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New Analytic Solutions of Fireball Hydrodynamics

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New , simple and exact solutions of fireball hydrodynamics:
Assuming non-relativistic , spherically symmetric expansion
and an ideal gas equation of state
generalization of Zimányi - Bondorf - Garpman solution
for arbitrary radial temperature profiles

Thanks to P. Csizmadia
B. Lukács,
B. Lörstad,

T. Cs., nucl-th/9809011
P. Csizmadia, T. Cs., B. Lukács, nucl-th/9805006, PLB in press
T. Cs., B. Lörstad, PRC 54 (1996) 1390
J. Bondorf, S. Garpman and J. Zimányi, Nucl. Phys. A296 (1978) 320

Motivation: 20 years old problem

Zimányi - Bondorf - Garpman :
Special solution of non-relativistic (NR) hydrodynamics
last stage of NR heavy ion collisions
obtained in Copenhagen 20 years ago.

The NR hydrodynamical problem
is specified by the continuity, Euler and energy equations:

$$\partial_t n + \nabla(\mathbf{v} n) \;=\; 0, \tag{1}$$
$$\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} \;=\; -(\nabla p)/(mn), \tag{2}$$
$$\partial_t \epsilon + \nabla(\epsilon \mathbf{v}) \;=\; -p\nabla\mathbf{v}, \tag{3}$$

where n is the number density of particles,
$\mathbf{v}$ stands for the NR flow field,
ϵ for the NR energy density,
p for the pressure and
T for the temperature field.

We assume a NR ideal gas equation of state,

$$p \;=\; nT, \tag{4}$$
$$\epsilon \;=\; \frac{3}{2}p = \frac{3}{2}nT, \tag{5}$$

which closes the set of equations for n, $\mathbf{v}$ and T.

The new family of exact solutions

A set of solutions for self-similar density, flow and temperature profile.

Non-trivial dependence of the solution through the *scaling variable*

$$x = \mathbf{r}^2 / R^2(t), \tag{6}$$

where $R(t)$ is a characteristic, time dependent radial scale.
The scaling variable x is not r_x, $\mathbf{r} = (r_x, r_y, r_z)$.
Rotational symmetry is assumed .
Dimensionless scaling functions $\nu(x)$, $\beta(x)$ and $\mathcal{T}(x)$
instead of $n = n(\mathbf{r}, t)$, $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ and $T = T(\mathbf{r}, t)$, respectively:

$$\begin{aligned} n &= n_0 \frac{R_0^3}{R^3} \nu(x), &(7)\\ \mathbf{v} &= f_v(t)\, \beta(x)\, \mathbf{r}, &(8)\\ T &= T_0\, f_T(t)\, \mathcal{T}(x), &(9) \end{aligned}$$

the time-dependence of the radial scale is

$$R^2 = R^2(t) = R_0^2\, \phi(t), \tag{10}$$

The unknown functions $f_v(t)$, $f_T(t)$, $\phi(t)$
are determined from the hydro equations.

Initial conditions

The parameters n_0, T_0 and R_0 are constants
given by the initial conditions .
The "turning point radius " R_0 is prescribed at a given instant t_0
and the central particle density and temperature at this time t_0 is
$n_0 = n(t_0, \mathbf{0})$ and $T_0 = T(t_0, \mathbf{0})$.

$$\nu(0) = \beta(0) = \mathcal{T}(0) = 1, \tag{11}$$

$$f_T(t_0) = \phi(t_0) = 1, \qquad R_0 = R(t_0). \tag{12}$$

The new hydro solutions

The new hydro solutions read in terms of physical variables as :

