LOCALIZED SOLUTION OF A SIMPLE NONLINEAR QUANTUM LANGEVIN EQUATION

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A simple nonlinear quantum Langevin equation is introduced as phenomenological equation for quantum brownian motion. Easy calculations yield a unique localized wave function in the stationary regime. The given example may encourage more general use of nonlinear quantum Langevin equations for damped quantum systems, e.g. in measurement theory, in heavy ion physics, etc.

1. Introduction

The classical Langevin equation reflects the obvious fact that the equation of motion of a brownian particle must contain a stochastic term representing the interaction with the reservoir. This feature of the equation of motion will survive for quantum brownian motion as well: the quantum state vector ψ of the massive brownian particle satisfies the free Schrödinger equation *except for the instants of random collisions* with the lighter particles of the reservoir.

Following our previous paper [1], we introduce the *nonlinear quantum Langevin equation* (QLE) for the pure state operator $\hat{\rho}\equiv\psi\psi^+$ of the brownian particle:

$$\mathrm{d}\hat{\rho}=(-\mathrm{i}[\hat{p}^2/2m,\hat{\rho}]-\tfrac{1}{4}\gamma[\hat{q},[\hat{q},\hat{\rho}]])\,\mathrm{d}t+\{\hat{q}-\langle\hat{q}\rangle,\hat{\rho}\}\,\mathrm{d}\xi\,, \tag{1.1}$$

$\hat{q}$, $\hat{p}$ are the canonical coordinate and momentum operators, respectively, m is the mass of the particle. The constant γ characterizes the strength of interaction with the reservoir.

This equation is a so-called *stochastic differential equation* a la Itô (see Arnold's excellent review [2]), where ξ is a c-number Wiener process whose Itô *differential* $\mathrm{d}\xi$ obeys the following algebra:

$$\langle\mathrm{d}\xi\rangle_{\mathrm{st}}=0,\quad \mathrm{d}\xi\,\mathrm{d}\xi=\tfrac{1}{2}\gamma\,\mathrm{d}t\,,$$
$$(\mathrm{d}\xi)^n=0,\quad\text{if}\quad n=3,4,\dots\,. \tag{1.2}$$

Throughout our paper $\langle\ \rangle$ and $\langle\ \rangle_{\mathrm{st}}$ denote quantum and stochastic averages, respectively; ρ stands for the current *pure* quantum state while the usual *density operator* will consequently be denoted by $\langle\hat{\rho}\rangle_{\mathrm{st}}$.

It is crucial to see that eq. (1.1) retains the pure state property $\hat{\rho}\equiv\psi\psi^+\equiv\hat{\rho}^2$. Assuming $\hat{\rho}=\hat{\rho}^2$ holds for a given moment we are going to prove that $\mathrm{d}\hat{\rho}=\mathrm{d}\hat{\rho}^2$. Let us substitute eq. (1.1) into the r.h.s. of the identity $\mathrm{d}\hat{\rho}^2\equiv\mathrm{d}\hat{\rho}\,\hat{\rho}+\hat{\rho}\,\mathrm{d}\hat{\rho}+\mathrm{d}\hat{\rho}\,\mathrm{d}\hat{\rho}$:

$$\mathrm{d}\hat{\rho}^2=(-\mathrm{i}/2m)\{\hat{\rho},[\hat{p}^2,\hat{\rho}]\}\,\mathrm{d}t-\tfrac{1}{4}\gamma\{\hat{\rho},[\hat{q},[\hat{q},\hat{\rho}]]\}\,\mathrm{d}t+\{\hat{\rho},\{\hat{q}-\langle\hat{q}\rangle,\hat{\rho}\}\,\mathrm{d}\xi+\tfrac{1}{2}(\{\hat{q}-\langle\hat{q}\rangle,\hat{\rho}\})^2\,\mathrm{d}t\,. \tag{1.3}$$

The last term on the r.h.s. comes from the specific Itô correction $\mathrm{d}\hat{\rho}\,\mathrm{d}\hat{\rho}=(\{\hat{q}-\langle\hat{q}\rangle,\ \hat{\rho}\}\,\mathrm{d}\xi)^2$; cf. eqs. (1.2). Now recall the assumption $\hat{\rho}=\hat{\rho}^2$. The r.h.s. of eq. (1.3) becomes then identical to the r.h.s. of eq. (1.1) which, in turn, proves $\hat{\rho}^2\equiv\hat{\rho}$.

