

Exact positivity of the Wigner and P -functions of a Markovian open system

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Abstract

We discuss the case of a Markovian master equation for an open system, as it is frequently found from environmental decoherence. We prove two theorems for the evolution of the quantum state. The first one states that for a generic initial state the corresponding Wigner function becomes strictly positive after a finite time has elapsed. The second one states that also the P -function becomes exactly positive after a decoherence time of the same order. Therefore, the density matrix becomes exactly decomposable into a mixture of Gaussian pointer states.

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1. Introduction

The study of Markovian open systems is of general interest. One of the reasons is that the coupling of open systems to their ubiquitous environment often leads to master equations which are local in time [1]. The interaction of dust particles with air molecules or radiation, for example, delocalizes any interference terms into correlations with the environmental degrees of freedom on a short decoherence timescale. Thereafter, the dust particle can be perfectly described by a Markovian master equation for its density matrix $\hat{\rho}(t)$:

$$\frac{d\hat{\rho}}{dt} \equiv \mathcal{L}\hat{\rho} = -\frac{i}{2m}[\hat{p}^2, \hat{\rho}] - \frac{D}{2}[\hat{x}, [\hat{x}, \hat{\rho}]]. \quad (1)$$

Such an equation results frequently from an interaction with the environment in cases where friction is negligible [1]. The strength of the coupling is given by the parameter D . The first term in (1) would lead to unitary spreading, whereas the second term would lead to nonunitary localization. For a wavefunction of characteristic width σ , these effects act on timescales $m\sigma^2$ and $1/D\sigma^2$, respectively. Both effects are thus balanced for an ‘equilibrium width’, given approximately by

$$\sigma_0 = (Dm)^{-1/4} \quad (2)$$

see [1, 2]. The corresponding timescale is

$$t_0 = \sqrt{m/D} \quad (3)$$

and will set the timescale for decoherence.

The oldest way to elucidate a quantum state in terms of a pseudo-classical distribution of phase-space variables $\Gamma \equiv (x, p)$ is due to Wigner. Instead of the density matrix, one discusses the Wigner function

$$W(\Gamma) \equiv \int \langle x - \frac{1}{2}r | \hat{\rho} | x + \frac{1}{2}r \rangle e^{ipr} dr \quad (4)$$

in order to study aspects of an open system [1]. Normalization holds with the choice $d\Gamma = dp dx / 2\pi$ of the phase-space volume element. The Wigner function $W(\Gamma; t)$ corresponding to (1) satisfies the Fokker–Planck equation

$$\frac{dW}{dt} = -\frac{p}{m} \frac{\partial W}{\partial x} + \frac{D}{2} \frac{\partial^2 W}{\partial p^2}. \quad (5)$$

In a general situation, the Wigner function is negative in some regions of phase space. For this reason it cannot be regarded as a probability distribution. If decoherence occurs, the general expectation is that these negative parts are smoothed out in the course of time. Many examples support this expectation. In section 2 of this paper we shall prove a much stronger statement: after a certain decoherence time, the Wigner function becomes *strictly positive*. This is very different from the behaviour of the density matrix whose nondiagonal terms (describing interferences) become zero only asymptotically, i.e. they remain nonzero at any finite time.

Through the process (1) or, equivalently, (5) the position basis is distinguished as the preferred basis with respect to which no interferences can be observed and which remains a robust basis in time. Such a classical basis is usually called the pointer basis. It is the basis which remains most stable against the influence of an environment. How can the pointer basis be determined? We have shown in [3] that this can be achieved by three different methods which all lead to the same results. The first method invokes the principle of ‘Hilbert–Schmidt robustness’ stating that the pointer states mimic the local nonunitary evolution as closely as possible with respect to the Hilbert–Schmidt norm. A unique set of Gaussian pointer states has thereby been found. The second method demands that the local production of entropy be minimal (‘predictability sieve’). Again, this has led to a set of unique Gaussian pointer states with practically the same width as the previous ones. The last method invokes quantum state diffusion and leads again to Gaussian pointer states with the same width.