$$
\begin{aligned}
R^2(t) &= R_0^2 + \langle u_t \rangle^2 (t - t_0)^2, && (13)\\
\mathbf{v}(t,\mathbf{r}) &= \frac{R'(t)}{R(t)}\,\mathbf{r} = \frac{\langle u_t \rangle^2 (t-t_0)}{R_0^2 + \langle u_t \rangle^2 (t-t_0)^2}\,\mathbf{r}, && (14)\\
T(t,\mathbf{r}) &= T_0\,\mathcal{T}(x)\,\frac{R_0^2}{R^2} = T\left[t_0, \mathbf{r} R_0/R(t)\right]\,\frac{R_0^2}{R^2(t)} && (15)\\
n(t,\mathbf{r}) &= n_0\,\frac{R_0^3}{R^3(t)}\,\frac{T(t_0,\mathbf{0})}{T\left[t_0, \mathbf{r} R_0/R(t)\right]} \times \\
&\times \exp\left[-\frac{m\langle u_t \rangle^2}{R_0^2} \int_0^{|\mathbf{r}| R_0/R(t)} \frac{r' dr'}{T(t_0, r')} \right]. && (16)
\end{aligned}
$$

A system that contracts for $t < t_0$, its minimum size is R_0 at $t = t_0$ when the flow field vanishes. For later times, $t > t_0$, the system expands again.

The temperature field is obtained by rescaling the initial temperature field.

The explicit solution for the density profile can be written into a simpler form :

$$
n(t,\mathbf{r}) = \frac{n_0}{\mathcal{T}(x)}\,\frac{R_0^3}{R^3(t)} \exp\left[-\frac{m\langle u_t \rangle^2}{2T_0} \int_0^{\mathbf{r}^2/R^2(t)} \frac{du}{\mathcal{T}(u)} \right]. \qquad (17)
$$

Initial conditions II:

$\langle u_t \rangle$ controls the *asymptotic* expansion,
as $\lim_{t\to\infty} R'(t) = \langle u_t \rangle$.
4 constants of integration, R_0, $\langle u_t \rangle$, T_0 and t_0,
$+$ one free dimensionless *function* $\mathcal{T}(x)$.
Three of constants of integration
$\rightarrow$ three integrated equations :
(the shperical continuity, Euler and energy equations).

The constant t_0 corresponds to the homogeneity
of the equations of NR hydrodynamics in time .
The freedom in choice of the dimensionless scaling function $\mathcal{T}(x)$
corresponds to the freedom in
choice of the initial radial temperature profile (boundary condition).

One solution is assinged to every integrable function
of one variable $1/\mathcal{T}(x)$.

$\rightarrow$ Infinitely many new solutions.

Uncountably infinite

Special cases:

Knudsen-gas
Zimányi - Bondorf - Garpman
Buda - Lund

Collisionless (Knudsen) gas limit

Simplest limiting case: $T(t_0, \mathbf{r}) = T(t_0) = T_0$ homogenous
Solution reads as:

$$\begin{aligned}
R^2(t) &= R_0^2 + \langle u_t \rangle^2 (t - t_0)^2, \quad \text{(same)} && (18)\\
\mathbf{v}(t, \mathbf{r}) &= \frac{R'(t)}{R(t)} \mathbf{r} \quad \text{(same)} && (19)\\
T(t, \mathbf{r}) &= T_0 \frac{R_0^2}{R(t)^2} && (20)\\
n(t, \mathbf{r}) &= \frac{N_0}{[2\pi R_G^2(t)]^{(3/2)}} \exp\left[-\frac{\mathbf{r}^2}{2R_G^2(t)} \right] && (21)\\
R_G^2(t) &= R^2(t) \frac{T}{m \langle u_t \rangle^2}. && (22)
\end{aligned}$$

Solution of NR hydro everywhere, all the time.

Surprisingly! The phase-space distribution of this hydro solution solves the collisionless Vlasov equation:

$$\begin{aligned}
f_k(t, \mathbf{x}, \mathbf{p}) &= C \exp\left[-\frac{[\mathbf{p} - m\mathbf{v}(t, \mathbf{r})]^2}{2mT(t)} - \frac{\mathbf{r}^2}{2R_G^2(t)} \right] && (23)\\
\partial_t f_k + (\mathbf{v}\nabla) f_k &= 0(!!) && (24)
\end{aligned}$$

and vice versa.

