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Evolution of a qubit under the influence of a succession of weak measurements with unitary feedback

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We investigate the evolution of a single qubit subject to a continuous unitary "free" dynamics and an additional interrupting influence which occurs periodically. One may imagine a dynamically evolving closed quantum system which becomes open at certain times. The interrupting influence is represented by an operation, which is assumed to equivalently describe a nonselective weak measurement. It may be decomposed into the action of a positive operator, which in the case of a measurement represents the pure measurement part, followed by a unitary backaction depending on the result of the measurement (feedback). Equations of motion for the state evolution are derived in the form of difference equations. In the lowest order, the stochastic feedbacks cause a modification of the "free" Hamiltonian by an additional appropriately averaged Hamiltonian. The positive operator specifies a decoherence rate and results in a decoherence term. Two higher-order terms are discussed. One shows decoherence induced by the stochastic feedbacks and the other represents generalized friction. The selective evolution is investigated. In order to bridge the gap between sequential and continuous measurements, the continuum limit to a master equation is performed. Additional correcting higher-order terms are worked out in the Appendixes.

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I. INTRODUCTION

Experimental and theoretical studies of the dynamics of single two-level systems have become very important in the context of quantum computation and quantum information. In this paper, we investigate the evolution of a single qubit subject to two influences. On the one hand, there is a continuously acting unitary dynamics (undisturbed or "free" dynamics) with operator U. On the other hand, the qubit is affected by an interrupting additional influence, which is nonunitary and acts periodically at times $t_n = t_0 + n\tau$, $n = 1,2 \ldots$. The duration $\delta\tau$ of this influence is assumed to be much shorter than τ, so that it can be neglected. One may imagine a dynamically evolving closed quantum system which becomes open at times t_n. The corresponding *single influence* is represented by an *operation* $\mathcal{E}$ which transforms the state of the qubit given by its reduced density operator ρ according to

$$\rho \to \mathcal{E}(\rho) = \sum_{k=\pm} M_k \rho M_k^\dagger \tag{1}$$

with *operation elements* M_k, which are sometimes also called Kraus operators. $\mathcal{E}$ is assumed to be trace preserving. The representation (1) of the single operation is called the *operator-sum representation* or Kraus representation.

Such a periodically occurring, nearly instantaneous change can be caused by a recurring interaction with a second system provided that this system does not "remember" the influence it may have experienced from the qubit at former times (Markov process). Typically the second system could be an environment or it could consist of a number of systems of the same kind which interact only once with the qubit, as is the case in a sequence of scattering processes.

Let us determine the single influence further. According to Eq. (1), we are not dealing with the most general form of such an influence on a qubit, which would correspond to an operation with four operation elements, but we restrict ourselves to interactions which may be represented by only two elements, M_+ and M_-. Pairs of operation elements can describe such important operations as, for example, amplitude damping and phase damping, bit flips, and phase flips, as well as projection measurements and weak measurements. The concept of a weak measurement will be explained below. We will restrict ourselves to an operation which is equivalent, as far as the map $\mathcal{E}$ is concerned, to a nonselective weak measurement. For simplicity, we will speak about the operation and its elements in terms of generalized measurements. Please note that this may still comprise many physical processes which at first glance do not look like a weak measurement.

To specify the operation elements and to reveal the different physical effects which are caused by the operation $\mathcal{E}$, it is useful to decompose $M_\pm$. According to the polar decomposition theorem, each operation element $M_\pm$ may be written as a product of a unitary operator and a positive operator

$$M_\pm = U_\pm |M_\pm|. \tag{2}$$

We are dealing with a class of generalized measurements with two outcomes $+$ and $-$. The unitary part $U_\pm$ extracts no information from the qubit. $U_\pm$ can reflect the coupling of the system with the meter (backaction of the meter) or can be due to an external field which is applied depending on the outcome of the measurement (thus $U_+ \neq U_-$). It then represents an *instantaneous feedback* [1,2]. We will allow both possibilities and speak uniformly of feedback. The case $U_+ = U_-$ indicates an additional unitary evolution independent of the measurement outcome. $U_\pm = \mathbb{1}$ implies the minimal disturbance. The set $\{|M_-|^2,|M_+|^2\}$ represents a positive-operator-valued measure (POVM). The demand of weakness of the single influence is introduced below by conditions on $U_\pm$ and $|M_\pm|$. A sequence of such generalized measurements are of practical importance because they can be employed to explore the original dynamics of the system [3,4] or, not less important, to control its dynamics by means of a specific feedback [1,5].

In the nonselective case of Eq. (1), when the measurement results $\pm$ are not read off, our total physical setup defined by U, τ, $U_\pm$, and $M_\pm$ may also be regarded as a particular noisy channel. Below, the unitary operators $U_\pm$ will not be neglected because they form an important part of the sequential operations in realistic situations. In general, it would need a nontrivial feedback procedure to eliminate their influence; compare [3] for an example.

It is our goal to derive equations of motions for the state of the system subject to the periodical influence in the form of *difference equations*. Difference equations take into account the discrete nature of the influence due to the finite time τ between its occurrences and grant therefore a more exact description. In different orders of weakness, a rich structure of physical influences appears which furthermore cannot be seen in the continuum limit.

