A UNIVERSAL MASTER EQUATION FOR THE GRAVITATIONAL VIOLATION OF QUANTUM MECHANICS

L. DIÓSI [1]
Department of Physics, Israel Institute of Technology, Haifa 32000, Israel

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In this paper we propose a master equation which contains a damping term universally violating the quantum mechanics of massive systems. From this equation follows that the quantum mechanical superposition principle breaks down if the states have radically different mass distributions. The damping of the coherence appears on a reasonable scale of masses and distances.

1. Introduction

Classical theories of gravitation [1,2] attribute sharply given functions to the gravitational field (or to the metric of the space–time) and therefore they do not apply to the microworld where measurable quantities must usually spread due to quantum effects. Conversely, the extension of the quantum theory to the macroworld also leads to contradictions, the best example of which was presented by Schrödinger in his cat paradox [3].

Thus, both theories, i.e. that of gravitation and of quantization, must be changed when we wish to extend their fields of application or even to unify them.

So far, most of the works [4] have attempted to cure the classival gravitation theory by applying some, more or less standard, quantization procedure to it. In our paper the opposite task is concerned. We impose a certain gravitational modification on the ordinary quantum mechanics in order to eliminate the illnesses of the macroscopic quantum theory as, e.g. the cat paradox is. The Schrödinger equation would then be substituted by the proper and unique master equation which shall be derived from gravitational considerations.

[1] Present address: Central Research Institute for Physics H-1525 Budapest 114, P.O. Box 49, Hungary.

2. Indeterminacy of the gravitational field

We shall confine ourselves to the newtonian limit of general relativity; the metric tensor g_{ab} has the following non-zero components:

$$g_{aa} = c^2 + 2\phi, \quad \text{for} \quad a=0\,,$$
$$= -1, \quad \text{for} \quad a=1,2,3\,, \tag{1}$$

c denotes the velocity of light, ϕ stands for the newtonian gravitational potential, $|\phi| \ll c^2$ is assumed.

As we mentioned above, the metric must possess a certain spread in a universal theory because of quantum effects. If a given quantum gravity theory were derived by the quantization of the classical field then these fluctuations would come from the non-vanishing commutators of canonical pairs of field operators. We, however, do not quantize the classical gravitational field; nevertheless, we shall propose a measure for its fluctuations.

Our method goes back to a famous paper [5] by Bohr and Rosenfeld. They investigated the principles of measuring the classical electromagnetic field by apparatuses obeying the quantum mechanics. For the optimal sensitivy of such measurements they were able to obtain a value just corresponding to the vacuum fluctuations of the quantized electromagnetic field.

Recently, Diósi and Lukács [6] estimated the measurability of the newtonian gravitational acceleration field

$$g(r,t) = -\nabla\phi(r,t)\,, \tag{2}$$

depending on coordinate r and on time t. They considered a probe of a given mass and extension. The gravitational acceleration of the probe was observed in a Bohr–Rosenfeld gedankenexperiment. On one hand, the probe satisfies the Schrödinger equation with the proper gravitational term and, on the other, its back reaction onto the gravitational field must be taken into the account. The optimal precision δg of the measurement is then universally bounded by

$$(\delta\tilde{g})^2 \gtrsim \hbar G/VT\,, \tag{3}$$

where G is the newtonian constant of gravity. A tilde denotes, here and after, averaging like

$$\tilde{g}(r,t) \equiv \frac{1}{VT} \iint\limits_{\substack{|r'-r|\leq R \\ |t'-t|\leq T/2}} g(r',t')\,\mathrm{d}^3r'\,\mathrm{d}t'\,, \tag{4}$$

over a given volume $V = \frac{4}{3}\pi R^3$ and time T as well [‡1].

