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Advancement of estimation fidelity in continuous quantum measurement

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Abstract

We estimate an unknown qubit from the long sequence of n random polarization measurements of precision Δ. Using the standard Ito stochastic equations of the *a posteriori* state in the continuous measurement limit, we calculate the advancement of fidelity. We show that the standard optimum value $2/3$ is achieved asymptotically for $n \gg \Delta^2/96 \gg 1$. We append a brief derivation of novel Ito equations for the estimate state.

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1. Introduction

The standard object of quantum inference is the value σ of some Hermitian observable $\hat{\sigma}$ of the given quantum system. The process of inference is called quantum measurement. One can consider the *a priori* quantum state $\hat{\rho}$ of the given system as an additional object of inference [1, 2]. The limitations as well as the optimization of state determination are the focus of recent investigations [3–5] especially in the field of quantum information and communication [6]. A completely unknown state $\hat{\rho}$ cannot be inferred from a *single* system: the fidelity of the estimate $\hat{\rho}'$ will be poor. If the *a priori* state $\hat{\rho}$ is pure then the estimate $\hat{\rho}'$ must also be pure, and the simple bilinear expression $F = \mathrm{Tr}[\hat{\rho}'\hat{\rho}]$ defines its fidelity. If we assume that the *a priori* pure $\hat{\rho}$ is completely random then lower and upper limits become analytically calculable for the average fidelity $\bar{F}$ [4]. For a single two-state system (qubit), one obtains

$$\tfrac{1}{2} \leqslant \bar{F} \leqslant \tfrac{2}{3}. \tag{1}$$

Any deliberate trial $\hat{\rho}'$, when completely unrelated to $\hat{\rho}$, will yield the same worst value $1/2$. The best value can be attained in many ways. Let us, for instance, measure the Pauli-polarization matrix $\hat{\sigma}$ along a single randomly chosen spatial direction. Let $\sigma = \pm 1$ be the results of the *projective* measurement. It is then natural to identify the estimate pure state $\hat{\rho}'(\sigma)$ with the standard *a posteriori* pure state $\hat{\rho}(\sigma)$ taught in textbooks:

$$\hat{\rho}'(\sigma) = \hat{\rho}(\sigma) \equiv \frac{\hat{I} + \sigma\hat{\sigma}}{2}. \tag{2}$$

Trivial calculation can prove that the average fidelity over random *a priori* pure states $\hat{\rho}$ is $2/3$.

No quantum measurement however involved could improve on $\bar{F} = 2/3$. In particular, it would make no sense to perform a second projective measurement on the given single qubit. We can, however, consider *non-projective* measurements [6, 7] from the beginning. A typical non-projective measurement yields less information than an ideal measurement does. Hence it makes sense to combine successive non-projective measurements on a single system [8] in order to improve fidelity. In what follows, we mean non-projective measurements unless we say otherwise.

The general case involving a sequence of repeated measurements is beyond the capacity of analytic calculations. There is, nonetheless, an effective theory for *long* sequences. Then the measured value σ, the *a posteriori* state $\hat{\rho}(\sigma)$, and the state estimate $\hat{\rho}'(\sigma)$ all become time dependent and satisfy coupled stochastic differential equations. The 'conditional' master equation of the *a posteriori* state [9] as well as its coupling to the measured value [10] have long been well known (see also [11]) as the ultimate formalism of earlier continuous measurement models [12, 13]. The equation of the estimate state has remained undefined and we outline its derivation in the appendix.

In section 2 we discuss state estimate from a single measurement. We succeed in expressing the average fidelity in terms of *a posteriori* states. In section 3 this result is generalized for a sequence of measurements. In section 4 the conditional 'master' equation is introduced for the *a posteriori* state. In section 5 we calculate the progression of fidelity for long sequences of very unsharp measurements and we prove how fidelity will saturate to $2/3$. Although we develop the concrete equations for two-state systems, most results can trivially be extended for higher dimensions *N*.