Although it is only a byproduct and not our central aim, we also look at the limit $\tau\rightarrow 0$ of continuous measurements and compute the corresponding master equation (nonselective description). The master equation can be understood as a means to compute approximately the dynamics of a system which is subject to a sequence of operations of the type (1). Since there are a lot more mathematical methods to solve differential equations than difference equations, it may be useful in some cases to approximate sequential measurements by continuous measurements as far as the study of the dominant physical influence is concerned.

To point out what is new in our approach, we describe the related literature in some detail: Master equations for special cases of measurements with a nonvanishing unitary part of the operation elements have been considered in the literature. A master equation for general feedback was derived by Wiseman [6]. However, [6] does not comprise our results since it deals with a special kind of continuous measurements. They have Poissonian statistics and allow finite-state changes during infinitesimal time intervals. More precisely, this means that only very seldom a certain measurement result occurs which is then connected to a finite-state change during an infinitesimal time while for other measurement results the state changes only infinitesimally. We are excluding Poissonian statistics and require the state change in the continuum limit to be infinitesimal during infinitesimal times. Thus Wiseman's and our studies do not overlap.

In a later paper, Wiseman [2] employed the operation formalism to analyze a homodyne measurement in quantum optics to apply instantaneous feedback in order to minimize disturbance, i.e., compensate the unitary part of the operation. Korotkov investigated a measurement with nonminimal disturbance in the context of continuous measurement of a qubit by means of a single electron transistor [7,8]. He noted that this nonminimal disturbance acts in the master equation like a change of the distance between the two energy levels of the qubit. We find this effect as a special case of our studies ($[U_\pm,H]=0$), cf. Eq. (57) and the text below. Korotkov also derives a modification of the stochastic master equation (selective regime) due to the nonminimal disturbance [7], which is averaged out in the nonselective case.

Many examples of the application of instantaneous feedback in continuous measurements by means of external changes of the Hamiltonian of the system can be found in the literature of quantum control (e.g., [1,5]). They correspond to special choices of the unitary part of the measurement operation and obey the equations of motions derived here, provided the measurements do not inflict finite changes during infinitesimal times.

In the context of quantum dissipation, the influence of a heat bath on an infinite-dimensional quantum system has been investigated by Caldeira and Leggett [9]. The coupling was such that in terms of the operation formalism, the corresponding operation of the system had a unitary part additional to the one stemming from its free original evolution. They derived a master equation for high temperatures which was later modified to also describe medium temperatures [10]. Although we are looking at a qubit, we find in the difference equation among others similar terms such as those which represent decoherence and dissipation, but only one of them survives in the continuum limit.

Barchielli *et al.* [11] studied the continuum limit of sequential measurements without unitary feedback in an infinite-dimensional Hilbert space. In the context of continuous position measurements, Caves and Milburn [12] presented a sequential measurement which possesses Gaussian-shaped effects. They also employed a special feedback, which served to compensate jumps in the mean position and mean momentum.

As compared to these papers, we study a quantum object in a two-dimensional Hilbert space (qubit) which is influenced in a genuinely sequential manner. We discuss Gaussian statistics and we include a general feedback ($U_\pm\neq\mathbb{1}$). Our focus on the discrete nature of the sequential process and the inclusion of general feedback are the points in which we differ from the literature. The related difference equation shows physical effects which are lost in a continuum limit.

As mentioned above, our results can be applied to weak influences. Examples are sequences of scattering processes (cf. [13] the experimental setup of Brune, Haroche *et al.* [14] cf. [3]), and similar experiments which are usually not discussed in terms of generalized measurements.

We proceed as follows. We first consider the single measurements of the sequence. Then we bundle the whole sequence in subsequences of N measurements ("N series"). The resulting operation has a Gaussian shape. Afterwards we integrate over all outcomes to obtain the average state change due to an N series (nonselective regime). We then discuss how to find the right continuum limit which conserves the physical characteristics of the N series and derive the master equation. Finally, we deal with the selective regime of measurement and write down the stochastic master equation. The Appendixes serve to derive the difference equation for the nonselective regime up to second order in the occurring small parameters.

II. THE SINGLE QUANTUM OPERATION

An example for a physical realization of the operation $\mathcal{E}$ from Eq. (1) is given by a qubit which interacts at times t_n unitarily for a short duration $\delta\tau$ with an environment and thus becomes an open system. The resulting change of its reduced density operator may formally be expressed with the help of operation elements $M_\pm$ as in Eq. (1). Just the same change of the reduced density operator results if (i) a projection measurement on the environment is performed with outcome $+$ or $-$ transferring the qubit to the states

$$\rho \rightarrow \rho_\pm = \frac{M_\pm \rho M_\pm^\dagger}{\mathrm{tr}[M_\pm \rho M_\pm^\dagger]} \tag{3}$$

respectively and (ii) the outcome is not read off (nonselective case). In the generic case, Eq. (3) describes thereby a generalized measurement of the qubit. For simplicity reasons the terminology we are going to use will refer to measurements, but the results apply equally to any operation with operator-sum representation (1) if the same specifications of $M_\pm$ are made. This is independent of how the operation is experimentally realized.