The main goal of the present paper is to show that the QLE (1.1) leads to a unique localized shape of the wave function.

2. Incidental remarks on the master equation

It is well known that, as far as one considers only observables belonging purely to the brownian par-

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ticle (but not to the reservoir), the knowledge of the density operator $\langle\hat{\rho}\rangle_{\rm st}$ provides full information. From eqs. (1.1) and (1.2) we see that the density operator fulfils the following linear *master equation* [3]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\rho}\rangle_{\rm st}=-\mathrm{i}[\hat{p}^2/2m,\langle\hat{\rho}\rangle_{\rm st}]$$
$$-\tfrac{1}{4}\gamma[\hat{q},[\hat{q},\langle\hat{\rho}\rangle_{\rm st}]]\,. \tag{2.1}$$

Hence, concerning the reduced dynamics of the brownian particle, the QLE (1.1) is physically equivalent to the master equation (2.1).

The QLE offers a certain pure state representation (see, e.g., ref. [4]) of the quantum brownian motion. It is intimately connected to quantum measurements optionally performed on the reservoir particles as we have clarified in our former paper [1]. The QLE (1.1) is equivalent to the equations of continuous position measurement theory [5–7].

Although the use of QLE (1.1) might seem to be redundant as compared to the master equation (2.1), it nevertheless helps us to recognize the physical characteristics of the quantum brownian motion which are not seen explicitly from eq. (2.1). Such feature is *wave function localization* which qualitatively follows [8,9] from eq. (2.1).

3. Stationary solution of the quantum Langevin equation

In this section we are going to show that the QLE (1.1) leads to a unique localized shape of the wave function in the stationary regime. The quantum expectation values of the position and momentum will move along a certain random path in the phase space.

In this section we use state vector formalism. Bearing in mind that $\hat{\rho}=\psi\psi^+$, we write the following QLE for the state vector:

$$\mathrm{d}\psi=\{[-\mathrm{i}(\hat{p}^2/2m)+\tfrac{1}{4}\gamma(\hat{q}-\langle\hat{q}\rangle)^2]\,\mathrm{d}t$$
$$+(\hat{q}-\langle\hat{q}\rangle)\,\mathrm{d}\xi\}\psi\,. \tag{3.1}$$

Sometimes it is more convenient to use the exponential form:

$$\psi+\mathrm{d}\psi=\exp\{[-\mathrm{i}(\hat{p}^2/2m)-\tfrac{1}{2}\gamma(\hat{q}-\langle\hat{q}\rangle)^2]\,\mathrm{d}t$$
$$+(\hat{q}-\langle\hat{q}\rangle)\,\mathrm{d}\xi\}\psi\,. \tag{3.2}$$

Exploiting the rules of the Itô algebra (1.2) for ξ, one can verify that eq. (3.1), as well as eq. (3.2), leads to the QLE (1.1).

Obviously, nobody expects a stationary solution of the above QLE itself. On physical grounds, however, we do expect it in the co-moving system of the particle. Therefore we are going to transform the QLE (3.2) into the co-moving system.

For each moment, let us transform the current state vector ψ as follows:

$$\tilde{\psi}=\exp(-\mathrm{i}\langle\hat{p}\rangle\hat{q})\,\exp(\mathrm{i}\langle\hat{q}\rangle\hat{p})\psi\,. \tag{3.3}$$

The new vector $\tilde{\psi}$ corresponds to the state viewed from the co-moving system of the particle. The state $\tilde{\psi}$ satisfies the identities

$$\tilde{\psi}^+\hat{q}\tilde{\psi}\equiv0,\quad \tilde{\psi}^+\hat{p}\tilde{\psi}\equiv0\,, \tag{3.4}$$

for all times.

If we perform the transformation (3.3) on both sides of the QLE (3.2) we obtain

$$\tilde{\psi}+\mathrm{d}\tilde{\psi}=\exp[-\mathrm{i}(\langle\hat{p}\rangle+\mathrm{d}\langle\hat{p}\rangle)\hat{q}]$$
$$\times\exp[\mathrm{i}(\langle\hat{q}\rangle+\mathrm{d}\langle\hat{q}\rangle)\hat{p}]$$
$$\times\exp\{[-\mathrm{i}(\hat{p}^2/2m)-\tfrac{1}{2}\gamma(\hat{q}-\langle\hat{q}\rangle)^2]\,\mathrm{d}t$$
$$+(\hat{q}-\langle\hat{q}\rangle)\,\mathrm{d}\xi\}\psi\,, \tag{3.5}$$