Thus this is a collisionless Knudsen-gas solution.

Recovering the Zimányi - Bondorf - Garpman solution

20 years ago, a famous analytic solution of NR hydrodynamics found by Zimányi, Bondorf and Garpman (ZBG) in Copenhagen:

Second simplest limiting case: $T(t_0, \mathbf{r})/T(t_0) \propto n(t_0, \mathbf{r})/n_0$: the density and the temperature profile is given by the same function.

$$
\begin{aligned}
R^2(t) &= R_0^2 + \langle u_t \rangle^2 (t - t_0)^2, \quad \text{(same)} && (25)\\
\mathbf{v}(t, \mathbf{r}) &= \frac{R'(t)}{R(t)} \mathbf{r} \quad \text{(same)} && (26)\\
T(t_0, \mathbf{r}) &= T_0 \left[1 - \frac{\mathbf{r}^2}{R^2(t)} \right] \qquad \text{if } |\mathbf{r}| \le R(t). && (27)\\
n(t, \mathbf{r}) &= n_0 \frac{R_0^3}{R^3(t)} \left[1 - \frac{\mathbf{r}^2}{R^2(t)} \right]^{\alpha} \quad \text{if } |\mathbf{r}| \le R(t), && (28)
\end{aligned}
$$

which corresponds to the ZBG solution with $\alpha = \dfrac{m \langle u_t \rangle^2}{2T_0} - 1.$

New element:

in the ZGB paper, $\alpha > 0$ assumed: decreasing density profiles.

However, if $m\langle u_t \rangle^2/2 < T_0$, a new class of solutions is found with $\alpha < 0$: $\rightarrow$ smoke-rings!

This is a special case of the general solution described in nuc-th/9809011.

Recovering the Buda - Lund hydro parameterization:

An analytic parameterization a hydrodynamically expanding source was developed recently by T. Cs. and B. Lörstad, to describe NA44 particle correlations and spectra in high energy heavy ion physics.

Temperature gradients (Sinyukov) and a non-relativistic transversal flow also utilized by the (Regensburg) group were considered simultaneously in the Buda - Lund formulation, in a Gaussian approximation: T. Cs and B. Lörstad Phys. Rev. **C54** (1996) 1390

This parameterization keeps only the means and variances of the density distribution, the inverse temperature profile and the flow profile.

$$T(\tau,\mathbf{r}) = \frac{T_0}{\left[1+\langle\frac{\Delta T}{T}\rangle_r\frac{\mathbf{r}^2}{2R^2}\right]\left[1+\langle\frac{\Delta T}{T}\rangle_\tau\frac{(\tau-\tau_0)^2}{2\Delta\tau^2}\right]}, \tag{29}$$

$$n(t_0,\mathbf{r}) = n_0\left(1+\langle\Delta T/T\rangle_r\mathbf{r}^2/(2R_0^2)\right)\times \exp\left[-\frac{m\langle u_t\rangle^2}{2T_0}\left(\frac{\mathbf{r}^2}{R_0^2}+\langle\frac{\Delta T}{T}\rangle_r\frac{\mathbf{r}^4}{R_G^4}\right)\right]. \tag{30}$$

The longitudinal proper-time, $\tau=\sqrt{t^2-r_z^2}$ is introduced for relativistic longitudinal expansions not described here.

$$\frac{1}{2\Delta\tau^2}\langle\Delta T/T\rangle_\tau = \frac{\langle u_t\rangle^2}{R_0^2}. \tag{31}$$

A non-trivial, exact, analytic solution of NR hydro .

$\mathbf{r}=(r_x,r_y)$, free parameters are $\langle\Delta T/T\rangle_r$, R, $\langle u_t\rangle$ and τ_0.