Because of the polar decomposition theorem, the operation elements $M_\pm$ may be written as products of a unitary operator and a positive operator

$$M_\pm = U_\pm |M_\pm|. \tag{4}$$

We introduce the POVM *effects*

$$E_\pm = |M_\pm|^2, \tag{5}$$

which obey the completeness relation

$$E_+ + E_- = \mathbb{1}. \tag{6}$$

The probability of the outcome $+$ or $-$ is given by

$$p_\pm = \langle E_\pm \rangle_\rho \tag{7}$$

with $p_+ + p_- = 1$.

Equation (4) represents a decomposition of the operation into a *pure measurement part* described by $|M_\pm|$, followed by a *unitary feedback* given by $U_\pm$ depending on the result $+$ or $-$. These denominations are justified for the following reasons: All the information which can be read off from the meter is related to $|M_\pm|$, which therefore represents the unavoidable minimal disturbance. The unitary operators leave the von Neumann entropy unchanged and therefore do not allow to export information to an observer. Because they depend on the result $+$ or $-$, they may be interpreted as a specific feedback caused by the measuring apparatus or by some other means inducing an additional Hamiltonian evolution of the qubit. We formally introduce the corresponding Hamiltonians $H_\pm$ according to

$$U_\pm =: \exp\left(-\frac{i}{\hbar} H_\pm \tau\right). \tag{8}$$

This unitary feedback represents an important part of the quantum operation and appears naturally in the generic situation.

According to Eqs. (5) and (6), $|M_+|^2$ and $|M_-|^2$ commute. Therefore, we can find orthonormal basis vectors $|1\rangle$ and $|2\rangle$ of the qubit Hilbert space with respect to which $|M_\pm|^2$ are diagonal. We introduce the eigenvalues p_1 and p_2 of $|M_+|^2$, which are positive and because of Eq. (6) obey $0 \leqslant p_{1,2} \leqslant 1$. Without restriction of generality, we choose $p_2 \geqslant p_1$. Reading off the eigenvalues of $|M_-|^2$ from Eq. (6) and taking the square root, we find

$$|M_+| := \sqrt{p_1}\,|1\rangle\langle 1| + \sqrt{p_2}\,|2\rangle\langle 2|, \tag{9}$$

$$|M_-| := \sqrt{1-p_1}\,|1\rangle\langle 1| + \sqrt{1-p_2}\,|2\rangle\langle 2|.$$

p_1 and p_2 are the probabilities to measure the outcome $+$ when the system is, respectively, in the state $|1\rangle$ or $|2\rangle$ immediately before the measurement.

The elements $|M_+|$ and $|M_-|$ commute. We will characterize the operation later on by the parameters

$$p_0 := \frac{1}{2}(p_1 + p_2), \quad \Delta p := p_2 - p_1, \tag{10}$$

with $0 \leqslant \Delta p \leqslant 1$. Introducing

$$\sigma_z := |1\rangle\langle 1| - |2\rangle\langle 2|, \tag{11}$$

the effects $E_\pm$ of Eq. (5) are rewritten in the form

$$E_+ = p_0 \mathbb{1} - \tfrac{1}{2}\Delta p\, \sigma_z, \quad E_- = (1-p_0)\mathbb{1} + \tfrac{1}{2}\Delta p\, \sigma_z. \tag{12}$$

In the limiting case $\Delta p = 1$, the pure part of the measurement (9) results in a projection on $|1\rangle$ or $|2\rangle$ depending on the measurement outcome $+$ or $-$. We call this a *sharp measurement* of an observable with eigenvectors $|1\rangle$ and $|2\rangle$, for example σ_z. Note that also for a sharp measurement the result of the quantum operation (1) will in general not be the state $|1\rangle$ or $|2\rangle$ because of the remaining influence of the unitary feedback.

Up to now no approximation has been made. We have only rewritten the most general operation with two operation elements. We now specify a weak single measurement as having a weak influence on the state by demanding for the pure measurement part

$$\frac{\Delta p}{2\min\{p_0, 1-p_0\}} \ll 1 \tag{13}$$

and for the feedback

$$U_\pm \approx \mathbb{1}. \tag{14}$$

Because of Eq. (13), the $|M_+|$ and $|M_-|$ are nearly proportional to the identity operator. Accordingly, the pure measurement part has only a weak influence on the state. Equation (12) shows that the probability p_+ (or p_-) to obtain the measurement result $+$ (or $-$) is then nearly independent of the initial state of the qubit. There is almost no state discrimination. Because of this low sensitivity, we are also dealing with an *unsharp measurement*. Note that in this limit the parameters p_0 and $1-p_0$ become, because of Eqs. (7) and (12), approximately the mean probabilities to obtain the measurement results $+$ or $-$, respectively. The constraint (14) demands a weak feedback so that also this influence of the operation element is weak.