In section 3 we introduce the overcomplete set of Gaussian pointer states $|\Gamma\rangle$. Then we define additional phase-space distributions: the (generalized) Q - and P -functions are related to the Wigner function by Gaussian coarse-grainings. In the discussion of the local entropy production [3] we have already used the important fact that the density matrix can be decomposed exactly into a mixture of Gaussian states after a finite decoherence time. In section 4 of the present paper we shall prove this theorem—the positivity of the P -function—which, in fact, holds for arbitrary initial states. The technical details are relegated to appendices A, B and C.

2. Strict positivity of the Wigner function

We shall now consider the Wigner function, $W(\Gamma; t)$, of our open quantum system, obeying equation (5). We shall prove the theorem that for *any* initial state, W will become positive after a finite time t_D , i.e.

$$W(\Gamma; t) \geq 0 \quad t \geq t_D. \quad (6)$$

W can of course be positive even earlier; the theorem states that it cannot be negative later. The Wigner function corresponding to a Gaussian wavefunction, for example, stays positive for all times. In fact, as shown for example in [4], Gaussian states are the only pure states that lead to a positive Wigner function. It must, however, be emphasized that the positivity is only a necessary requirement for classicality, not a sufficient one. Squeezed Gaussian states of harmonic oscillators, for example, are genuine nonclassical states, but correspond to a positive Wigner function.

Since Fokker–Planck equations such as (5) preserve the positivity of distribution functions, one might guess that the set of positive solutions attracts the indefinite ones, so that any indefinite distribution becomes positive in the course of time. This is, however, not true. Equation (5) is a linear equation. Since it also allows solutions that remain negative, the superposition principle will allow indefinite solutions at all times.

The proof, therefore, has to rely on specific properties of the Wigner function. It is known that W possesses, in fact, very special features, see e.g. [5]. For example, it obeys $|W| \leq 1/\pi$. Moreover, its negative parts are always restricted to small regions in phase space.

To be concrete, in our proof we shall make use of two properties for the Wigner function. The first one exploits its connection with the Q -function (see e.g. [5]). Introducing the normalized Gaussian

$$g(\Gamma; C) = \frac{1}{C} \exp\left[-\frac{x^2 + p^2}{2C}\right] \quad (7)$$

its convolution with the Wigner function, $g \star W$, is always positive provided $C \geq 1/2$ (and may be indefinite otherwise). In the marginal case $C = 1/2$ this convolution gives the Q -function which yields the probability distribution for finding the coherent states in the density operator $\hat{\rho}$, and is therefore manifestly positive [5]. This is why the negative regions of W are so restricted: the convolution of W with the Gaussian $g(\Gamma; 1/2)$ would not yield a positive function if W contained negative regions that are spread out over regions with areas much bigger than about $1/2$. Since the case $C > 1/2$ represents a stronger coarse-graining of W than $C = 1/2$, it is clear that this positivity remains true.

The second property of the Wigner function, which will serve as an ingredient in our proof, is its invariance under *linear* canonical transformations. To be precise, if we make such a transformation for both the quantum operators and the classical canonical variables, one has

$$(\hat{x}, \hat{p}) \rightarrow (\hat{x}', \hat{p}') \Rightarrow W(x, p) \rightarrow W'(x', p') = W(x, p). \quad (8)$$

This feature is not obvious from the usual expressions defining W . There exists, however, an alternative equivalent form given by [6]:

$$W(x, p) = \text{tr}[\hat{\rho}\{\delta(x - \hat{x})\delta(p - \hat{p})\}_{\text{sym}}] \quad (9)$$

where we refer to a simple symmetrization process regarding the order of \hat{x} and \hat{p} :

$$\{\hat{\Gamma} \hat{O}\}_{\text{sym}} \equiv \frac{1}{2}(\hat{\Gamma} \hat{O} + \hat{O} \hat{\Gamma}) \quad \text{for } \hat{\Gamma} = (\hat{x}, \hat{p}) \text{ and for all } \hat{O}. \quad (10)$$