2. Fidelity from single measurements

We approximate the exact eigenstates of a given Hermitian observable $\hat{\sigma}$ by approximate Gaussian projectors of precision Δ:

$$\hat{\Pi}(\sigma) = \frac{1}{\sqrt{2\pi\Delta^2}} \exp\left[-\frac{(\hat{\sigma}-\sigma)^2}{2\Delta^2}\right]. \tag{3}$$

They satisfy the completeness condition

$$\int \hat{\Pi}(\sigma)\,\mathrm{d}\sigma = \hat{I} \tag{4}$$

and form a POVM [6, 7]. In the simplest case, the corresponding (non-projective) measurement of $\hat{\sigma}$ will transform the *a priori* state $\hat{\rho}$ into the following *a posteriori* state:

$$\hat{\rho} \longrightarrow \hat{\rho}(\sigma) = \frac{\hat{\Pi}^{1/2}(\sigma)\hat{\rho}\hat{\Pi}^{1/2}(\sigma)}{\mathrm{Tr}\,[\hat{\Pi}(\sigma)\hat{\rho}]} \tag{5}$$

where σ is the random outcome of the measurement. It may take any real value with the normalized probability density

$$p(\sigma) = \mathrm{Tr}\,[\hat{\Pi}(\sigma)\hat{\rho}]. \tag{6}$$

The theory of (non-projective) measurements does not imply a theory for the estimate $\hat{\rho}'$. One could mistakenly think the *a posteriori* state $\hat{\rho}(\sigma)$ a reasonable estimate for the *a priori* state $\hat{\rho}$. Unfortunately, the experimenter has no access to it. He/she infers the measured value σ and it is, contrary to the projective measurement (2), not enough to derive the *a posteriori* state. It is only sufficient to identify the approximate projector $\hat{\Pi}(\sigma)$. Its normalized form can be a reasonable estimate:

$$\hat{\rho}'(\sigma) = \frac{\hat{\Pi}(\sigma)}{\operatorname{Tr}\hat{\Pi}(\sigma)}. \tag{7}$$

This is a mixed state. If the *a priori* states $\hat{\rho}$ are unknown pure states then the estimate should also be pure. To this end, the experimenter must refine his/her first choice (7). The estimate will be one of the pure eigenstates of the mixed state estimate (7), chosen randomly with probability equal to the corresponding eigenvalue. (The optimum estimate would be the most probable eigenstate [5].)

In our work, we discuss pure *a priori* states and, accordingly, we use the above-mentioned pure state estimates. In other words, the pure state estimate will be an eigenstate of $\hat{\Pi}(\sigma)$, with probability proportional to the corresponding eigenvalue of $\hat{\Pi}(\sigma)$. By construction, the average of these pure state estimates is identical with the mixed state estimate (7). This has a useful consequence in fidelity calculations. The bilinearity of fidelity $\operatorname{Tr}[\hat{\rho}'\hat{\rho}]$, valid originally between two pure states, will be preserved for the expected fidelity of our estimates:

$$F = \int \operatorname{Tr}[\hat{\rho}'(\sigma)\hat{\rho}]p(\sigma)\,\mathrm{d}\sigma \equiv E\operatorname{Tr}[\hat{\rho}'(\sigma)\hat{\rho}] \tag{8}$$

where $\hat{\rho}'$ is defined by (7) and E stands for stochastic expectation value.

We benefit from the bilinearity. We are going to find a simpler expression for F. While we retain the notation $\hat{\rho}$ for the pure *a priori* state, we imagine a *hypothetical a priori* state $\hat{\rho}^? = \hat{I}/N$ as well, which is totally mixed. We apply the non-projective measurement (5), (6) to $\hat{\rho}^?$. This yields the simple relationship $\hat{\Pi}(\sigma) = Np^?(\sigma)\hat{\rho}^?(\sigma)$ where $p^?(\sigma) = N^{-1}\operatorname{Tr}\hat{\Pi}(\sigma)$ is the probability distribution of the outcomes for the measurement on the hypothetical *a priori* state $\hat{\rho}^?$. Substituting these relationships into (7) and inserting the result into (8), we obtain the following new form:

$$F = N\int (\operatorname{Tr}[\hat{\rho}^?(\sigma)\hat{\rho}])^2 p^?(\sigma)\,\mathrm{d}\sigma \equiv NE(\operatorname{Tr}[\hat{\rho}^?(\sigma)\hat{\rho}])^2. \tag{9}$$

Note that the stochastic average is to be taken with the hypothetical probability distribution $p^?(\sigma)$ instead of the true $p(\sigma)$. The new expression (9) contains the (hypothetical) *a posteriori* state while the old formula (8) contained the (true) estimate state. It pays because the *a posteriori* states will satisfy simpler equations than the estimate states, see section 4 and the appendix.