It may be instructive to see how, for example, amplitude damping is excluded by the conditions above. The two-level systems decay process (quantum jump) is described by the operation

$$\mathcal{E}_{\mathrm{AD}}(\rho)=\tilde{M}_+\rho\tilde{M}_+^\dagger+\tilde{M}_-\rho\tilde{M}_-^\dagger \tag{15}$$

with

$$\tilde{M}_+=\sqrt{p}\,|1\rangle\langle 2|, \tag{16}$$

$$\tilde{M}_-=|1\rangle\langle 1|+\sqrt{1-p}\,|2\rangle\langle 2|. \tag{17}$$

The polar decomposition (4) leads to the diagonal $|M_\pm|$ of Eq. (9) with

$$\tilde{U}_+=-i\sigma_x,\quad \tilde{U}_-=\mathbb{1}, \tag{18}$$

$$|\tilde{M}_+|=\sqrt{p}|2\rangle\langle 2|,\quad |\tilde{M}_-|=\tilde{M}_-, \tag{19}$$

so that $p_1=0$, $p_2=p$, $\Delta p=p$, and $p_0=p/2$. Neither of the weakness conditions (13) and (14) are fulfilled for quantum jumps even for arbitrarily small p. The single jump, when it occurs, has a strong influence on the state of the qubit. Note that U_+ is not close to 1. This is a Poisson process that contradicts our conditions.

We are mainly interested in the nonselective case where information about the results $\pm$ or the corresponding states of the environment is not available. The influence on the qubit at times t_n may then be written in the operator-sum representation as in Eq. (1),

$$\rho\rightarrow\mathcal{E}(\rho)=\sum_{k=\pm}M_k\rho M_k^\dagger, \tag{20}$$

whereby

$$\sum_{k=\pm}M_k^\dagger M_k=\mathbb{1} \tag{21}$$

because of Eq. (6). The quantum operation $\mathcal{E}$ is trace-preserving.

III. *N* SERIES AND RELATED OPERATION

The time between consecutive measurements is τ. We assume that the duration $\delta\tau$ of a measurement is much shorter than τ. The undisturbed or "free" dynamics of the system between the measurements is given by the Hamiltonian H. We bundle N consecutive measurements to an N *series* of duration $\Delta t=N\tau$ as we have done in [3] (cf. also [15]). This procedure has several advantages: We will obtain effects of explicit Gaussian structure for the N series. This enables us to work out the operator sum explicity. A comparison with the results in the literature regarding continuous measurements becomes more evident. And finally the discussion of the selective case is simpler.

We require

$$N\gg 1. \tag{22}$$

We relate N to the unsharpness of the measurement by

$$\frac{N\Delta p}{2\min\{p_0,1-p_0\}}\ll 1 \tag{23}$$

and demand in addition

$$\Delta t\|H\|\ll\hbar \tag{24}$$

and

$$\Delta t\|H_\pm\|\ll\hbar. \tag{25}$$

This sharpens the conditions (13) and (14). It means that the influence of the undisturbed dynamics of the qubit and the unitary feedback dynamics due to the measurements are both small over the duration Δt of an N series. With $\Delta t=N\tau$ we have obtained the above restrictions for N,τ,H, and $H_\pm$.

The density operator resulting at the end of an N series of measurements with results $m_1,\ldots,m_N$, each of which can assume the values "$+$" and "$-$," reads

$$\rho(t+\Delta t)=M_{m_N}U\cdots M_{m_1}U\rho(t)U^\dagger M_{m_1}^\dagger\cdots U^\dagger M_{m_N}^\dagger \tag{26}$$

with

$$U:=\exp\left\{-\frac{i}{\hbar}H\tau\right\}. \tag{27}$$

The influences of the system's dynamics and the measurement will in general not commute. The following relation is derived in Appendix A:

$$M_{m_N}UM_{m_{N-1}}U\cdots M_{m_1}U=M_{m_N}M_{m_{N-1}}\cdots M_{m_1}U^N(1+C_1) \tag{28}$$

with

$$\|C_1\|\leqslant O(N\Delta p\Delta t\|H\|/\hbar)+O(\Delta t^2\|H\|\max\{\|H_\pm\|\}/\hbar^2). \tag{29}$$

Also the operations $M_\pm$ will not commute. Based on the decomposition (4), we show in Appendix A that

$$\begin{aligned}U_{m_N}|M_{m_N}|\cdots U_{m_1}|M_{m_1}|U^N\\=U_+^{N_+}U_-^{N-N_+}|M_+|^{N_+}|M_-|^{N-N_+}U^N(1+C_2)\end{aligned} \tag{30}$$

with

$$\|C_2\| \leqslant O(N\Delta p \Delta t \|\max\{\|H_\pm\|\}/\hbar) + O(\Delta t^2 \|H_+\| \|H_-\|/\hbar^2). \tag{31}$$

N_+ and $N-N_+$ are the total numbers of measurement results "$+$" and "$-$" in the N series, respectively. Because of the assumptions (23), (24), and (25), we may neglect C_1 and C_2. The calculation which takes into account C_1 and C_2 is given in Appendix C.