This ordering is explicitly invariant for linear canonical transformations. The invariance of the Wigner function follows immediately. Note, however, that the coarse-graining is *not* invariant and coarse-grained Wigner functions, Q -functions in particular, will be non-invariant even for linear canonical transformations. In the simple case of the transformation $\hat{x}' = a\hat{x}$, $\hat{p}' = a^{-1}\hat{p}$, and assuming a pure state with wavefunction $\psi(x)$, the invariance can be seen directly from the standard integral expression. Using

$$\psi'(x') = \frac{1}{\sqrt{a}} \psi\left(\frac{x'}{a}\right) \quad (11)$$

one has

$$W'(x', p') = \frac{1}{\pi} \int dy' \exp(2ip'y') \psi^{*'}(x' + y') \psi'(x' - y') \quad (12)$$

where W' denotes the Wigner function with respect to the same state in the transformed basis, and therefore

$$W'(x', p') = W(x, p). \quad (13)$$

Using this invariance, one can extend the previous coarse-graining (7) to general Gaussians with correlation matrix \mathbf{C} :

$$g(\Gamma; \mathbf{C}) = |\mathbf{C}|^{-\frac{1}{2}} \exp \left[-\Gamma^T \frac{1}{2\mathbf{C}} \Gamma \right] \quad (14)$$

(Γ and Γ^T stand for column and row vectors, respectively). Application of a linear canonical transformation to the convolution $g \star W$ rendering $\mathbf{C} = \sqrt{|\mathbf{C}|\mathbf{I}}$ then demonstrates that the sufficient and necessary condition for the positivity of the coarse-grained Wigner function reads

$$g(\Gamma; \mathbf{C}) \star W(\Gamma) \geq 0 \quad \text{iff} \quad |\mathbf{C}| \geq 1/4. \quad (15)$$

With this lemma, the proof of the theorem (6) becomes straightforward. The Fokker-Planck equation (5) imposes a progressive Gaussian coarse-graining (14) on the initial Wigner function:

$$W(\Gamma; t) = g(\Gamma; \mathbf{C}_W(t)) \star W(x - pt/m, p; 0) \quad (16)$$

where the time-dependent correlation matrix of the coarse-graining is

$$\mathbf{C}_W(t) = Dt \begin{pmatrix} t^2/3m^2 & t/2m \\ t/2m & 1 \end{pmatrix} \quad (17)$$

as can be found from equations (5), (14) and (16). The determinant yields

$$|\mathbf{C}_W(t)| = \frac{D^2 t^4}{12m^2}. \quad (18)$$

It follows from condition (15) that the Wigner function is indeed positive for

$$\frac{t}{t_0} \geq 3^{1/4} \approx 1.32 \quad (19)$$

which is of the same order as the decoherence timescale of section 1. This completes our proof.

We would like to mention that, based on more complicated mathematics [7], lemma (15) has been stated earlier [8]. Similarly to the spirit of [10] and the present paper, it was also suggested in [7, 8] that particular quantum features of open systems would disappear after an estimated finite time.

3. Pointer states, Q - and P -functions

We are going to consider the overcomplete set of normalized pure Gaussian pointer states $|\Gamma\rangle$ obeying

$$\int |\Gamma\rangle\langle\Gamma| d\Gamma = \hat{I}. \quad (20)$$

In appendix A we discuss the position representation of these states. The overlap of two pointer states is a Gaussian:

$$|\langle \Gamma | \Gamma' \rangle|^2 = \exp \left[-(\Gamma - \Gamma')^T \frac{1}{4\mathbf{C}_{1/4}} (\Gamma - \Gamma') \right]. \quad (21)$$

The matrix $\mathbf{C}_{1/4}$ is positive and has determinant $1/4$. This value for the determinant makes the above Gaussian normalized and assures the consistency of the completeness relation (20) with the normalization of the pointer states. In fact, $\mathbf{C}_{1/4}$ can be directly identified as the matrix of quantum uncertainties of the canonical pair of operators in the Gaussian pointer states, cf (34) in appendix A. We can calculate $\mathbf{C}_{1/4}$ for simplicity in the fiducial state $|\Gamma = (0, 0)\rangle \equiv |0\rangle$:

$$\mathbf{C}_{1/4} \equiv \langle 0 | \hat{\Gamma} \hat{\Gamma}^T + \text{h.c.} | 0 \rangle. \quad (22)$$