If we follow the example of section 1, we have to average fidelity (9) over random pure qubit states $\hat{\rho}$:

$$\bar{F} = \tfrac{1}{3} + \tfrac{1}{3}E\operatorname{Tr}[\hat{\rho}^?(\sigma)]^2. \tag{10}$$

This formula of the average fidelity will be generalized for the continuous estimation of random pure states in section 3.

3. Fidelity from sequential measurements

We start from the sequence $\hat{\Pi}_1(\sigma_1), \ldots, \hat{\Pi}_n(\sigma_n)$ of n measurements (3)–(6). The measured observables $\hat{\sigma}_1, \ldots, \hat{\sigma}_n$ need not be identical. Thus our measurements may not commute. It is well known that a sequence of measurements is formally equivalent to a single (though complicated) measurement. Applying equation (5) n times repeatedly, the *a posteriori* state becomes

$$\hat{\rho} \longrightarrow \hat{\rho}(\sigma_.) = \frac{\hat{G}_n(\sigma_.)\,\hat{\rho}\hat{G}_n^\dagger(\sigma_.)}{\operatorname{Tr}[\hat{\Pi}_n(\sigma_.)\hat{\rho}]}. \tag{11}$$

The sequentially composed *Kraus-operator* [6, 7] reads

$$\hat{G}_n(\sigma_.) = \hat{\Pi}_n^{1/2}(\sigma_n)\cdots\hat{\Pi}_1^{1/2}(\sigma_1) \tag{12}$$

where the shorthand notation $(\sigma_1, \ldots, \sigma_n) = (\sigma_.)$ is being used. The new POVM elements

$$\hat{\Pi}_n(\sigma_.) = \hat{G}_n^\dagger(\sigma_.)\hat{G}_n(\sigma_.) \tag{13}$$

are normalized for all n:

$$\int \hat{\Pi}_n(\sigma_.)\,\mathrm{d}\sigma_1\cdots\sigma_n = \hat{I} \tag{14}$$

as follows from equations (4), (12) and (13). The probability of the whole sequence of outcomes can be written in the compact form,

$$p_n(\sigma_.) = \mathrm{Tr}\,[\hat{\Pi}_n(\sigma_.)\hat{\rho}]. \tag{15}$$

Equations (11)–(15) constitute a single (complicated) measurement. We invoke all considerations of state estimate from section 2. In such a way we shall introduce the mixed state estimate

$$\hat{\rho}_n'(\sigma_.) = \frac{\hat{\Pi}_n(\sigma_.)}{\mathrm{Tr}\,\hat{\Pi}_n(\sigma_.)} \tag{16}$$

whose eigenstates, as in case of (7), will be the pure state estimates. The same considerations that led to fidelities (8) and (10) in section 2 apply invariably. We can, for instance, write the average fidelity in terms of the *a posteriori* state (11) emerging from a hypothetical *a priori* qubit state $\hat{\rho}^? = \hat{\rho}_0^? = \hat{I}/2$:

$$\bar{F}_n = \tfrac{1}{3} + \tfrac{1}{3}E\,\mathrm{Tr}\,[\hat{\rho}_n^?(\sigma_.)]^2. \tag{17}$$

It is obvious that $\bar{F}_0 = 1/2$, and we expect $\bar{F}_n$ to be a monotone function of n. In section 5 we prove that $\bar{F}_n$ achieves the upper limit (1) even when each individual measurement is very unsharp. Actually, we shall prove that $\hat{\rho}_n^?$ tends to be pure for large n. Section 4 prepares the mathematical tool of the proof.