where $\mathrm{d}\langle\hat{p}\rangle=\mathrm{tr}(\hat{p}\,\mathrm{d}\hat{\rho})$ and, similarly, $\mathrm{d}\langle\hat{q}\rangle=\mathrm{tr}(\hat{q}\,\mathrm{d}\hat{\rho})$. If we take $\mathrm{d}\hat{\rho}$ from, e.g., eq. (1.1) we get the following results:

$$\mathrm{d}\langle\hat{p}\rangle=\langle\{\hat{p}-\langle\hat{p}\rangle,\hat{q}-\langle\hat{q}\rangle\}\rangle\,\mathrm{d}\xi\equiv2R\,\mathrm{d}\xi\,,$$
$$\mathrm{d}\langle\hat{q}\rangle=m^{-1}\langle\hat{p}\rangle\,\mathrm{d}t+2\langle(\hat{q}-\langle\hat{q}\rangle)^2\rangle\,\mathrm{d}\xi$$
$$\equiv m^{-1}\langle\hat{p}\rangle\,\mathrm{d}t+2\sigma^2\,\mathrm{d}\xi\,, \tag{3.6}$$

with obvious shorthand notations R and σ^2.

Let us choose a special coordinate system where $\langle\hat{p}\rangle=\langle\hat{q}\rangle=0$ at the given instant t when ψ is considered. In this frame ψ coincides with $\tilde{\psi}$ for the given moment t (but $\mathrm{d}\tilde{\psi}\neq\mathrm{d}\psi$, of course). Using the equality $\tilde{\psi}=\psi$ and substituting eqs. (3.6) into eq. (3.5) one gets

$$\tilde{\psi}+\mathrm{d}\tilde{\psi}=\exp(-2\mathrm{i}R\hat{q}\,\mathrm{d}\xi)\,\exp(2\mathrm{i}\sigma^2\hat{p}\,\mathrm{d}\xi)$$
$$\times\exp\{[-\mathrm{i}(\hat{p}^2/2m)-\tfrac{1}{2}\gamma\hat{q}^2]\,\mathrm{d}t+\hat{q}\,\mathrm{d}\xi\}\tilde{\psi}\,. \tag{3.7}$$

This is already a closed QLE for $\tilde{\psi}$ since R and σ^2 can be calculated in the state $\tilde{\psi}$ as well (cf. eqs. (3.6)):

$$R=\tfrac{1}{2}\tilde{\psi}^{+}\{\hat{p},\hat{q}\}\tilde{\psi}, \quad \sigma^2=\tilde{\psi}^{+}\hat{q}^2\tilde{\psi}. \tag{3.8}$$

Let us observe the following important point. All $\tilde{\psi}$'s are always invariant under galilean transformations of the frame. Hence the validity of eqs. (3.7) and (3.8) will be independent of the special frame where $\langle\hat{p}\rangle=\langle\hat{q}\rangle=0$ has been required. These equations are valid in arbitrary inertial frames.

What is left is to write the product of the three exponential factors in eq. (3.7) into a more convenient form. Up to irrelevant phase factors one obtains

$$\begin{aligned}\tilde{\psi}+\mathrm{d}\tilde{\psi}=&\exp\{[-\mathrm{i}(\hat{p}^2/2m)-\tfrac{1}{2}\gamma(\hat{q}^2-\sigma^2)]\,\mathrm{d}t\\&+(\hat{q}-2\mathrm{i}R\hat{q}+2\mathrm{i}\sigma^2\hat{p})\,\mathrm{d}\xi\}\tilde{\psi}.\end{aligned} \tag{3.9}$$

This is the wanted QLE in the co-moving system of the observed particle. Remember that R and σ^2 are $\tilde{\psi}$-dependent quantities, cf. eq. (3.8).