Given the above overcomplete set of Gaussian states, one can introduce the generalized Q - and P -functions, related to the density operator respectively by

$$Q(\Gamma) = \langle \Gamma | \hat{\rho}(\Gamma) \rangle \quad (23)$$

and

$$\hat{\rho} = \int P(\Gamma) |\Gamma\rangle \langle \Gamma| d\Gamma. \quad (24)$$

The Q -function is the probability distribution of the value Γ . In a generalized quantum measurement it can be inferred from the *positive operator valued measure* (see e.g. [9]) formed by $|\Gamma\rangle \langle \Gamma| d\Gamma$. The P -function has a different meaning. When we expand the density matrix as a sum of the pointer states, the weighting function is called the P -function. While the Q -function is non-negative by construction, the P -function may be indefinite (even ill-defined) for generic states. The Q - and P -functions are related to the Wigner function by the same Gaussian coarse-graining but in opposite senses [5]:

$$W(\Gamma) = g(\Gamma; \mathbf{C}_{1/4}) * P(\Gamma) \quad (25)$$

$$Q(\Gamma) = g(\Gamma; \mathbf{C}_{1/4}) * W(\Gamma). \quad (26)$$

The correlation matrix $\mathbf{C}_{1/4}$ of coarse-grainings is the one that appeared earlier as the matrix of canonical quantum uncertainties in the Gaussian pointer states, see (22). All sets of Gaussian pure pointer states are classified in appendix A. Details of derivations for (25), (26) are given in appendix B.

The Q -function satisfies a Fokker–Planck equation which is, according to (26), the coarse-grained version of the Fokker–Planck equation (5) for the Wigner function. For the evolution equation of the P -function we are going to present a more direct derivation in appendix C.

4. Strict positivity of the P -function

In section 3 we have expanded the density operator as a weighted sum of Gaussian pointer states, see (24). The weight function is called the P -function and it is indefinite for a generic quantum state $\hat{\rho}(0)$. We shall now prove that, due to the open system dynamics (1), the P -function becomes *exactly* positive,

$$P(\Gamma; t) \geq 0 \quad t \geq t'_D. \quad (27)$$

The density operator $\hat{\rho}(t)$ can thus be decomposed *exactly* into a statistical mixture of Gaussian pointer states $|\Gamma\rangle$ (see (32) below) after a *finite* decoherence time t'_D has elapsed. We have already emphasized that this holds for a *generic* $\hat{\rho}(0)$ (not necessarily Gaussian). This theorem

generalizes the corresponding statement made in [10] for a single choice of Gaussian pointer states (i.e. of $\mathbf{C}_{1/4}$) as well as the asymptotic statements proved in [11, 12].

The proof will be reduced to lemma (15) used in the proof of the positivity of the Wigner function, see (6) in section 2. Consider solution (16) for the Wigner function and substitute expression (25) into its left-hand side. As for the right-hand side, assume that enough time has elapsed so that $\mathbf{C}_W(t) - \mathbf{C}_{1/4}$ is a non-negative matrix. Then the convolution factorizes as

$$g(\Gamma; \mathbf{C}_W(t)) = g(\Gamma; \mathbf{C}_{1/4}) * g(\Gamma; \mathbf{C}_W(t) - \mathbf{C}_{1/4}) \quad (28)$$

and can be substituted into (16). We obtain

$$g(\Gamma; \mathbf{C}_{1/4}) \star P(\Gamma; t) = g(\Gamma; \mathbf{C}_{1/4}) \star g(\Gamma; \mathbf{C}_W(t) - \mathbf{C}_{1/4}) \star W(x - pt/m, p; 0). \quad (29)$$

The identical convolutions on both sides cancel each other and leave us with the explicit solution for the P -function as the coarse-grained Wigner function. The lemma (15) tells us that a generic initial P -function becomes non-negative after a time t ,

$$P(\Gamma; t) \geq 0 \quad \text{iff} \quad |\mathbf{C}_W(t) - \mathbf{C}_{1/4}| \geq 1/4. \quad (30)$$

Calculating the determinant from (17) and (36) below, this condition leads to a cubic equation for t , yielding the numeric estimate

$$\frac{t}{t_0} \geq 1.97. \quad (31)$$

As expected, therefore, the decoherence timescale t'_D coincides approximately with t_0 . This completes the proof.