Just at a similar level of accuracy and care (besides, it is mostly accepted; see, e.g. refs. [7,8]) Unruh [9] proposed a formally relativistic gedankenexperiment where a given massive and extended clock was thought to measure a given time interval. Unruh's bound on the maximal precision of such a measurement was represented by the following limit on the covariance between the zero–zero components of the metric g and the Einstein tensor G,

$$\delta\tilde{g}_{00}\,\delta\tilde{G}^{00} \gtrsim \hbar G/c^4 VT\,. \tag{5}$$

In the newtonian limit (1), the l.h.s. of the above inequality will be equal to $-c^{-4}(\delta\tilde{\phi}\,\Delta\delta\tilde{\phi})$; this covariance can be rewritten in the explicitly symmetric form $c^{-4}(\delta\nabla\tilde{\phi}\,\delta\nabla\tilde{\phi})$ thus, applying eq. (2), Unruh's eq. (5) will coincide with the bound (3) proposed via intrinsic nonrelativistic arguments by Diósi and Lukács.

Of course, the bound (3) might be irrelevant if, e.g. our measuring apparatuses were oversimplified. Nevertheless, in this paper we assume that the relation (3) represents an absolute indeterminacy of the gravitational field.

3. Universal gravitational white noise

We shall require that the gravitational field possesses universal fluctuations with spread equal, up to a constant factor of O(1), to the indeterminacy represented by the r.h.s. of formula (3):

$$\langle[\nabla\tilde{\phi}(r,t)]^2\rangle - [\langle\nabla\tilde{\phi}(r,t)\rangle]^2$$

$$= \mathrm{const}\times\hbar G/VT\,, \tag{6}$$

where the symbols $\langle\;\rangle$ stand for the expectation values the operational meaning of which is a delicate question .

Remind that we refuse to quantize the gravitational potential but, nevertheless, we accept the need of universal fluctuations for it. Actually, we are going to assume the gravitational potential $\phi(r,t)$ to be a c-number stochastic variable [6,10]. Thus symbols $\langle\;\rangle$ have to be specified as stochastic average of the quantity enclosed.

We are going to derive the probability distribution of the potential ϕ. Obviously, the mean $\langle\phi(r,t)\rangle$ should be identified by the classical newtonian potential originating from the actual mass densities. For simplicity, however, we neglect the mean gravitational field and take

$$\langle\phi(r,t)\rangle \equiv 0\,, \tag{7}$$

but we do not claim anyhow that the inclusion of the mean field would be a trivial task, cf. ref. [9].

In ref. [6] we showed that eq. (6) determines the correlation function of $\phi(r,t)$ almost uniquely:

$$\langle\phi(r,t)\phi\hbar(r',t')\rangle = \hbar G|r-r'|^{-1}\delta(t-t')\,, \tag{8}$$

where also eq. (7) has been meant, of course. The numeric factor on the r.h.s. of eq. (1) was set equal to 4π.

The distribution of the potential $\phi(r,t)$ will be completely specified by the moments (7) and (8) if we assume the distribution is gaussian. Then, in other words, $\phi(r,t)$ is called a gaussian white noise.

Our treatment is nonrelativistic, therefore formu-

[‡1] An outline of arguments of ref. [6] follows. Let δv, δx denote the quantum uncertainties of the velocity and the position, respectively, of the probe. The average acceleration of the probe measures the value of $\tilde{g}$ (4) with an error $(\delta\tilde{g})_1 \sim \delta v/T$; the field of the probe yields another uncertainty $(\delta\tilde{g})_2 \sim (GM/R^3)\delta x$. By the choice $\delta x \sim R$ (here not explained) and via Heisenberg relation $\delta x\delta v \sim \hbar/M$, we obtain $(\delta\tilde{g})_1 \sim \hbar/MRT$ and $(\delta\tilde{g})_2 \sim GM/R^2$. By varying M, the simultaneous minimization of $(\delta\tilde{g})_1$, $(\delta\tilde{g})_2$ results in eq. (3).

lae like (3), (6) or (8) are not valid for too short distances either in space or time. For example, ideal point-like massive objects are beyond our scope.