4. Conditional master equation

There is a particular class of sequential measurements which is treatable with good accuracy in terms of Markovian stochastic differential equations. We assume long sequences of very unsharp measurements:

$$n \gg 1 \qquad \Delta \gg 1. \tag{18}$$

The asymptotic limit [10, 12]

$$n, \Delta \longrightarrow \infty \qquad \frac{n}{\Delta^2} = \mathrm{const} \tag{19}$$

will be called the 'continuum limit'. In the case of two-state systems, we assume that the measured observables $\hat{\sigma}_1, \ldots, \hat{\sigma}_n$ are Pauli-polarizations chosen independently along random directions. Formally, let us count the succession of measurements as if they happened at constant rate $\nu = 12/\Delta^2$. Accordingly, we replace the discrete parameter n by the continuous time:

$$t = \frac{12n}{\Delta^2}. \tag{20}$$

We consider all quantities as continuous functions of t, coarse-grained on scales $\gg 1/\nu$ involving many measurements. In this limit, an approximate theory emerges in the form of

Markovian stochastic differential equations. (The theory becomes exact in the continuum limit.) The *a posteriori* state (see equation (31) of the appendix) satisfies the conditional (or selective) master equation:

$$\frac{\mathrm{d}\hat{\rho}_t}{\mathrm{d}t} = -\frac{1}{2}[\hat{\vec{\sigma}}, [\hat{\vec{\sigma}}, \hat{\rho}_t]] + \{\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle_t, \hat{\rho}_t\}\vec{w}_t \tag{21}$$

where $\langle\hat{\vec{\sigma}}\rangle_t = \mathrm{Tr}\,[\hat{\vec{\sigma}}\hat{\rho}_t]$. We have suppressed denoting the functional dependence of $\hat{\rho}_t$ on the outcomes $\{\sigma_\tau; 0 \leqslant \tau \leqslant t\}$. The $\vec{w}_t$ is the standard isotropic white noise and the equation must be interpreted in the sense of the Ito stochastic calculus. There is a second stochastic differential equation for the outcome:

$$\vec{\sigma}_t = \langle\hat{\vec{\sigma}}\rangle_t + \tfrac{1}{2}\vec{w}_t. \tag{22}$$

The features of the above equations have been well understood. In particular, the solution $\hat{\rho}_t$ becomes asymptotically pure for long times [14, 15]. This assures the saturation of average fidelity (17), as proved in the next section. So far, the stochastic differential equation governing the estimate $\hat{\rho}'_t$ has been missing. We construct it in the appendix.

5. Saturation of fidelity

We are going to discuss the time dependence of the average fidelity $\bar{F}_t$. Remember that it corresponds to the (coarse-grained) n-dependent fidelity $\bar{F}_n$ (17) via $t = 12n/\Delta^2$. The latter requires the knowledge of the hypothetical *a posteriori* state which, for a qubit, we shall parametrize by the polarization vector $\vec{s}_t \equiv \langle\hat{\vec{\sigma}}\rangle_t$:

$$\hat{\rho}^?_t = \frac{\hat{I} + \vec{s}_t\hat{\vec{\sigma}}}{2} \qquad |\vec{s}_t| \equiv s_t \leqslant 1. \tag{23}$$

Recall that the initial state must be the hypothetical state $\hat{I}/2$ implying the initial value $\vec{s}_0 = 0$. The stochastic 'master' equation (21) yields the following stochastic differential equation for the polarization vector:

$$\frac{\mathrm{d}\vec{s}_t}{\mathrm{d}t} = -4\vec{s}_t - 2\,(\vec{s}_t\vec{w}_t)\,\vec{s}_t + 2\vec{w}_t. \tag{24}$$

This is an isotropic inhomogeneous spatial diffusion process. A stochastic differential equation for the squared norm (purity) follows from it:

$$\frac{\mathrm{d}s_t^2}{\mathrm{d}t} = 4\big(3 - s_t^2\big)\big(1 - s_t^2\big) + 4\big(1 - s_t^2\big)s_t w_t \tag{25}$$

where w_t is the standard white noise. This is a one-dimensional inhomogeneous diffusion. For long times the norm will approach unity, therefore the *a posteriori* state becomes asymptotically pure. The Monte-Carlo calculations by the author have shown that the purity s_t^2 is dominated by the drift term. Ignoring diffusion, the error remains within 2% and the analytic solution is possible:

$$Es_t^2 \equiv s_t^2 = \frac{\mathrm{e}^{8t} - 1}{\mathrm{e}^{8t} - 1/3}. \tag{26}$$