One might wonder how much of our results do depend on our particular Markovian master equation (1). We expect that our proofs of positivity can be extended to all master equations of this kind which are at most quadratic in position and momentum, e.g. for the damped harmonic oscillator. As far as corrections to the Markovian approximation are concerned, we do not expect that they render our results—exact positivity of Wigner and P -functions after a finite time—obsolete; because of their exponential smallness they should not spoil an exact positivity. Our results should thus be stable with respect to post-Markovian corrections, but a formal proof would go beyond the scope of this paper. We hope to return to these issues elsewhere.

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Appendix A. Classes of Gaussian pointer states

We choose the following form for the wavefunctions $\langle q|\Gamma\rangle$ of pure Gaussian pointer states $|\Gamma\rangle$:

$$\psi_\Gamma(q) = (\alpha_R/2\pi)^{1/4} \exp(-\alpha(q-x)^2/4 + ip(q-x)). \quad (32)$$

This is the most general form of a normalized Gaussian wave packet shifted and boosted uniformly from the fiducial state with $\psi_0(q)$ with $\Gamma = (0, 0)$,

$$\psi_0(q) = (\alpha_R/2\pi)^{1/4} \exp(-\alpha q^2/4). \quad (33)$$

The correlation matrix $\mathbf{C}_{1/4}$, defined in (22), takes the following form:

$$\langle 0| \begin{pmatrix} \hat{x}^2 & (\hat{x}\hat{p} + \text{h.c.})/2 \\ (\hat{x}\hat{p} + \text{h.c.})/2 & \hat{p}^2 \end{pmatrix} |0\rangle = \frac{1}{\alpha_R} \begin{pmatrix} 1 & -\alpha_I/2 \\ -\alpha_I/2 & |\alpha|^2/4 \end{pmatrix}. \quad (34)$$

Indeed, its determinant is $1/4$ for all α . In [3] we have found a distinguished value of the complex parameter α for the class of robust pointer states, given by

$$\alpha_0 \equiv \alpha_R + i\alpha_I = (1 - i)\sqrt{2Dm}. \quad (35)$$

For this value (34) gives

$$\mathbf{C}_{1/4} = \begin{pmatrix} \sigma_0^2/\sqrt{2} & 1/2 \\ 1/2 & \sigma_0^{-2}/\sqrt{2} \end{pmatrix}. \quad (36)$$

This is being used in the proof for the positivity of the P -function, equation (30). The proof can be performed for any α (it always leads to a cubic equation), but the basic object is the matrix $\mathbf{C}_{1/4}$ with determinant $1/4$.

It is worthwhile to add the following. Let us assume that we had constructed no pointer states first. Instead, assume that we had taken an arbitrary real positive matrix of determinant $1/4$ to perform a coarse-graining on the Wigner function such as in (26). This matrix would then define a complex parameter α as on the right-hand side of (34), and we would be able to construct a unique overcomplete set of Gaussian wavefunctions (32) whose $\mathbf{C}_{1/4}$ -matrix is just our chosen one. Thus it would turn out that our coarse-grained Wigner function was just the corresponding Q -function which is always positive. In this way we have obtained an alternative proof of theorem (6) of section 2.

Appendix B. Symplectic Fourier transform

This appendix is devoted to the proofs of (25) and (26). To facilitate the mathematical derivations we use the Fourier representation. For instance, the Fourier-transformed Wigner function reads

$$\tilde{W}(\tilde{\Gamma}) = \int W(\Gamma) \exp[-i\tilde{\Gamma}^T \Gamma] d\Gamma \quad (37)$$

where $\tilde{\Gamma} = (\tilde{p}, -\tilde{x})$. We use the symplectic product $\tilde{\Gamma}^T \Gamma = \tilde{p}x - \tilde{x}p$ of the original and the Fourier-transformed variables in (37). In the Fourier representation, the coarse-graining relations (25), (26) reduce to algebraic relations:

$$\tilde{W}(\tilde{\Gamma}) = \exp\left[-\frac{1}{2}\tilde{\Gamma}^T \mathbf{C}_{1/4} \tilde{\Gamma}\right] \tilde{P}(\tilde{\Gamma}) \quad (38)$$

$$\tilde{Q}(\tilde{\Gamma}) = \exp\left[-\frac{1}{2}\tilde{\Gamma}^T \mathbf{C}_{1/4} \tilde{\Gamma}\right] \tilde{W}(\Gamma). \quad (39)$$

