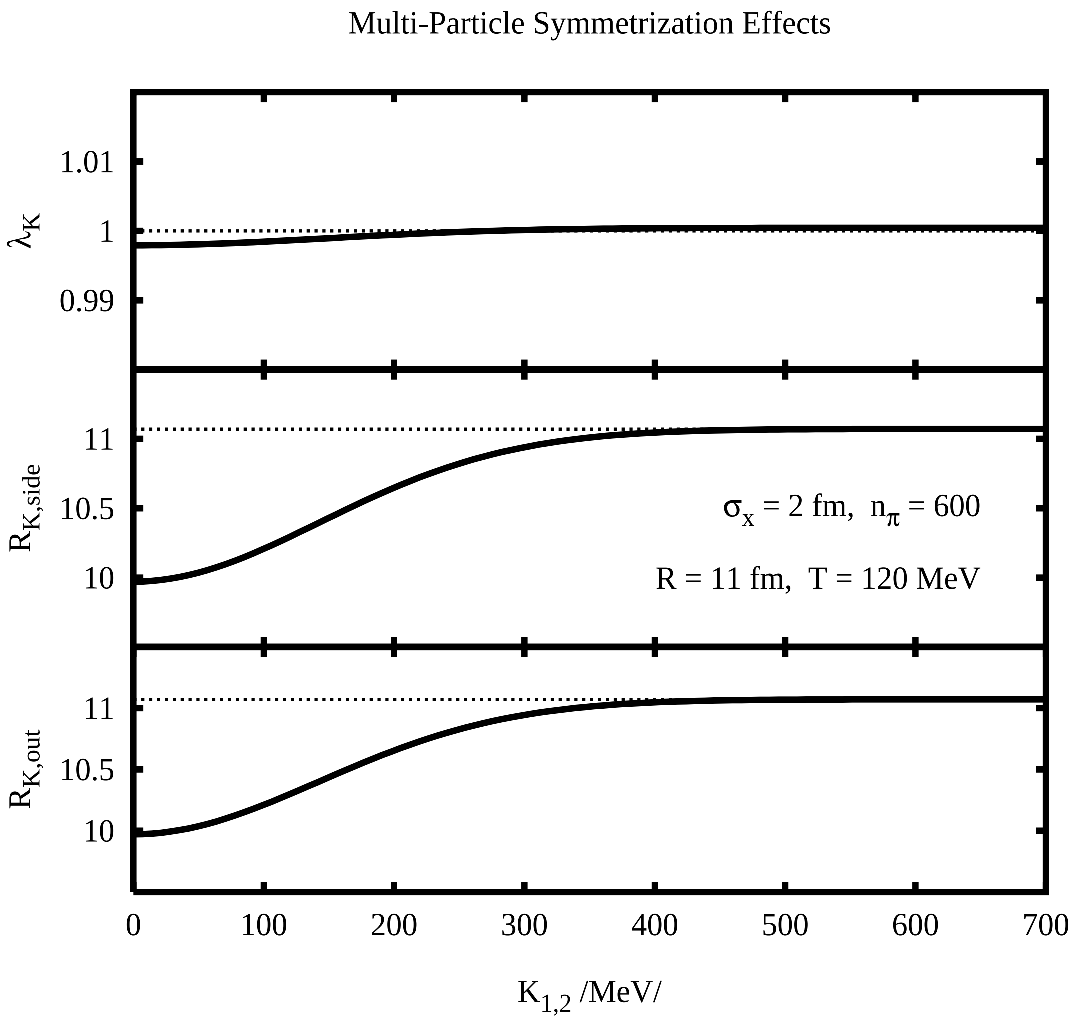
Two-particle exclusive correlation function,for wave packet sizes $\sigma_x = 1.0$, 2.5 and 10.0, (solid, dashed and dotted lines). $|\mathbf{k}_1 + \mathbf{k}_2|/2 = 100$ MeV, out direction. The number of pions was fixed to $n = 700$.