Let us restore the original variable $n = t\Delta^2/12$ and substitute the above result into expression (17):

$$\bar{F}_n = \frac{1}{2} + \frac{1}{6}E\,s_t^2 = \frac{1}{2} + \frac{1}{6}\frac{\mathrm{e}^{96n/\Delta^2} - 1}{\mathrm{e}^{96n/\Delta^2} - 1/3}. \tag{27}$$

The average fidelity approaches the optimum value 2/3 after a characteristic number $n \sim \Delta^2/96$ of unsharp measurements. Recalling conditions (18) we conclude that our result is valid for *very* unsharp measurements, i.e. Δ must be much greater than $\sqrt{96} \approx 10$.

6. Discussion

We have discussed single quantum state determination via sequential non-projective (POVM) measurements in the limit of very unsharp measurements. We have proved that the known optimum average fidelity of estimating random qubits can be approached gradually with many successive random unsharp measurements. Whether this is true for non-random qubits is an open issue, but it is certainly tractable with the method of the present work. It may for instance turn out that one has to replace the strategy of random unsharp measurements by some adaptive strategy.

We profited from analytic tools. We used the standard theory of (Markovian) continuous quantum measurement and we completed it with the novel concept of continuous state estimation. The recent work [15] has already coined a similar concept. It has, however, been restricted to the particular case of Gaussian states. Although we have detailed the concept for a single qubit, most of the equations are valid for any higher dimension N. The standard theory of continuous quantum measurement treats discrete and continuous observables on an equal footing with the same formalism. We guess that our continuous estimation formalism can also be applied to the tomography of light quanta [16], particularly to its Gaussian POVM formulation [17].

Stochastic differential equations, used so far for continuous measurement, will apply to optimum state determination as well. Continuous state determination is of interest every time one is accumulating and analysing information from low rate quantum inference. These conditions are typical for an eavesdropper of secret quantum communication, a cloner of $n \gg 1$ identical qubits into $n+1$ identical qubits, or in tomography with low detection efficiency.

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Appendix. Continuous measurement and estimation

In the continuum limit (19), the outcome $\vec{\sigma}_.$ of sequential measurement (section 3) becomes a (vectorial) function $\vec{\sigma}_t$ of time. The basic mathematical objects will be functionals of the outcome. First of all, we define the continuum limit of the sequential Kraus-operators (12) in terms of the time-ordered exponentials:

$$\hat{G}_t[\vec{\sigma}] = \mathrm{T}\exp\left[-\int_0^t |\hat{\vec{\sigma}} - \vec{\sigma}_\tau|^2 \,\mathrm{d}\tau\right]. \tag{28}$$

The normalizing pre-factor of the exponential has been omitted and, as usual, will be incorporated in the functional measure $\mathrm{d}[\vec{\sigma}]$. The above operators yield the continuum limit of the sequential POVM (13):

$$\hat{\Pi}_t[\vec{\sigma}] = \hat{G}_t^\dagger[\vec{\sigma}]\,\hat{G}_t[\vec{\sigma}]. \tag{29}$$

It describes the isotropic continuous polarization measurement in the period $[0, t]$. The POVM satisfies the completeness relation at any time, with respect to the functional integration:

$$\int \hat{\Pi}_t[\vec{\sigma}]\,\mathrm{d}[\vec{\sigma}] \equiv \hat{I}. \tag{30}$$

The operators $\hat{\Pi}_t[\vec{\sigma}]$, a kind of time-ordered Gaussian projectors, form a functional POVM for all time t. Given the random outcome $\{\vec{\sigma}_\tau; 0 \leqslant \tau \leqslant t\}$, the *a posteriori* state at time t takes this form:

$$\hat{\rho} \longrightarrow \hat{\rho}_t[\vec{\sigma}] = \frac{\hat{G}_t[\vec{\sigma}]\hat{\rho}\hat{G}_t^\dagger[\vec{\sigma}]}{\mathrm{Tr}\,[\hat{\Pi}_t[\vec{\sigma}]\hat{\rho}]} \tag{31}$$

with the normalized functional probability distribution

$$p_t[\vec{\sigma}] = \mathrm{Tr}\,[\hat{\Pi}_t[\vec{\sigma}]\hat{\rho}]. \tag{32}$$

