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Coarse graining and decoherence translated into von Neumann language

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The notions of coarse graining and decoherence, introduced recently by Gell-Mann and Hartle for quantum cosmology, have been reformulated in usual terms of the formal von Neumann measurement theory. The formalism of continuous measurement has shown to be equivalent to coarse graining, with the soft version of decoherence.

The so-called quantum measurement problem culminates in quantum cosmology [1]. After a longstanding (though often metaphysical) discontent with the quantum measurement theory of von Neumann [2], it seems inevitable to generalize the quantum theory itself, as done most recently by, e.g., Gell-Mann and Hartle (GMH) [3] applying the concept of coarse graining and decoherence.

There have, however, been a great deal of similar attempts from the part of measurement theorists too, motivated by the aforementioned (and incriminated) sort of discontent. In particular, modified quantum theories [4–9] have been based on generalizations of the notion of the von Neumann measurement, concluding to a concept of continuous spontaneous measurement [10–14].

One nearly does not dare use the term "measurement" because of its anthropocentric look. All over this paper, however, measurement is only a historic term to denote the mathematically well defined collapse of the quantum state. Furthermore, the frequently blamed "measurement problem" is a historic term for the longstanding pseudoproblem which has finally been crystallized into a definite physical problem in quantum cosmology, as emphasized in ref. [1].

GMH [3] do not use the notion of measurement and of collapse either. They, nevertheless, use a complete time-dependent set $\{P_\alpha(t);\alpha=1,2,...\}$ of orthogonal hermitian projectors. Consider an increasing sequence $t_1, t_2, ..., t_N$. A certain coarse grained history h is defined by a given sequence $h=(\alpha_1, \alpha_2, ..., \alpha_N)$. Then GMH introduce the corresponding history-dependent coarse grained quantum state as follows:

$$\Psi(h)=P_{\alpha_N}(t_N)\,...P_{\alpha_2}(t_2)P_{\alpha_1}(t_1)\Psi_0$$
$$\equiv T\left(\prod_{n=1}^{N} P_{\alpha_n}(t_n)\right)\Psi_0\,, \tag{1}$$

here Ψ_0 is the original (fine grained) Heisenberg state and T denotes time ordering of the operators. To a given history one assigns the probability

$$w(h)\equiv D(h|h)=\|\Psi(h)\|^2\,. \tag{2}$$

Such probabilities are compatible with each other provided all pairs of considered histories decohere. Two (nonidentical) histories h and h' are said to decohere if their decoherence functional D vanishes, i.e.,

$$D(h'|h)\equiv\Psi^+(h')\Psi(h)=0,\quad h'\neq h\,. \tag{3}$$

Eqs. (1)–(3) represent GMH's proposal [3], perhaps not in its most general version, for the quantum theory of decohering coarse grained histories.

No doubt, the above model is a radical generalization of the standard quantum theory. One can, nevertheless, translate it into the familiar language of the von Neumann measurements. It deserves to perform the translation because the standard language makes elementary discussions easier.

Consider the time dependent observable

$$Q(t)=\sum_{\alpha}\alpha P_{\alpha}(t) \tag{4}$$

composed of GMH's projectors. Observe that *coarse graining (1), (2) is mathematically equivalent to N subsequent measurements of $Q(t_1)$, $Q(t_2)$, ..., $Q(t_N)$ to perform à la von Neumann.* The history h is given by the sequence $(\alpha_1, \alpha_2, ..., \alpha_N)$ of the individual measurement outcomes. The history-dependent state (1) turns out to be the resulting state after the N measurements and, furthermore, the probability (2) of the corresponding history can be recognized as the standard probability of the N subsequent wavefunction collapses.

We owe to translate the crucial decoherence criterion (3) as well. Let us consider the most simple case by taking $N=2$, and consider decoherence between histories $h=(\alpha_1, \alpha_2)$ and $h'=(\beta_1, \beta_2)$, respectively. Invoking eq. (3), the decoherence functional

$$D(h'|h)\equiv D(\beta_1,\beta_2|\alpha_1,\alpha_2)$$
$$\equiv \Psi_0^+ P_{\beta_1}(t_1)P_{\beta_2}(t_2)P_{\alpha_2}(t_2)P_{\alpha_1}(t_1)\Psi_0 \tag{5}$$

must vanish for nonidentical histories:

$$D(\beta_1,\beta_2|\alpha_1,\alpha_2)=0\,, \quad \text{if } \alpha_1\neq\beta_1 \text{ or } \alpha_2\neq\beta_2\,. \tag{6}$$

It is now straightforward to prove the following statement: if eq. (6) is satisfied then the expectation value of $Q(t_2)$ in a von Neumann measurement at t_2 will be independent of whether $Q(t_1)$ was earlier (at $t_1<t_2$) measured or was not measured at all. Let us see the proof.