4. Master equation with gravitational damping term

Let us now turn to the effect of the stochastic white noise ϕ on the quantum state ψ of a given system. Formally, the state vector satisfies the Schrödinger equation

$$\mathrm{i}\hbar\dot{\psi}(t)=\left(\hat{H}_0+\int\phi(\boldsymbol{r},t)\hat{f}(\boldsymbol{r})\,\mathrm{d}^3r\right)\psi(t)\,, \tag{9}$$

where $\hat{H}_0$ is the nongravitational part of the hamiltonian, $\hat{f}(\boldsymbol{r})$ stands for the operator of the local mass density of the system.

Since the total hamiltonian is stochastic the state vector $\psi(t)$ becomes also a stochastic variable governed by the proper stochastic process. It is well known, however, that the physical relevant quantity is the density operator

$$\hat{\rho}(t)\equiv\langle\psi(t)\psi^{\dagger}(t)\rangle\,, \tag{10}$$

and it obeys a certain deterministic equation, of course.

Actually, for gaussian white noise (7), (8), eqs. (9) and (10) lead to the following master equation:

$$\dot{\hat{\rho}}(t)=\frac{-\mathrm{i}}{\hbar}[\hat{H}_0,\hat{\rho}(t)]$$

$$-\frac{G}{2\hbar}\iint\frac{\mathrm{d}^3r\,\mathrm{d}^3r'}{|\boldsymbol{r}-\boldsymbol{r}'|}[\hat{f}(\boldsymbol{r}),[\hat{f}(\boldsymbol{r}'),\hat{\rho}(t)]]\,. \tag{11}$$

The technics of deriving such markovian master equations is discussed in many places, see e.g. in refs. [11,12].

The above master equation is our central result. The second (damping) term on its r.h.s. represents a universal violation of ordinary quantum mechanics.

5. The nature of violation

Let X stand for the coordinates (both classical and spin ones) of the dynamical system in question. Given a configuration X the corresponding mass density at point $\boldsymbol{r}$ will be denoted by $f(\boldsymbol{r}|X)$. It is worthwhile to introduce the motion of characteristic time $\tau_{\mathrm{d}}(X,X')$ of damping by

$$[\tau_{\mathrm{d}}(X,X')]^{-1}$$

$$=\frac{G}{2\hbar}\iint\frac{[f(\boldsymbol{r}|X)-f(\boldsymbol{r}'|X')]^2\,\mathrm{d}^3r\,\mathrm{d}^3r'}{|\boldsymbol{r}-\boldsymbol{r}'|}\,, \tag{12}$$

for a given couple of configurations X and X'.

Observe that $\tau_{\mathrm{d}}=\infty$ if X and X' coincide; the larger the difference between the mass distribution represented by the configurations X and X', the shorter the characteristic time $\tau_{\mathrm{d}}(X,X')$ of damping.

Introducing the coordinate eigenstates $|X\rangle$ for our quantum system and by using the obvious relation $f(r|X)\delta(X'-X)\equiv\langle X'|\hat{f}(\boldsymbol{r})|X\rangle$, one can rewrite the master equation (11) as follows:

$$\langle X|\dot{\hat{\rho}}(t)|X'\rangle=(-\mathrm{i}/\hbar)\langle X|[\hat{H}_0,\hat{\rho}(t)]|X'\rangle$$

$$-[\tau_d(X,X')]^{-1}\langle X|\hat{\rho}(t)|X'\rangle\,. \tag{13}$$

We try to cast into words how violation of the quantum mechanics works. Due to the second term on the r.h.s. of eq. (13) the off-diagonal terms $\langle X|\hat{\rho}|X'\rangle$ of the density operator will tend to be damped according to the characteristic time (12). Consequently, the interference between states, say $|X\rangle$ and $|X'\rangle$ will be destroyed if the difference between the corresponding mass distributions $f(\boldsymbol{r}|X)$ and $f(\boldsymbol{r}|X')$ is essential.