Buda - Lund II.

The time-evolution of this density can be obtained from simple re-scaling.

The density distribution in a Gaussian approximation is the BL Gaussian ansatz with

$$\frac{1}{R_G^2} = \frac{1}{R_0^2}\left(\frac{m\langle u_t\rangle^2}{T_0} - \langle \Delta T/T\rangle_r\right) \geq 0. \tag{32}$$

with constrained values of the cooling parameter $\langle \Delta T/T\rangle_r$.

If

$$\frac{m\langle u_t\rangle^2}{T_0} < \langle \Delta T/T\rangle_r \tag{33}$$

the radial density profile will increase first before decreasing at large distances.

This type of solution corresponds to a kind of *expanding "smoke-ring"* in the transversal plane, or, to an *expanding spherical shell* if the expansion is three dimensional.

See Figure 1 for $R_0 = 5$ fm, $m\langle u_t\rangle^2/T_0 = 0.125$.

Smoke -rings were found in 50 - 70 MeV/A heavy ion reactions first theoretically by W. Bauer et al, PRL **69** (1992) 1888, observed experimentally in N. T. B. Smoke et al, PRL **78** (1997) 2084.

Smoke - rings are special cases of the general solution described in nuc-th/9809011.

Donuts versus muffins

Smoke-ring or donut shape
$\rightarrow$ decreasing pressure and stalling of matter on the surface.

Muffin solutions (ordinary density profile)
$\rightarrow$ pressure and expansion on the surface is
fast enough to take away the matter.

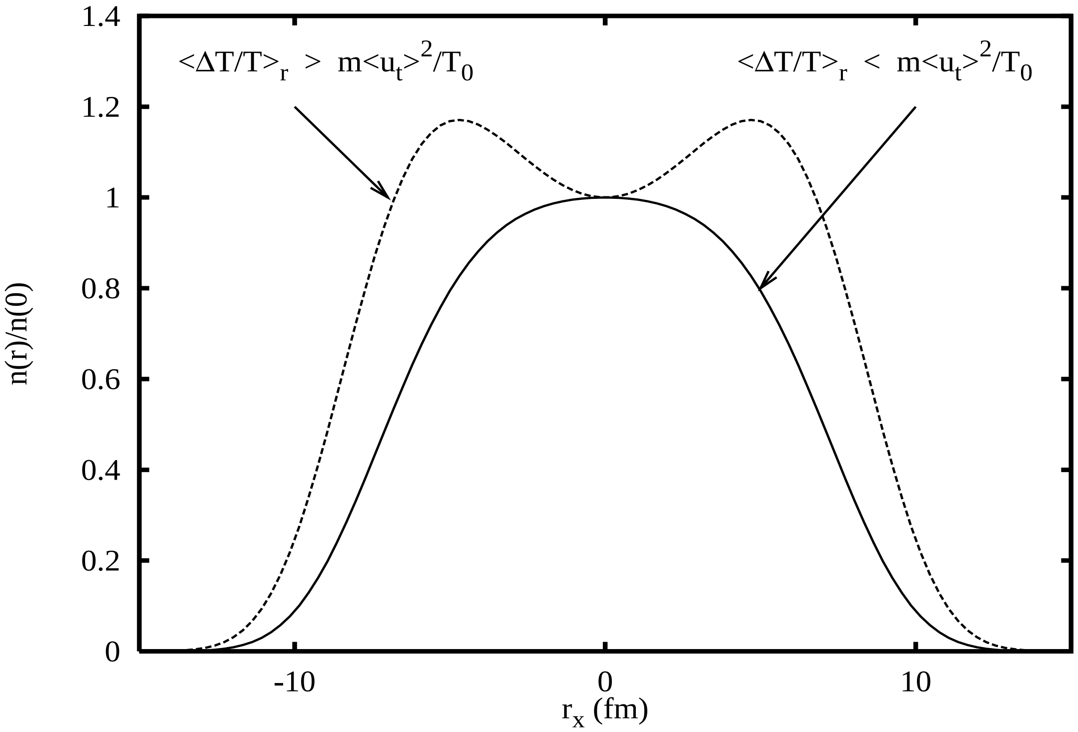
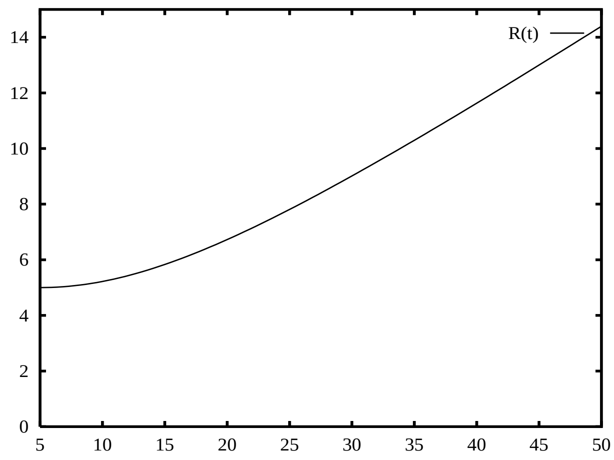
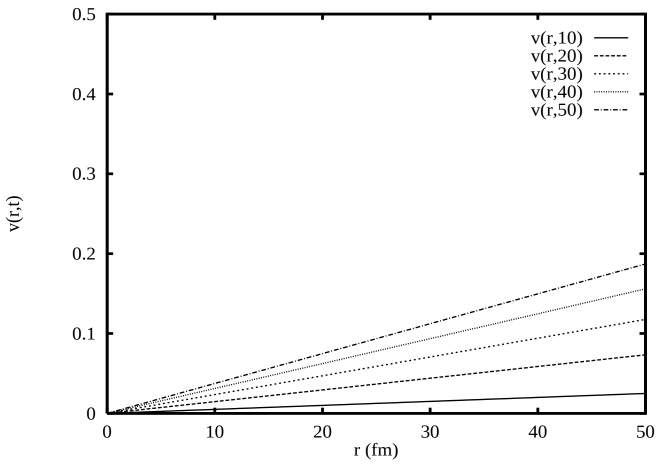