If $Q(t_1)$ was not measured at all then the von Neumann theory yields the following expectation value for the measurement of $Q(t_2)$ at t_2:

$$\langle Q(t_2)\rangle=\Psi_0^+ Q(t_2)\Psi_0\,. \tag{7}$$

On the other hand, if $Q(t_1)$ was measured then, according to von Neumann,

$$\langle Q(t_2)\rangle_{Q(t_1)\text{measured}}$$
$$=\sum_{\alpha_1}\Psi_0^+ P_{\alpha_1}(t_1)Q(t_2)P_{\alpha_1}(t_1)\Psi_0\,, \tag{8}$$

where the spectral expansion (4) has been applied to $Q(t_1)$. Inserting the similar spectral expansion of $Q(t_2)$ as well, and invoking the definition (5) of the decoherence functional, the RHS of eq. (8) reads

$$\sum_{\alpha_1}\Psi_0^+ P_{\alpha_1}(t_1)Q(t_2)P_{\alpha_1}(t_1)\Psi_0$$
$$=\sum_{\alpha_2}\sum_{\alpha_1}\alpha_2 D(\alpha_1,\alpha_2|\alpha_1,\alpha_2)\,. \tag{9}$$

Now, from the decoherence condition (6) follows the identity:

$$\sum_{\alpha_2}\sum_{\alpha_1}\alpha_2 D(\alpha_1,\alpha_2|\alpha_1,\alpha_2)$$
$$=\sum_{\alpha_2}\sum_{\beta_1}\sum_{\alpha_1}\alpha_2 D(\alpha_1,\alpha_2|\beta_1,\alpha_2)\,. \tag{10}$$

Considering again eq. (5) as well as invoking the completeness and idempotentness of the projectors $\{P_\alpha(t)\}$, the RHS of eq. (10) reads

$$\sum_{\alpha_2}\sum_{\beta_1}\sum_{\alpha_1}\alpha_2 D(\alpha_1,\alpha_2|\beta_1,\alpha_2)$$
$$=\sum_{\alpha_2}\alpha_2\Psi_0^+ P_{\alpha_2}(t_2)\Psi_0\,. \tag{11}$$

Eq. (4) shows the RHS of eq. (11) is equal to $\Psi_0^+ Q(t_2)\Psi_0$. Hence, collecting eqs. (7)–(11) one obtains the identity which was to be verified:

$$\langle Q(t_2)\rangle=\langle Q(t_2)\rangle_{Q(t_1)\text{measured}}\,. \tag{12}$$

Certainly we can generalize the above result as follows. *The decoherence condition (3) is equivalent to require that the von Neumann measurements $Q(t_1)$, $Q(t_2)$, ..., $Q(t_N)$ do not disturb each other* in a sense of eq. (12).

Since the translation of the quantum theory of decohering coarse grained histories [3] into von Neumann terms has been successfully performed, one can discuss it on the language of the standard measurement theory [2]. In particular, the decoherence condition (3) will provoke some doubts when confronted with conventions of measurement theory. Namely, we think that constructing a sequence of nondisturbing observables is not a trivial task at all. On the contrary, it has been the central subject of the historical measurement problem in quantum mechanics. And it is still open. Nevertheless, a special progress has been achieved.

The concept of nondisturbing measurements seems hopeless restrictive; we have to soften it. In particular, we can specify fuzzy measurements instead of von Neumann's ones. As a reward, the concept of (time-) continuous measurement (i.e., of permanent coarse graining) can be introduced, cf. refs. [10–14]. Let us summarize it in short.

If a certain quantized variable $Q(t)$ is to be coarse grained, a formal continuous measurement will be assumed for it. The c-number function $\bar{Q}(t)$ of measurement outcomes will represent the measured history, according to GMH's philosophy. We shall write $\bar{Q}$ for history, instead of h. The history dependent state is then equal to

$$\Psi(\bar{Q}) = T\exp\left(-\tfrac{1}{2}\gamma \int [Q(t)-\bar{Q}(t)]^2\,\mathrm{d}t\right)\Psi_0\,, \quad (13)$$

where γ characterizes the strength of the continuous measurement, i.e., the strength of continuous coarse graining.

One may observe that expression (13) is a smoothed time-continuous version of GMH's one (1). For completeness, let us write down the counterpart of GMH's equation (2) for the probability of a given history:

$$w(\bar{Q}) \equiv D(\bar{Q}|\bar{Q}) = \|\Psi(\bar{Q})\|^2\,. \quad (14)$$

It is shown [13] that eqs. (12) and (13) of continuous measurement theory can be cast into stochastic differential equations which offer very flexible mathematical tools for explicit calculations, cf. refs. [8,9,15–18].

The properly softened version of GMH's decoherence criterion (3) will be the following:

$$D(\bar{Q}'|\bar{Q}) \equiv \Psi^+(\bar{Q}')\Psi(\bar{Q}) \approx 0\,, \quad (15)$$

if $\bar{Q}'$ and $\bar{Q}$ differ "much". The fulfillment of such a criterion follows from the very nature of continuous measurement. Exact results are obtained, e.g., for a free particle subjected to continuous position measurement [13,18].

We claim that quantum theories with spontaneous continuous measurement are quantum theories with (continuous) coarse graining and vice versa.

The concept of continuous measurement has a few formal results in quantum cosmology cf., e.g., refs. [19–21]. Of course, one expect more applications in the future. Still, we mention a nonrelativistic caricature of the quantum cosmology with coarse graining (or, in alternative terminology, with formal continuous measurement).

According to the author's recent proposal [8,14], the coarse grained (i.e. spontaneously measured) history variable Q (4) is, by assumption, the newtonian gravitational field strength. The strength parameter of coarse graining (13) is chosen to be inverse proportional to the product of Planck's and Newton's constants: $\gamma = \mathrm{const.}/\hbar G$. Then eqs. (13), (14) leads to a generalized (coarse grained, if you like) quantum mechanics in stochastic differential equation form.

Perhaps the general lesson of the present paper is that the mathematical and logical machinery of the von Neumann measurement theory cannot be ignored by any generalization of the standard quantum theory. Even if one make a radical trial of a new formulation, the elements of the von Neumann measurements will be, nolens volens, recognizable in it. This is probably valid also for the most recent formulation of coarse grained histories [22] which appeared during the completion of the present work.

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