Equations (28)–(32) constitute the model of isotropic continuous measurement of the polarization $\hat{\vec{\sigma}}$. Similar to the case of a single measurement, the choice of the estimate $\hat{\rho}'_t$ is not unique. Following (7) and (16), as well as for mathematical convenience, we take

$$\hat{\rho}'_t[\vec{\sigma}] = \frac{\hat{\Pi}_t[\vec{\sigma}]}{\mathrm{Tr}\,\hat{\Pi}_t[\vec{\sigma}]} \tag{33}$$

and, as in section 2, we interpret it as the random average of its pure eigenstates.

In contrast to the *a posteriori* state $\hat{\rho}_t$, the estimate state $\hat{\rho}'_t$ does not satisfy an autonomous stochastic differential equation. Neither does the composite object $\hat{\rho}_t \otimes \hat{\rho}'_t$. To construct a closed set of stochastic differential equations, we introduce the state $\hat{\rho}^?_t$ where a hypothetical initial state $\hat{I}/2$ would have evolved under the *true* operations $\hat{G}_t[\vec{\sigma}]$ which the *true a priori* state $\hat{\rho}_0 = \hat{\rho}$ had undergone:

$$\hat{\rho}^?_t = \frac{\hat{G}_t[\vec{\sigma}]\,\hat{G}_t^\dagger[\vec{\sigma}]}{\mathrm{Tr}\,[\hat{\Pi}_t[\vec{\sigma}]]}. \tag{34}$$

Note in contrast to the preceding sections, in particular to section 2, that here we retain for $\hat{\rho}^?_t$ probability (32) of the *true* continuous measurement. (Actually, we could have modified the notation $\hat{\rho}^?_t$.) We introduce two normalized variants of the Kraus-operators (28):

$$\hat{g}_t = \frac{\hat{G}_t}{[\mathrm{Tr}\,\hat{\Pi}_t\hat{\rho}]^{1/2}} \qquad \hat{g}'_t = \frac{\hat{G}_t}{\left[\frac{1}{2}\mathrm{Tr}\,\hat{\Pi}_t\right]^{1/2}}. \tag{35}$$

They will build up the time-dependent *a posteriori* (31), estimate (33) and hypothetic state (34), respectively:

$$\hat{\rho}_t = \hat{g}_t\hat{\rho}\hat{g}_t^\dagger \qquad \hat{\rho}'_t = \tfrac{1}{2}(\hat{g}'_t)^\dagger\hat{g}'_t \qquad \hat{\rho}^?_t = \tfrac{1}{2}\hat{g}'_t(\hat{g}'_t)^\dagger. \tag{36}$$

The normalizations $\mathrm{Tr}\,\hat{\rho}_t = \mathrm{Tr}\,\hat{\rho}'_t = \mathrm{Tr}\,\hat{\rho}^?_t \equiv 1$ of these states follow from normalizations (35). Two time-dependent expectation values will be defined in function of the normalized operators (35):

$$\langle\hat{\vec{\sigma}}\rangle_t = \mathrm{Tr}\,[\hat{\rho}(\hat{g}_t)^\dagger\hat{\vec{\sigma}}\hat{g}_t] \qquad (=\mathrm{Tr}\,[\hat{\rho}_t\hat{\vec{\sigma}}]) \tag{37}$$

$$\langle\hat{\vec{\sigma}}\rangle^?_t = \tfrac{1}{2}\mathrm{Tr}\,[(\hat{g}'_t)^\dagger\hat{\vec{\sigma}}\hat{g}'_t] \qquad (=\mathrm{Tr}\,[\hat{\rho}^?_t\hat{\vec{\sigma}}]). \tag{38}$$