In our approximation, the influence (26) of the N series becomes a function of N_+ only, independent of the $\binom{N}{N_+}$ different orderings of the "$+$" and "$-$" results [cf. Eq. (30)]. Therefore, the total N series of duration Δt including the "free" dynamics given by H corresponds to a quantum operation with operation elements

$$M(N_+,N) = U_+^{N_+} U_-^{N-N_+} |M(N_+,N)| \exp\left\{-\frac{i}{\hbar} H \Delta t\right\} \tag{32}$$

with

$$\begin{aligned}|M(N_+,N)| &= \sqrt{\binom{N}{N_+}} |M_+|^{N_+} |M_-|^{N-N_+} \\ &= \sqrt{\binom{N}{N_+}} [p_1^{N_+/2}(1-p_1)^{(N-N_+)/2} |1\rangle\langle 1| \\ &\quad + p_2^{N_+/2}(1-p_2)^{(N-N_+)/2} |2\rangle\langle 2|].\end{aligned} \tag{33}$$

The square root in front ensures the completeness relation of the effects

$$\sum_{N_+=0}^{N} M(N_+,N)^\dagger M(N_+,N) = 1. \tag{34}$$

The unitary feedback part caused by the N-series measurements can be expressed as

$$\begin{aligned}U(N_+,N) &:= U_+^{N_+} U_-^{N-N_+} \\ &= \exp\left\{-\frac{i}{\hbar}[N_+ H_+ + (N-N_+)H_-]\tau\right\}.\end{aligned} \tag{35}$$

We now make use of the condition that N is large, so that $|M(N_+,N)|$ of Eq. (33) may approximately be written in the form of a Gaussian,

$$|M(N_+,N)| = \frac{1}{\sqrt[4]{2\pi N E_+ E_-}} \exp\left\{-\frac{(NE_+ - N_+)^2}{4NE_+E_-}\right\}, \tag{36}$$

which contains the operators $E_\pm$ of Eq. (12). Because we assumed that the single measurements are unsharp and therefore Eq. (13) is fulfilled, the "spread" of the Gaussian becomes in lowest order a c number,

$$\begin{aligned}E_+E_- &= [p_0(1-p_0) - \tfrac{1}{4}(\Delta p)^2]\mathbb{1} + \Delta p(p_0 - \tfrac{1}{2})\sigma_z \\ &= p_0(1-p_0),\end{aligned} \tag{37}$$

where we have ignored terms of order Δp and higher on the right-hand side of Eq. (37). The error thus committed in the Gaussian in Eq. (36) is of order $N\Delta p^3$, which can be seen by inserting E_+ from Eq. (12) and N_+/N from Eq. (38) and expanding Gaussian (36) in powers of Δp. A more detailed calculation can be found in Appendix C.

We introduce a new variable s to replace the readout N_+/N of the N series according to

$$\frac{N_+}{N} =: p_0 - \tfrac{1}{2}\Delta p s. \tag{38}$$

Because N is large, we may approximately regard s as continuous. Its range is limited by

$$0 \leqslant p_0 - \tfrac{1}{2}\Delta p s \leqslant 1. \tag{39}$$

In addition, we introduce the new quantity

$$\gamma := \frac{(\Delta p)^2}{4p_0(1-p_0)\tau}, \tag{40}$$

which will turn out to be the *decoherence rate* in lowest order. It contains apart from Δp and p_0 also the time interval τ between two measurements. These three parameters characterize completely the influence of the sequence of pure measurements. γ increases when the pure measurements become sharper and accordingly have a stronger influence on the qubit. A decreasing time difference τ between two measurements results as well in an increase of γ. This reflects a *Zeno-type effect* which also happens for sequential weak measurements.

Installing γ, we get from Eq. (36) the ultimate expression for the pure measurement part, valid for $\Delta p/[p_0(1-p_0)] \ll \tau/\Delta t \ll 1$, which follows from Eq. (23):

$$|M_s| = \frac{1}{\sqrt[4]{2\pi/(\gamma\Delta t)}} \exp\left\{-\gamma \frac{(\sigma_z - s)^2}{4} \Delta t\right\}. \tag{41}$$

The resulting effects $E_s = |M_s|^2$ have Gaussian form. They show the characteristics which are known, for instance, from an unsharp position measurement as investigated, e.g., in [12]. Instead of a continuous observable, however, we are dealing here with a discrete observable.

With reference to s, the complete operation elements of the N series including the feedback and the "free" evolution are given by

$$M_s = U_s |M_s| \exp\left\{-\frac{i}{\hbar} H \Delta t\right\}, \tag{42}$$

where, using Eqs. (35) and (38), we obtain for U_s which replaces $U(N_+,N)$ of Eq. (35)

$$U_s = \exp\left\{-\frac{i}{\hbar} H_{AV} \Delta t - \frac{i}{2\hbar} \Delta H s \Delta p \Delta t\right\}. \tag{43}$$

We have thereby introduced the *averaged feedback Hamiltonian* H_{AV} and the difference ΔH of the feedback Hamiltonians, respectively:

$$H_{AV} := p_0 H_+ + (1-p_0) H_- , \tag{44}$$

$$\Delta H := (H_- - H_+). \tag{45}$$

M_s of Eq. (42) replaces $M(N_+,N)$ of Eq. (32) for the continuous variable s.

The N-series operation elements above correspond to a continuous set of effects with the Gaussian distribution function

$$p_s = \langle M_s^\dagger M_s \rangle_\rho . \tag{46}$$

The completeness relation is satisfied if we extend the range of s to the whole real axis:

$$\int_{-\infty}^{\infty} M_s^\dagger M_s ds = \mathbb{1}. \tag{47}$$

The statistical weight of the unphysical values of s will be negligible, if condition (13) is satisfied. In fact, Eq. (13) was designed to exclude that the binomial distributions in Eq. (33) do peak at $N_+ = 0$ or $N_+ = N$. This in turn leads to negligible weights for the unphysical values of s and thus justifies the formal extension of the values of s beyond their physical range (39).