Illustration of the development of smoke-ring solutions for large temperature gradients in exact analytic solutions of NR hydro

Expansion of scale and decrease of flow

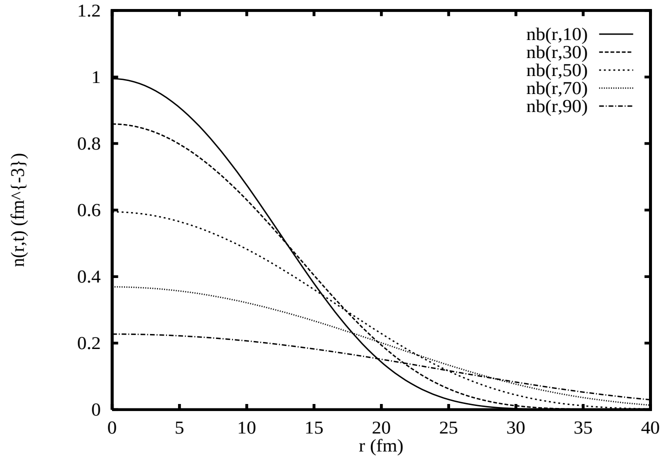


Expansion in time of the scale parameter $R(t)$.

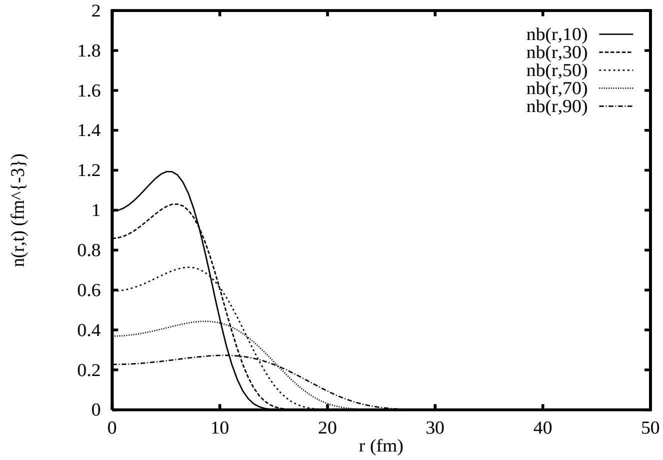


Decrease in time of the radial flow $v(r,t) = rR'/R$.

Expansion of “muffin” and “donut” shapes

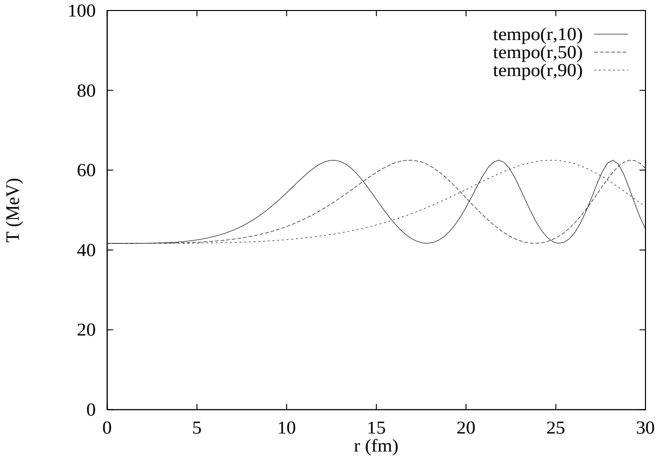


Expansion in time of the density profile for small temperature gradients.



Expansion in time of the density profile for large temperature gradients.

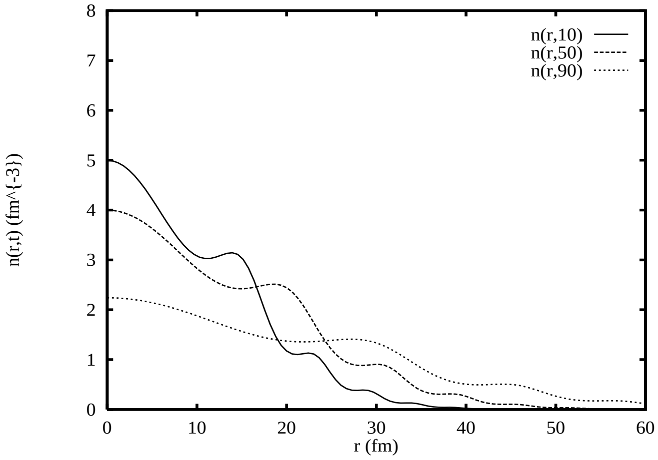
An oscillating temperature and density profile



Expansion in time of an initially oscillating temperature profile.

$$T_{osc}(t_0, r) = T_0/[1 + a\cos(r^2/R^2(t)/l)]$$

for $T_0 = 50$ MeV, $a = 0.5$ and $l = 2$



The oscillating density from the above temperature profile.

Given by one elementary integration

Summary

Limitations : Non-rel hydro
ideal gas EOS
radial expansion.

Pro's : Exact solutions in time and space
Simple, analytic forms
Infinitely many new solutions
Expansion in d dimensions
Generalization of Zimányi - Bondorf -Garpman
after 20 years

To come: Elliptic flows
Non-ideal EOS .

Open problems: Relativistic expansions.
Calculation of observables
for various freeze-out scenario.

Experimental implications:
FOPI used Knudsen-gas solution in data analysis
Look for donuts
Opacity effects in high energy heavy ions
(Buda-Lund, H. Heiselberg).