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Critique of proposed limit to space-time measurement, based on Wigner's clocks and mirrors

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Abstract. – Based on a relation between inertial time intervals and the Riemannian curvature, we show that the space-time uncertainty derived by Ng and van Dam appears to imply unrealistic uncertainties of the Riemannian curvature.

Recently, Ng and van Dam [1], [2] presented a proof of the intrinsic quantum uncertainty $\delta\ell$ of *any* geodetic length ℓ being proportional to the one-third power of the length itself:

$$\delta\ell = \ell_{\mathrm{P}}^{2/3}\ell^{1/3}\,, \tag{1}$$

where ℓ_{P} is the Planck length. In addition, they claim that an intrinsic uncertainty of space-time metric has been derived in ref. [1], [2]. Now, the problem deserves a discussion since, a few years ago, the present authors [3] pointed out that the formula (1) would certainly overestimate the uncertainty of the space-time. We suggest [3] that this formula *would be* the uncertainty of a distincted world line whose length is measured at the price of total ignorance about the lengths of any other neighbouring world lines. In a sense, the uncertainties of all neighbouring world lines within about a tube of diameter ℓ will charge the uncertainty of the distincted one.

Calculate, for instance, the mass m of the clock when adjusted according to eqs. (3) and (4) of Ng and van Dam:

$$m = m_{\mathrm{P}}\left(\frac{\ell}{\ell_{\mathrm{P}}}\right)^{1/3}, \tag{2}$$

where m_{P} is the Planck mass. We note that the optimum measurement of a length $\ell \approx 1\,\mathrm{cm}$ requires a clock of mass $m \approx 10^6\,\mathrm{g}$ and, similarily, the optimum measurement of a time-like distance $t \approx 1\,\mathrm{s}$ needs a clock with $m \approx 10^{16}\,\mathrm{g}$ (*i.e.* 10^{10} metric tons!). Of course, the large mass of the clock needed to reach the limit of accuracy is not a proof against the proposed

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fluctuation formula. But we can show that eq. (1) leads to drastic effects in the space-time continuum, strongly affecting macroscopy. That eq. (1) appears to seriously overestimate the uncertainty of the space-time can now be shown by an independent elementary consideration.

Let us start with formula (6) of Ng and van Dam:

$$\delta t = t_{\mathrm{P}}^{2/3} t^{1/3}\,, \tag{3}$$

where δt is the proposed uncertainty of the time t along an arbitrarily chosen time-like geodesic and t_{P} is the Planck time. This uncertainty implies a certain uncertainty of the physical space-time geometry. One expects that the corresponding fluctuations of the local Riemann curvature are fairly small.

Fortunately, there exists a simple relation between a subtle triplet of time intervals on the one hand and the average Riemannian curvature on the other. We recapitulate this relation according to Wigner [4].

Assume space-time is flat on average. Take a clock and at a distance $\ell/2$ a mirror; for simplicity's sake let them be at rest relative to each other. Let us emit a light signal from the clock to the mirror, and let the clock measure the total flight time t_1 as the signal has got back to it. Repeat the same experiment immediately after, for the flight time t_2, and similarly for a third one t_3. Then, the average $\overline{R}_{0101}$ of the 0101 component of the Riemann curvature tensor in the space-time region swept by the light pulses is

$$\overline{R}_{0101} = 2\frac{t_1\, t_3 - t_2^2}{c^2\, t_2^4}\,, \tag{4}$$

provided both clock and mirror lay along the first coordinate axis [4].