We are going to prove the first relationship (the second one can easily be proved along the same lines). From (4) and (37) we obtain

$$\tilde{W}(\tilde{\Gamma}) = \int e^{i\tilde{p}x} \langle x + \frac{1}{2}\tilde{x} | \hat{\rho} | x - \frac{1}{2}\tilde{x} \rangle dx. \quad (40)$$

Substituting the P -function expansion (24) of $\hat{\rho}$ on the right-hand side yields

$$\tilde{W}(\tilde{\Gamma}) = \int e^{-i\tilde{p}x} \langle x + \frac{1}{2}\tilde{x} | \Gamma' \rangle \langle \Gamma' | x - \frac{1}{2}\tilde{x} \rangle P(\Gamma') d\Gamma'. \quad (41)$$

Invoking the expression (32) of the pointer state wavefunctions we can perform the above integral, yielding

$$\tilde{W}(\tilde{\Gamma}) = \exp\left[-\frac{\alpha_R \tilde{x}^2}{8} - \frac{1}{2\alpha_R} \left(\tilde{p} - \frac{1}{2}\alpha_I \tilde{x}\right)^2\right] \tilde{P}(\tilde{\Gamma}). \quad (42)$$

The quadratic form in the above exponent can be written as $-\frac{1}{2}\tilde{\Gamma}^T \mathbf{C}_{1/4} \tilde{\Gamma}$. This completes the proof.

Appendix C. ‘Fokker–Planck’ equation for the P -function

In this part we return to the notation of our recent paper [3] and derive directly the evolution equation for the P -function. We introduce the projectors on the states $\psi_\Gamma(q) \equiv \langle q|\Gamma\rangle$, $\hat{P}(\Gamma) = |\Gamma\rangle\langle\Gamma|$. We also denote the P -function $P(\Gamma; t)$ of the open system by $f(\Gamma; t)$. Application of (1) to (24) yields

$$\frac{d}{dt}\hat{\rho} = \int f(\Gamma; t)\mathcal{L}\hat{P}(\Gamma) d\Gamma = \int \dot{f}(\Gamma; t)\hat{P}(\Gamma) d\Gamma. \quad (43)$$

To derive from this the equation for f , we first observe that the action of \mathcal{L} on $\hat{P}(\Gamma)$ in coordinate representation yields

$$\begin{aligned} \langle q|\mathcal{L}\hat{P}|q'\rangle &= \frac{i}{2m} \left\{ -i\alpha_I - ip[\alpha^*(q' - x) + \alpha(q - x)] - \left(\frac{\alpha^*}{2}\right)^2 (q' - x)^2 \right. \\ &\quad \left. + \left(\frac{\alpha}{2}\right)^2 (q - x)^2 \right\} \langle q|\hat{P}|q'\rangle - \frac{D}{2}(q - q')^2 \langle q|\hat{P}|q'\rangle. \end{aligned} \quad (44)$$

Using (32) in the identity $\langle q|\hat{P}(\Gamma)|q'\rangle = \psi_\Gamma(q)\psi_\Gamma^*(q')$, the above action can be expressed as an operator containing derivative terms in x and p , acting on \hat{P} . After partial integration in (43) one then finds the following equation for f :

$$\frac{df(\Gamma; t)}{dt} = -\frac{p}{m}\partial_x f(\Gamma; t) + \frac{1}{2} [D_{pp}\partial_{pp}^2 + D_{xx}\partial_{xx}^2 + 2D_{px}\partial_{px}^2] f(\Gamma; t) \quad (45)$$

where the elements of the diffusion matrix are given by

$$\mathbf{D} \equiv \begin{pmatrix} D_{xx} & D_{xp} \\ D_{px} & D_{pp} \end{pmatrix} = \begin{pmatrix} -\alpha_I/m\alpha_R & |\alpha|^2/4m\alpha_R \\ |\alpha|^2/4m\alpha_R & D \end{pmatrix}. \quad (46)$$

Equation (45) can be interpreted as a Fokker–Planck equation for $f(\Gamma, t)$ provided the diffusion matrix \mathbf{D} is non-negative. As usual, the first term in (45) will then describe a drift according to the free-particle dynamics, while the second term will describe a diffusion of the state of the system over the pointer states $\hat{P}(\Gamma)$. In [3] we have implemented the principle of minimal local entropy production by minimizing the width of the Gaussian pointer states. This leads to a value for α of the order of (35). We have shown that the relation $\mathbf{C} = \mathbf{D}t_0^2/2$ holds for the distinguished value (35) of α .

