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On the maximum number of decoherent histories

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Abstract

It is shown that the upper limit for the number of consistent histories of an N-state quantum system is N^2 in the theory of Gell-Mann and Hartle as well as in the theory of Griffiths.

Several versions of consistent history theories [1–5] have been proposed earlier to incorporate the interpretation of quantum state into the basic equations of the formalism. Here we are going to prove that the maximum number of consistent histories is equal in both the *decoherent history* and *consistent quantum trajectory* theories of Gell-Mann and Hartle (GMH) [3], and of Griffiths [5], respectively. (Further algebraic properties have been explored independently by Dowker and Kent [6].)

Let us consider a closed N-state quantum system whose state space is the Hilbert space $\mathcal{H}$ of N dimensions. Introduce a time-ordered sequence of events described by the Heisenberg operators

$$P_{\alpha_n}(t_n), P_{\alpha_{n-1}}(t_{n-1}), \dots, P_{\alpha_1}(t_1), \tag{1}$$

where the $P_{\alpha_k}(t_k)$ form complete orthogonal sets of Hermitian projectors for each $k = 0, 1, \dots, n$ in turn. A subset of sequences (1) forms a family of *consistent trajectories* à la Griffiths provided (i) all projectors $P_{\alpha_k}(t_k)$ are one-dimensional, (ii) nonvanishing overlaps between arbitrarily given pairs of initial and final events of respective labels α_1, α_n factorize,

$$\begin{aligned} &\mathrm{tr}[P_{\alpha_n}(t_n)P_{\alpha_1}(t_1)] \\ &\quad = \mathrm{tr}[P_{\alpha_n}(t_n)P_{\alpha_{n-1}}(t_{n-1})] \\ &\quad \times \mathrm{tr}[P_{\alpha_{n-1}}(t_{n-1})P_{\alpha_{n-2}}(t_{n-2})] \dots \\ &\quad \times \mathrm{tr}[P_{\alpha_2}(t_2)P_{\alpha_1}(t_1)], \end{aligned} \tag{2}$$

and (iii) the r.h.s. of this equation vanishes except for at most one choice of the labels $\alpha_2\alpha_3\dots\alpha_{n-1}$, for fixed α_1 and α_n. The overlap between $P_{\alpha_1}(t_1)$ and $P_{\alpha_n}(t_n)$ is then attributed to the trajectory as its (statistical) weight,

$$w(\alpha) = \mathrm{tr}[P_{\alpha_n}(t_n)P_{\alpha_1}(t_1)]. \tag{3}$$

Obviously, the number of consistent trajectories in a given family is equal to the number of nonvanishing overlaps which is N^2 when maximum.

To define *decoherent histories* of GMH we start again from the sequences of events (1) and introduce the corresponding Heisenberg operators

$$C_\alpha \equiv P_{\alpha_n}(t_n)\dots P_{\alpha_2}(t_2)P_{\alpha_1}(t_1). \tag{4}$$

The GMH construction depends on the Heisenberg state ρ of the system, too. The C_α are said to generate a family of *decoherent histories* [3] provided the so-called *decoherence functional* is diagonal,

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$$D(\alpha,\beta) \equiv \mathrm{tr}(C_\beta^\dagger C_\alpha \rho) = 0 \quad \text{for all } \alpha \neq \beta. \tag{5}$$

Its diagonal elements will then be assigned to decoherent histories as their probabilities p,

$$p(\alpha) = D(\alpha,\alpha). \tag{6}$$

In GMH theory the projectors $P_{\alpha_k}(t_k)$ in Eq.(4) may be of dimensions greater than 1. To make a comparison with Griffiths' theory natural we consider *fine-grained* decoherent histories [3], i.e. we require that $P_{\alpha_k}(t_k)$ be pure-state projectors. We show that N^2 is the upper limit for the number of fine-grained decohering histories in GMH families, too. Let us denote the rank of ρ by R. An orthogonal expansion $\rho = \sum_{r=1}^R w_r |\psi_r\rangle\langle\psi_r|$ is always possible with nonnegative normalized weights w_r. Consider the trivial embedding of our system into a larger one whose state space is $\mathcal{H}\otimes\mathcal{H}'$, where $\mathcal{H}'$ is an R-dimensional Hilbert space. Construct the following state vector in $\mathcal{H}\otimes\mathcal{H}'$,

$$|\Psi\rangle = \sum_{r=1}^{R} \sqrt{w_r}|\psi_r\rangle \otimes |\psi_r'\rangle, \tag{7}$$

where $\{|\psi_r'\rangle, r = 1,\dots,R\}$ form a complete orthonormal system in $\mathcal{H}'$. Introduce the following (unnormalized) vectors in $\mathcal{H}\otimes\mathcal{H}'$,

$$|\Phi_\alpha\rangle = (C_\alpha \otimes 1)\,|\Psi\rangle. \tag{8}$$

From Eqs. (5), (7), (8) one can prove that these vectors are also orthogonal to each other,

$$\langle\Phi_\beta|\Phi_\alpha\rangle = 0 \quad \text{for all } \alpha \neq \beta. \tag{9}$$

In the Hilbert space $\mathcal{H}\otimes\mathcal{H}'$, the maximum number of orthogonal vectors is equal to the number $N \times R$ of dimensions. In such a way we have proven that, for nondegenerate ρ, the maximum number of histories in a given decohering family is N^2. (Observe that the above proof would equally have been valid had the constituting events been described by higher (than 1) dimensional orthogonal projectors $P_{\alpha_k}(t_k)$. Of course, the bound N^2 applies also to the multiplicity of coarser-grained histories *within* a GMH decoherent family.)

In general, the consistency conditions of GMH and of Griffiths lead to different families of quantum histories. We have nevertheless shown that the maximum number of consistent quantum histories is identical at both GMH and Griffiths conditions. This result raises the general question: do ever GMH and Griffiths conditions allow completely identical families of quantum histories? The author has obtained a partial answer [8] and the issue deserves further investigations certainly.

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