Correlations at a Lower Mean Momentum:



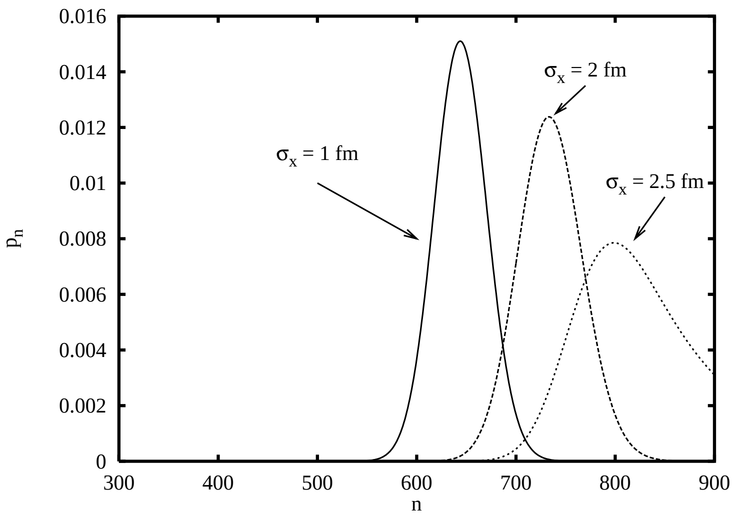
$C_2^{(800)}(\mathbf{k}_1, \mathbf{k}_2)$ for wave packet sizes $\sigma_x = 1.0$, 2.5, 4.0 and 15.0 fm (solid, dashed, dotted and dense-dotted lines). $|\mathbf{k}_1 + \mathbf{k}_2|/2 = 50$ MeV, out component. The number of pions was set to $n = 800$.

Dependence on Mean Momentum :

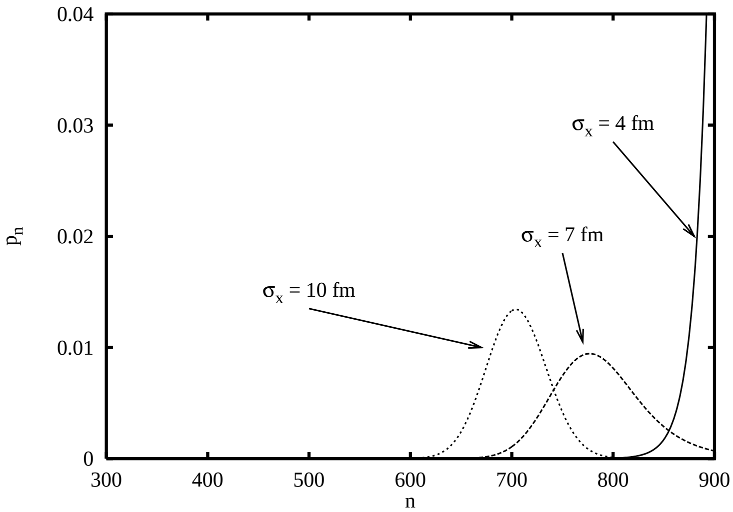


Momentum-dependent reduction of the intercept parameter $\lambda_{\mathbf{K}}$, the side-wards and the outwards radius parameters, $R_{\mathbf{K},s}$ and $R_{\mathbf{K},o}$ from their static values of 1 and R_{eff}, respectively.

Modification of pion number distribution:



As the size of the wave packets is changed, the overlap changes and stimulated emission results in larger multiplicities.



As the size of wave packets approaches 4 fm, the induced emission over-compensates , and the pion laser is set on . If the wave-packet size is increased further, the overlap decreases and lasing disappears.

Highlights:

Consequent quantum mechanical description.

Overlapping multi-particle wave-packet states describing stimulated emission

A weight that fluctuates between $[1, 10^{2323}]$.

Analitical solution

Momentum-dependent radius and intercept parameters even for static sources.

A special directional dependence of BECF.

Enhancement of the wave-packets in the low momentum modes, due to multi-particle Bose-Einstein symmetrization effects, formation of a pion- laser .

Kleppner's test of lasing, suggested in Physics Today, Aug. 1997 p 11, would fail in our case .