IV. NONSELECTIVE EVOLUTION

In the nonselective case, the state change during an N series can be expressed in the operator-sum representation as

$$\rho(t+\Delta t) = \int_{-\infty}^{\infty} M_s \rho M_s^\dagger ds. \tag{48}$$

We are going to expand the right-hand side (r.h.s.) up to linear terms in Δt.

The unitary parts of the operation M_s which are generated by H, H_{AV}, and ΔH lead to

$$\Delta\rho := \rho(t+\Delta t) - \rho(t) = -\frac{i}{\hbar}[H + H_{AV}, \rho(t)]\Delta t + D(\rho(t)) - \rho(t). \tag{49}$$

The integral

$$D(\rho) := \int_{-\infty}^{\infty} \exp\left\{ -\frac{i}{2\hbar}\Delta H s \Delta p \Delta t \right\} |M_s|\rho|M_s| \times \exp\left\{ \frac{i}{2\hbar}\Delta H s \Delta p \Delta t \right\} ds \tag{50}$$

over the parts which depend on s will first be calculated and then expanded.

Introducing operators which act from the left and are denoted with L (e.g., $\sigma_z^L \rho := \sigma_z \rho$) as well as operators which act from the right and are denoted with R (e.g., $\sigma_z^R \rho := \rho\, \sigma_z$), we rewrite the integrand of $D(\rho)$,

$$D(\rho) = \frac{1}{\sqrt{2\pi\gamma/\Delta t}} \times \int_{-\infty}^{\infty} \exp\left\{ -\frac{i}{2\hbar}(\Delta H^L - \Delta H^R) s \Delta p \Delta t \right\} \times \exp\left\{ -\frac{\gamma}{4}((\sigma_z^L - s)^2 + (\sigma_z^R - s)^2)\Delta t \right\} ds\, \rho. \tag{51}$$

It is important to take into account a further operator ordering for the integrand, namely that $\Delta H^L, \Delta H^R$ should remain leftmost and rightmost, respectively. The resulting integral is Gaussian in s. It may be solved in a closed form,

$$D(\rho) = \exp\left\{ -\left[\frac{\gamma}{8}(\sigma_z^L - \sigma_z^R)^2 + \frac{\Delta p^2}{8\gamma\hbar^2}(\Delta H^L - \Delta H^R)^2 + i\frac{\Delta p}{4\hbar}(\sigma_z^L + \sigma_z^R)(\Delta H^L - \Delta H^R) \right] \Delta t \right\} \rho. \tag{52}$$

Now we expand it up to the leading linear term in Δt and restore the usual operator formalism according to the rules for L and R. This leads, for example, to

$$(\sigma_z^L - \sigma_z^R)^2 \rho = (\sigma_z^L - \sigma_z^R)(\sigma_z^L - \sigma_z^R)\rho = (\sigma_z^L - \sigma_z^R)[\sigma_z, \rho] = [\sigma_z, [\sigma_z, \rho]] \tag{53}$$

and all together to

$$D(\rho) - \rho = \left(-\frac{\gamma}{8}[\sigma_z, [\sigma_z, \rho]] - \frac{(\Delta p)^2}{8\gamma\hbar^2}[\Delta H, [\Delta H, \rho]] - i\frac{\Delta p}{4\hbar}[\Delta H, \{\sigma_z, \rho\}] \right) \Delta t. \tag{54}$$

While the second and the third term on the r.h.s. are proportional to small quantities [cf. Eqs. (23)–(25)], the first contains the ratio of the two small quantities Δp^2 and τ [cf. Eq. (40)]. We assume γ not to be small. We thus obtain as a final result in lowest order for the state evolution during one N series the difference equation

$$\Delta\rho = \left(-\frac{i}{\hbar}[H + H_{AV}, \rho] - \frac{\gamma}{8}[\sigma_z, [\sigma_z, \rho]] \right) \Delta t. \tag{55}$$

The first term on the r.h.s. represents the unitary dynamical evolution related to the "free" Hamiltonian H and to the averaged Hamiltonian $H_{AV} = p_0 H_+ + (1-p_0) H_-$. The quantities p_0 and $1-p_0$ are approximately the state-independent probabilities that the single feedback causes a Hamiltonian development with H_+ or H_-, respectively (cf. Sec. II). Thus the overall influence of the feedback has in lowest order an intuitive interpretation. The second term on