For the sake of symmetry, I propose the normalized operators $\hat{g}_t$ and $\hat{g}'_t$, yielding $\hat{\rho}_t$ and $\hat{\rho}'_t$ via (36), to formulate a convenient couple of equations. An autonomous stochastic differential equation will exist for $\hat{g}_t$:

$$\frac{\mathrm{d}\hat{g}_t}{\mathrm{d}t} = \left[-\tfrac{1}{2}|\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle_t|^2 + (\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle_t)\vec{w}_t\right]\hat{g}_t. \tag{39}$$

This equation is equivalent to the well-known conditional master equation (21). A new equation can be written down for $\hat{g}'_t$:

$$\frac{\mathrm{d}\hat{g}'_t}{\mathrm{d}t} = \left[-|\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle_t|^2 + \tfrac{1}{2}|\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle^?_t|^2 + |\langle\hat{\vec{\sigma}}\rangle_t - \langle\hat{\vec{\sigma}}\rangle^?_t|^2 + (\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle^?_t)\vec{w}_t\right]\hat{g}'_t. \tag{40}$$

This equation couples to the previous equation via $\langle\hat{\vec{\sigma}}\rangle_t$ in addition to the white noise $\vec{w}_t$. The initial conditions are $\hat{g}_0 = \hat{g}'_0 = \hat{I}$. It is straightforward to show that the above equations preserve the normalization of $\hat{\rho}_t$ and $\hat{\rho}'_t$.

We outline the proof of equations (39) and (40). The proof will reside on the equation $\vec{\sigma}_t = \langle\hat{\vec{\sigma}}\rangle_t + \frac{1}{2}\vec{w}_t$ of continuous measurement theory (22). Let us substitute it into definition (28) of the Kraus-operator $\hat{G}_t[\vec{\sigma}]$. It yields

$$\mathrm{T}\exp\left[-\int_0^t |\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle_\tau|^2\,\mathrm{d}\tau + \int_0^t (\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle_\tau) w_\tau\,\mathrm{d}\tau\right] \tag{41}$$

times a numeric factor which will be irrelevant for the normalized operators $\hat{g}_t$ and $\hat{g}'_t$. It turns out that the above exponential *is* already the properly normalized $\hat{g}_t$:

$$\hat{g}_t = \mathrm{T}\exp\left[-\int_0^t |\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle_\tau|^2\,\mathrm{d}\tau + \int_0^t (\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle_\tau)\vec{w}_\tau\,\mathrm{d}\tau\right]. \tag{42}$$

Indeed, differentiating the above equation yields exactly equation (39).

Derivation of the novel equation (40) for $\hat{g}'_t$ is a bit more complicated. In addition to the exponential in equation (42), we assume a further c-number differential for the sake of normalization (35):

$$\hat{g}'_t = \exp\left[\int_0^t \alpha_\tau\,\mathrm{d}\tau + \vec{\beta}_\tau \vec{w}_\tau\,\mathrm{d}\tau\right] \mathrm{T}\exp\left[-\int_0^t |\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle_\tau|^2\,\mathrm{d}\tau + \int_0^t (\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle_\tau)\vec{w}_\tau\,\mathrm{d}\tau\right]. \tag{43}$$

We calculate $\mathrm{d}\hat{g}'_t/\mathrm{d}t$ and insert it into the normalization condition $\mathrm{Tr}\,\mathrm{d}\hat{\rho}'_t/\mathrm{d}t = 0$. This will yield the unique solutions $\vec{\beta}_t = \langle\hat{\vec{\sigma}}\rangle_t - \langle\hat{\vec{\sigma}}\rangle^?_t$ and $\alpha_t = |\beta_t|^2$. Inserting these results back into the equation of $\mathrm{d}\hat{g}'_t/\mathrm{d}t$ we obtain equation (40).

The evolution of the *a posteriori* $\hat{\rho}_t$ and the estimate state $\hat{\rho}'_t$ is indirectly described by the coupled stochastic differential equations (39) and (40). We mentioned that $\hat{\rho}$ obeys a closed equation but $\hat{\rho}_t \otimes \hat{\rho}'_t$ does not. From the above results, it would be trivial to show that $\hat{\rho}_t \otimes \hat{g}'_t \otimes (\hat{g}'_t)^\dagger$ contains all information on $\hat{\rho}_t \otimes \hat{\rho}'_t$ and it *does* satisfy a closed stochastic differential equation.

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