6. The scale of violation

Our last task is to estimate the critical scale where the gravitational breakdown of the quantum mechanics is to take place. As the simplest choice we consider the dynamical system consisting of a single rigid spherical ball of homogeneously distributed mass m and of radius R. The ball is assumed to be free and we investigate its translational motion. Therefore the configuration X of the system is represented by the c.m. coordinate $\boldsymbol{x}$, solely.

Then the mass distribution function of the ball is $f(\boldsymbol{r}|\boldsymbol{x})=mV^{-1}\theta(R-|\boldsymbol{r}-\boldsymbol{x}|)$ where $V=\frac{4}{3}\pi R^3$ and θ is the step function. The characteristic time (12) of damping turns out to be

$$\tau_d(\boldsymbol{x},\boldsymbol{x}')=\hbar[U(|\boldsymbol{x}-\boldsymbol{x}'|)-U(0)]^{-1}, \tag{14}$$

here U stands for the gravitational pair potential between homogeneous spheres of mass m and of radius R:

$$U(r)\equiv -Gm^2 \iint_{z,z'\leqslant R} \frac{d^3z\, d^3z'}{|\boldsymbol{z}-\boldsymbol{z}'+\boldsymbol{r}|}$$

$$\sim -(Gm^2/R)(\tfrac{6}{5}-\tfrac{1}{2}r^2/R^2), \quad r\ll R,$$

$$\sim -Gm^2/r, \quad r\gg R. \tag{15}$$

Using expression (14), the master equation (13) takes the following form for the density operator $\hat{\rho}$ of the free ball:

$$\frac{d}{dt}\langle \boldsymbol{x}|\hat{\rho}|\boldsymbol{x}'\rangle = \frac{i\hbar}{2m}(\Delta-\Delta')\langle \boldsymbol{x}|\hat{\rho}|\boldsymbol{x}'\rangle$$

$$-\frac{1}{\hbar}[U(|\boldsymbol{x}-\boldsymbol{x}'|)-U(0)]\langle \boldsymbol{x}|\hat{\rho}|\boldsymbol{x}'\rangle. \tag{16}$$

For the special case $|\boldsymbol{x}-\boldsymbol{x}'|\ll R$, this equation has recently been analysed by Joos and Zeh [13], cf. also ref. [14].

Let us define the coherent width l of a given state $\hat{\rho}$ as follows: l is the characteristic distance $|\boldsymbol{x}-\boldsymbol{x}'|$ above which the off-diagonals $\langle \boldsymbol{x}|\hat{\rho}|\boldsymbol{x}'\rangle$ become negligibly small.

A crude estimation of the characteristic time of the kinetic changes of $\hat{\rho}$ coming from the first term on the r.h.s. of eq. (15) yields $ml^2/\hbar$. Note that the kinetic term usually increases the coherent width l of the state while the damping term tends to decrease it. The two effects become balanced when l equals a critical value l_{crit} satisfying

$$ml_{\text{crit}}^2/\hbar \sim \tau_d(l_{\text{crit}})$$

$$\equiv \hbar[U(l_{\text{crit}})-U(0)]^{-1}. \tag{17}$$

If the coherent width l of the actual quantum state is much smaller than the critical value l_{crit} then the standard quantum kinetics dominates and damping is not effective. On the other hand, if $l\gg l_{\text{crit}}$ then the coherence of the state will heavily be destroyed by the gravitational damping term in the master equation (13).

By means of the asymptotic expansions given on the very right of eq. (15), we can evaluate relation (17). We obtain

$$l_{\text{crit}} \sim (\hbar^2/Gm^3)^{1/4}R^{3/4}, \quad \text{if} \quad Rm^3\gg \hbar^2/G, \tag{18a}$$

$$\sim (\hbar^2/Gm^3)^{1/2}R^{1/2}, \quad \text{if} \quad Rm^3\ll \hbar^2/G. \tag{18b}$$

Table 1
Critical coherence width versus particle radius.

l_{crit} (cm)	R (cm)
10^3	10^{-10}
1	10^{-8}
10^{-3}	10^{-6}
10^{-6}	10^{-4}
10^{-9}	10^{-2}
10^{-12}	1

Similar estimations for critical coherence width were obtained in refs. [15,16].