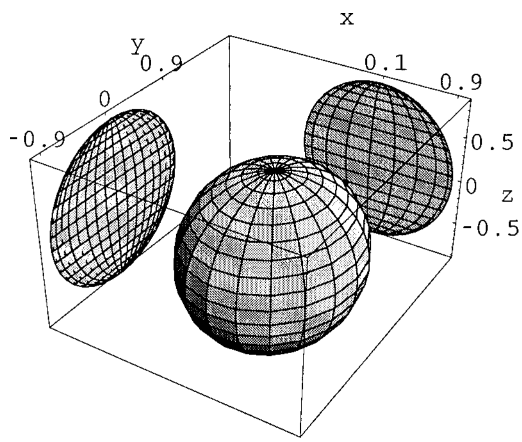

FIG. 1. Translation of the Bloch sphere in the x direction. The translation is generated by generalized friction (see text), which is connected to the feedback of the single measurements. In order to clearly demonstrate the effect, we have chosen the following parameters. Feedback: $U_{\pm}=\exp\{\mp i\sigma_y\pi/40\}$, pure measurement part: $\Delta p=-2/25$ and $p_0=\frac{1}{2}$. The N series which leads to the translation contains $N=10$ measurements. We assumed $U=\mathbb{1}$. Along with the translation, there comes a contraction of the Bloch sphere in the x and z directions which reflects the dephasing caused by the feedback and an additional contraction in the x and y directions due to the unsharp σ_z measurement. In order to recognize translation and contractions, consider the different scalings on the axes.

the r.h.s. reflects a pure decoherence in the eigenbasis of $|M_{\pm}|$, cf. Eq. (9). Since it also appears for $U_{\pm}=\mathbb{1}$, this decoherence is induced only by the pure measurement part $|M_{\pm}|$ of the single operations.

Already in the next order of the difference equation an abundant physical structure appears. This is shown in Appendix C. To demonstrate what type of effects are typically to be expected, we pick out the two higher-order terms which we have already obtained in Eq. (54). The second term on the r.h.s. represents additional decoherence in the basis in which ΔH is diagonal. This decoherence goes back to the fact that different feedbacks U_+ and U_- are acting in a random process on the qubit and thus cause a dephasing.

The third term on the r.h.s. of Eq. (54) is induced by the pure measurement part $|M_{\pm}|$ and by feedbacks with $U_+ \neq U_-$. We refer to its action as "generalized friction." It is of similar form as a friction-inducing term discovered in quantum Brownian motion [9]. Generalized friction can comprise energy dissipation as well as energy absorption, but in general it shifts the expectation value of all observables which do not commute with $|M_{\pm}|$ or with ΔH. This corresponds to a translation of the Bloch sphere perpendicular to the z direction and the direction associated with ΔH, cf. Fig. 1. Equation (49) together with Eq. (54) or the lowest-order approximation (55) describe the state change due to an N series. It can be employed to a sequential measurement by dividing the sequence of elementary measurements into a succession of N series. Since the r.h.s. of Eq. (55) is proportional to Δt, the given approximation is not sensitive to the division as long as N fulfills the conditions stated above.

This discrete-time analysis is the most natural approach to sequential measurements. An example of a sequential measurement where the difference equation can be applied is given in [4]. There an experiment is suggested to measure the oscillation of a photon between two coupled cavities. The energy of Rydberg atoms which pass through one of the cavities is detected. The detections constitute a sequence of weak measurements of the number of photons in the cavity which is passed by the Rydberg atoms. If the unitary part $U_{\pm}$ of the backaction is not compensated by a special feedback mechanism, effects like the modification of the unitary dynamics, generalized friction, and decoherence induced by the positive measurement part $|M_{\pm}|$ as well as the unitary part $U_{\pm}$ should be reflected in the outcome of the measurements.

V. CONTINUUM LIMIT

There are elaborated schemes for the treatment of permanently open quantum systems by continuous-time descriptions. Master equations are an example. One may profit from these schemes as approximations in the sequentially open case too, if a physically reasonable continuum limit $\tau\rightarrow 0$ is carried out. The corresponding demand for such a limit is that the physical characteristics of the sequential situation as they are found in the lowest-order difference equation (55) have to be taken over. Evidently we will lose all physical effects which only show up in higher order as, for example, the feedback-induced decoherence and generalized friction. This limits the usefulness of the continuum limit in our case.

On the other hand, the continuum limit connects sequential to continuous measurements. While this connection has been established for measurements of systems with infinite-dimensional Hilbert space without feedback [11] or with very special feedback [12], in the case of a qubit the continuum limit for our broad class of $U_{\pm}$ has not yet been discussed.

We proceed as follows: The quantity p_0 is the mean probability to obtain the measurement result $+$. We leave the value of p_0 unchanged in the continuum limit. In order not to change the decoherence behavior in the continuum limit, we next demand for the decoherence rate

$$\lim_{\tau\rightarrow 0}\frac{(\Delta p)^2}{4p_0(1-p_0)\tau}=\gamma=\text{const.} \tag{56}$$

The smaller is Δp, the weaker is the single measurement. With $\tau\rightarrow 0$ and the strength Δp of the single measurement unchanged, a Zeno effect would be obtained. This is prevented by appropriately diminishing the strength Δp of the measurement according to Eq. (56). This demand can also be found in the literature [11].

If in a given sequential physical situation the H_{AV} is nonvanishing, then the total Hamiltonian dynamics is, according to Eq. (55), governed by the Hamiltonian $H+H_{AV}$. We want to keep this dynamics in the continuum limit on physical grounds and demand, therefore, that H_{AV} remains unchanged. Performing the limit $\tau\rightarrow 0$ as specified above results in the master equation

$$\dot{\rho}=-\frac{i}{\hbar}[H+H_{AV},\rho]-\frac{\gamma}{8}[\sigma_z,[\sigma_z,\rho]], \tag{57}$$

which describes approximately the discontinuous situation in the noisy channel characterized above. In the special case of $[H_{AV},H]=0$, the modification of the Hamiltonian by H_{AV} reflects an effective shift of the energy levels.