To find a formal solution of (45), we use the Fourier representation $\tilde{f}(\tilde{\Gamma}; t)$. Equation (45) then leads to the following evolution equation for the Fourier components:

$$\frac{d\tilde{f}(\tilde{\Gamma}; t)}{dt} = -\frac{\tilde{p}}{m}\partial_{\tilde{x}}\tilde{f}(\tilde{\Gamma}; t) - \frac{1}{2}[\tilde{\Gamma}^T\mathbf{D}\tilde{\Gamma}]\tilde{f}(\tilde{\Gamma}; t). \quad (47)$$

The solution assumes the form

$$\tilde{f}(\tilde{\Gamma}; t) = \exp\left[-\frac{t}{2}\tilde{\Gamma}^T\mathbf{D}(t)\tilde{\Gamma}\right]\tilde{f}(\tilde{x} - \tilde{p}t/m, \tilde{p}; 0). \quad (48)$$

By substitution into (47) one obtains explicitly the matrix of time-dependent coefficients³

$$\mathbf{D}(t) = \begin{pmatrix} D_{xx} + D_{xp}t/m + D_{pp}t^2/3m^2 & D_{xp} + D_{pp}t/2m \\ D_{xp} + D_{pp}t/2m & D_{pp} \end{pmatrix}. \quad (49)$$

³ Our matrix $\mathbf{D}(t)$ corresponds to the matrix $\mathbf{G}(t)$ in [3]. Equation (17) there has misprints: the signs of the terms $D_{pp}t/2m$ must be reversed.

This matrix becomes more and more positive for $t > 0$ provided $\mathbf{D} \equiv \mathbf{D}(0)$ was chosen positive. One obtains the solution of the evolution equation (45) by the inverse Fourier-transform of expression (48). It takes the form of the convolution

$$f(\Gamma; t) = g(\Gamma; t\mathbf{D}(t)) * f(x - pt/m, p; 0). \quad (50)$$

The solution, therefore, emerges as the progressive time-dependent Gaussian coarse-graining of the free kinematic evolution. Note the close similarity of this equation with the time evolution (16) for the Wigner function.

References

- [1] Giulini D, Joos E, Kiefer C, Kupsch J, Stamatescu I-O and Zeh H D 1996 *Decoherence and the Appearance of a Classical World in Quantum Theory* (Berlin: Springer)
- [2] Diósi L 1987 *Phys. Lett. A* **120** 377
- [3] Diósi L and Kiefer C 2000 *Phys. Rev. Lett.* **85** 3552
- [4] Tatarskii V I 1983 *Sov. Phys.-Usp.* **26** 311
- [5] Leonhardt U 1997 *Measuring the Quantum State of Light* (Cambridge: Cambridge University Press)
Schleich W 2001 *Quantum Optics in Phase Space* (Berlin: Wiley-VCH)
- [6] Diósi L 1996 *Quantum Semiclass. Opt.* **8** 309
- [7] Khalfin L A and Tsirelson B S 1987 *Symp. on the Foundations of Modern Physics* ed P Lahti *et al* (Singapore: World Scientific)
- [8] Khalfin L A and Tsirelson B S 1992 *Found. Phys.* **22** 879
- [9] Peres A 1993 *Quantum Theory: Concepts and Methods* (Dordrecht: Kluwer)
- [10] Diósi L 1987 *Phys. Lett. A* **122** 221
- [11] Halliwell J J and Zoupas A 1995 *Phys. Rev. A* **52** 7294
- [12] Halliwell J J and Zoupas A 1997 *Phys. Rev. A* **55** 4697