Let us apply this result to the proton ($m\sim 10^{-24}$ g, $R\sim 10^{-13}$ cm), the typical form of massive matter in the microworld. Eq. (18b) yields $l_{\text{crit}}\sim 10^6$ cm and one can thus conclude that the coherence of the quantum states would be violated only for hugely large wave packets. So atomic systems are unaffected by damping.

Looking for violations we obviously need massive objects much more massive than some atom is. We have to regard objects which are big enough to have a large mass but are still small so that they had no internal excitation in effect. Let us choose a small rigid grain of normal (1 g cm^{-3}) density and assume for the extension $R\gg 10^{-12}$ cm. Then we can apply eq. (18a) resulting in

$$l_{\text{crit}} \sim 10^{-12}R^{-3/2} \text{ (cm)}, \tag{19}$$

see table 1.

Hence considering e.g. a typical colloid grain ($R\sim 10^{-5}$ cm) the critical coherent width l_{crit} will be of the order of the extension R of the grain. For larger objects the critical scale l_{crit} becomes even smaller, i.e. it gets microscopic. Therefore any wave packets of macroscopic extension will be destroyed by the proposed gravitational mechanism.

7. Conclusion

In section 4 we have proposed a universal quantum mechanical equation with a given gravitational damping term violating the ordinary quantum mechanics.

Markovian master equations, like ours is, were proposed recently by Ellis et al. [17] as a possible model for the violation of the quantum mechanics. The authors took Hawking's gravitational indeterminacy [18] as the theoretical implication for their work. Since Hawking indeterminacy was indicated on Planck scale ref. [18] looked for violations of the quantum mechanics in the microworld, first of all; as e.g. in long base line neutron interferometry.

From our master equation it follows that the violations act against high quantum fluctuations of the mass density and this would coincide well with our trivial macroscopic experiences. Then the breakdown of the quantum mechanics would be expected for massive systems rather than in the microworld and we should elaborate experimental tests in that direction [15].

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References

[1] I. Newton, Philosophiae naturalis principia mathematica (Streater, London, 1687).
[2] A. Einstein, Ann. Phys. (Leipzig) 49 (1916) 898.
[3] E. Schrödinger, Naturwissenschaften 23 (1935) 844.
[4] J.A. Wheeler, in: Battelle recontres, eds. C. DeWitt and J.A. Wheeler (Benjamin, New York, 1968).
[5] N. Bohr and L. Rosenfeld, K. Dans. Vidensk. Selsk. Mat. Fys. Medd. 12 (1933).
[6] L. Diósi and B. Lukács, KFKI-1985-46 preprint, Ann. Phys. (Leipzig) (1986), to be published.
[7] E.P. Wigner, Rev. Mod. Phys. 29 (1957) 255.
[8] A. Peres and N. Rosen, Phys. Rev. 118 (1960) 335.
[9] W.G. Unruh, in: Quantum theory of gravity, ed. S.M. Christensen (Hilger, Bristol, 1984).
[10] L. Smolin, Class. Quantum Grav. 3 (1986) 347.
[11] V. Gorini et al., Rep. Math. Phys. 13 (1978) 149.
[12] L. Diósi, Phys. Lett. A 112 (1985) 288.
[13] E. Joos and H.D. Zeh, Z. Phys. B 59 (1985) 223.
[14] E. Joos, Phys. Lett. A 116 (1986) 6;
H.D. Zeh, Phys. Lett. A 116 (1986) 9.
[15] F. Károlyházy, A. Frenkel and B. Lukács, in: Physics as natural philosophy, eds. A. Shimony and H. Feschbach (MIT Press, Cambridge, 1982).
[16] L. Diósi, Phys. Lett. A 105 (1984) 199.
[17] J. Ellis et al., Nucl. Phys. B 241 (1984) 381.
[18] S. Hawking, Commun. Math. Phys. 87 (1982) 395.