Let us find the stationary solutions of the above QLE. They must be of the form $\tilde{\psi}_\infty\exp(-\mathrm{i}Et)$ where $\tilde{\psi}_\infty$ is the time-independent part and E is a real number. Substituting this ansatz into the QLE (3.9) yields two equations for the stationary solution $\tilde{\psi}_\infty$:

$$[(\hat{p}^2/2m)-\tfrac{1}{2}\mathrm{i}\gamma(\hat{q}^2-\sigma^2)]\tilde{\psi}_\infty=E\tilde{\psi}_\infty, \tag{3.10a}$$

$$[(1-2\mathrm{i}R)\hat{q}+2\mathrm{i}\sigma^2\hat{p}]\tilde{\psi}_\infty=0. \tag{3.10b}$$

Surprisingly, it is very easy to solve this set of nonlinear equations. If one introduces the wave function $\tilde{\psi}_\infty(q)$ eq. (3.10b) turns into the corresponding ordinary differential equation yielding solutions

$$\tilde{\psi}_\infty(q)=\mathrm{const}\times\exp[-(1-2\mathrm{i}R)(q/2\sigma)^2]$$

with arbitrary real R and σ. Then, if we substitute this solution into the *frictional Schrödinger equation* [4] (3.10a) we find the *unique* solution $E=\tfrac{1}{2}\sqrt{\tfrac{1}{2}\gamma/m}$ while for R and σ^2 we obtain the following stationary values:

$$R_\infty=\tfrac{1}{2}, \quad \sigma_\infty^2=1/\sqrt{2\gamma m}. \tag{3.11}$$

Therefore the stationary solution of the QLE (3.9) takes the following unique form:

$$\begin{aligned}\tilde{\psi}_\infty(q)=&(2\pi\sigma_\infty^2)^{-1/4}\\&\times\exp[-(1-\mathrm{i})(q/2\sigma_\infty)^2].\end{aligned} \tag{3.12}$$

As expected, this solution is localized. Applying the stationary values (3.11) of R and σ^2 to the classical stochastic differential equation (3.6) we get

$$\begin{aligned}&\mathrm{d}\langle\hat{p}\rangle=\mathrm{d}\xi,\\&\mathrm{d}\langle\hat{q}\rangle=(1/m)\langle\hat{p}\rangle\,\mathrm{d}t+2\sigma_\infty^2\,\mathrm{d}\xi.\end{aligned} \tag{3.13}$$

These equations govern the stationary random walk of the wave packet (3.12) through the phase space spanned by the quantum expectation values $\langle\hat{p}\rangle$ and $\langle\hat{q}\rangle$ viewed from the laboratory system.

The stationary solution (3.12) was first derived in ref. [10]. The same paper presented the following Fokker–Planck equation for the phase space distribution $\rho(\langle\hat{q}\rangle,\langle\hat{p}\rangle,t)$ of quantum expectation values in the stationary regime:

$$\begin{aligned}\frac{\partial\rho}{\partial t}=&-\frac{p}{m}\frac{\partial\rho}{\partial q}\\&+\frac{1}{2}\left(\frac{1}{m}\frac{\partial^2}{\partial q^2}+\sqrt{\frac{2\gamma}{m}}\frac{\partial^2}{\partial p\partial q}+\tfrac{1}{2}\gamma\frac{\partial^2}{\partial p^2}\right)\rho.\end{aligned} \tag{3.14}$$

(Here we used the shorthand notations p, q instead of $\langle\hat{p}\rangle$, $\langle q\rangle$.) This equation is equivalent to our stochastic differential equations (3.13) recalling that ξ obeys the Itô algebra (1.2).

One would obviously expect that, for $t=\infty$, each solution of the QLE (3.9) tends to the single stationary one (3.12). Although we are still unable to prove the global stability of the solution (3.12), we conjecture its local stability. If $\tilde{\psi}$ is close to $\tilde{\psi}_\infty$, i.e. $\tilde{\psi}=\tilde{\psi}_\infty+\delta\tilde{\psi}$ is assumed where $\delta\tilde{\psi}$ is small and orthogonal to $\tilde{\psi}_\infty$ then, via the QLE (3.9), not too lengthy calculations lead to the result

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\|\delta\psi\|^2\rangle_{\mathrm{st}}=-2\gamma\sigma_\infty^{-2}|\delta\tilde{\psi}^{+}\hat{q}\tilde{\psi}|^2\leqslant 0 \tag{3.15}$$

in the lowest non-vanishing order of $\delta\tilde{\psi}$.

4. Conclusion

A simple nonlinear quantum Langevin equation has been introduced to describe the evolution of the quantum state of a brownian particle. This phenomenological equation yields analytical localized solutions of gaussian shape in the stationary regime. The stability of this shape needs further investigations. Nonlinear QLEs to be new alternative tools to formulate quantum damping in more complicated systems too, including fields as measurement theory, heavy ion physics etc.

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