Let us obtain the quantum uncertainty $\delta\overline{R}_{0101}$ of the above curvature. Of course, each period t_i ($i = 1, 2, 3$) has the same average value ℓ/c. Their quantum uncertainties δt_i are also equal. According to Ng and van Dam, any time-like geodesic length possesses the ultimate uncertainty (3); so do ours, too:

$$\delta t_i = \left(\frac{\ell_{\mathrm{P}}}{c}\right)^{2/3}\left(\frac{\ell}{c}\right)^{1/3}. \tag{5}$$

If $\ell \gg \ell_{\mathrm{P}}$ the periods t_i are much larger than their fluctuations (5) and, consequently, we can approximate the uncertainty of the curvature (4) by an expession linear in δt_i:

$$\delta\overline{R}_{0101} = \frac{2c}{\ell^3}\delta(t_1 - 2t_2 + t_3)\,. \tag{6}$$

To calculate the squared average value of $\delta\overline{R}_{0101}$, one rewrites the above equation in the following equivalent form:

$$\begin{aligned}[\delta\overline{R}_{0101}]^2 &= \left(\frac{2c}{\ell^3}\right)^2 \big(3[\delta t_1]^2 + 9[\delta t_2]^2 + 3[\delta t_3]^2 - \\ &\quad -3[\delta(t_1+t_2)]^2 - 3[\delta(t_2+t_3)]^2 + [\delta(t_1+t_2+t_3)]^2\big)\,.\end{aligned} \tag{7}$$

Each term on the r.h.s. is then evaluated by means of eq. (3). After extracting a root, one obtains

$$\delta\overline{R}_{0101} = 2\sqrt{15 - 6\times 2^{2/3} + 3^{2/3}}\,\frac{1}{\ell^2}\left(\frac{\ell_{\mathrm{P}}}{\ell}\right)^{2/3}. \tag{8}$$

It seems plausible to assume that the r.h.s. of eq. (8) yields the order of magnitude not only for the Riemann-tensor components but for the components of Ricci tensor as well as for the

Riemann scalar $\overline{R}$ unless special statistical correlation is shown or at least assumed between the various components of the Riemann tensor. So, eq. (8) yields the following estimation for the Riemann scalar $\overline{R}$ averaged in a 4-volume $\sim \ell^4/c$:

$$\delta\overline{R} \sim \frac{1}{\ell^2}\left(\frac{\ell_{\mathrm{P}}}{\ell}\right)^{2/3}. \tag{9}$$

Basically, one would expect with Ng and van Dam that these fluctuations are *small*. There is at least one good criterion to test their smallness. According to the Einstein theory of general relativity, the non-zero scalar curvature R assumes non-zero energy density. If we assume that the energy-momentum tensor is dominated by the energy density ρ, then the fluctuation (9) of the Riemann scalar would imply

$$\delta\bar{\rho} \sim (c^2/G)\delta\overline{R} \sim (\hbar/c)\ell_{\mathrm{P}}^{-2/3}\ell^{-10/3}, \tag{10}$$

where $\delta\bar{\rho}$ denotes the universal fluctuation of the energy density ρ averaged in a 4-volume $\sim \ell^4/c$. This fluctuation would be *extremly high* at small length scales. At $\ell \sim 10^{-5}$ cm, for instance, the uncertainty $\delta\bar{\rho}$ would be in the order of water density; that is trivially excluded by experience. According to recent cosmological estimations, *e.g.* from galaxy counts, the average mass density of our Universe should not exceed $10^{-29}\,\mathrm{g\,cm^{-3}}$. Then, eq. (10) yields $\ell \gg 10^4$ cm which in turn means that the proposal of Ng and Dam for the uncertainty of geodesic length may not be applied for lengths shorter than some 100 metres, otherwise one might get another universe due to the additional cosmologic mass density generated by the short-range metric fluctuations.

According to all these arguments, we think that Ng and van Dam in [1], [2] have in fact derived an *unconditional* uncertainty for a *single* geodesic. However, the uncertainty of a single geodesic length should not be used to calculate the intrinsic uncertainty of the space-time metric: it would need the simultaneous uncertainties of *all* geodesics or at least of a subtle subset of all. We pointed out that to ignore the correlations of those uncertainties would lead to high uncertainties of the space-time curvature. Finally, it is worth mentioning that a detailed account of the present authors' alternative to replace eq. (1) can be found in ref. [3].

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