VI. SELECTIVE EVOLUTION

In Sec. III, we calculated the Gaussian form (41) of effective operation elements (42) valid for an N series. In Sec. V, we derived the master equation (57) valid exactly in the continuous limit (56). As a matter of fact, the master equation describes the nonselective evolution. Selective evolution is, on the contrary, conditioned on the random measurement results (readout) and described by stochastic equations. In our case, the readout is s. It is the continuously measured unsharp value of the observable σ_z obtained in the N series in the limit (56).

The theory of the selective evolution has been available since long ago [16]. From the Gaussian operation elements (41) in the limit $\Delta t \to 0$, it has been proved that the selective evolution of the quantum state, conditioned on the measurement result s, satisfies the conditional master equation:

$$\dot{\rho}=-\frac{i}{\hbar}[H+H_{AV},\rho]-\frac{\gamma}{8}[\sigma_z,[\sigma_z,\rho]] + w\frac{\sqrt{\gamma}}{2}\{\sigma_z-\langle\sigma_z\rangle,\rho\}. \tag{58}$$

The function $w(t)$ is the standard white-noise and the equation should be understood in the Ito-stochastic sense. The state evolution couples to the readout s by

$$s=\langle\sigma_z\rangle+\frac{1}{\sqrt{\gamma}}w. \tag{59}$$

Obviously the stochastic mean of the conditional master equation (58) reduces to the unconditional master equation (57), as it should. Of course Eq. (58) applies to pure initial states as well. Then the pure state property $\rho^2=\rho$ is preserved. The derivation may be completely identical to that in Ref. [16]. In the continuum limit (56) the value of Δp must vanish and the feedback U_s is thus deterministic, given by H_{AV} alone.

The above equations of selective evolution are exact in the following sense. Elementary operations are being applied with frequency growing to infinity and strength decreasing to zero as given by Eq. (56), i.e., at fixed γ. We read out the rate N_+/N averaged over time Δt which should go to zero in such a way that $N=\Delta t/\tau$ still goes to infinity. The elementary time τ goes ‘‘faster’’ to zero than the time Δt to calculate the rate N_+/N. The calculated current rate N_+/N is related to s by Eq. (38):

$$s=\frac{p_0-N_+/N}{\sqrt{\gamma p_0(1-p_0)\tau}}. \tag{60}$$

The continuous limit (56) of s exists. As follows from Eq. (59), it is centered around a state-dependent part $\langle\sigma_z\rangle$ and superposed by the white-noise of constant intensity $1/\gamma$.

VII. CONCLUSIONS

We have derived in an approximation scheme the difference equation for the evolution of a qubit under the following circumstances: A continuously acting ‘‘free’’ unitary dynamics is periodically interrupted by a disturbance of negligible duration which acts weakly on the qubit. It may for example be caused by the scattering of a photon. The respective single influence can be described in terms of generalized measurements as a weak pure measurement with an additional unitary feedback. Turning to the sequence of influences, we have shown that the underlying statistics is approximately Gaussian. In lowest order, the nonselective evolution is governed (in addition to the ‘‘free’’ Hamiltonian) by both an averaged Hamiltonian originating from the feedback and decoherence caused by the pure measurement part only. This decoherence depends also on the shortness of the time interval between the influences. Two higher-order terms are analyzed. One term shows that the randomly acting feedbacks cause decoherence as well, although with respect to a different basis. The second term represents generalized friction. The master equation, which is obtained in an appropriate continuum limit, reflects only the influences of lowest order. It establishes the connection between sequential and continuous measurements of a qubit with unitary feedback. It is worth mentioning that in our discrete quantum system (qubit), contrary to continuous ones [10], feedback will only generate a new Hamiltonian term H_{AV} and no friction will survive in the continuum limit. Master equations without a modified Hamiltonian describe a very restricted class of continuous measurements. In order to complete the study, the selective evolution is discussed on the basis of the Gaussian structure obtained for the effective operation.

ACKNOWLEDGMENTS

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APPENDIX A

In the Appendixes we sketch the calculation of the change of state in the nonselective regime including all terms up to order $O(\Delta t\Delta p^2)$ and $O(\Delta t^2)$, where Δt occurs in products with either $H/\hbar, H_\pm/\hbar$, or γ.

We start with the exact operation element Ω for an N series with unitary development U between consecutive measurements,

$$\Omega_{(m_i)}:=U_{m_N}\cdots U_{m_1}|M_{m_N}|\cdots|M_{m_1}|U^N+R_1+R_2, \tag{A1}$$

where R_1 and R_2 are the terms which arise from commuting out the evolution operators U and the feedback operators

$U_\pm$, respectively, cf. Eqs. (28) and (30). The relation of R_1 and R_2 to C_1 and C_2 and an estimation of the order of magnitude of C_1 and C_2 is described below. Since the commutators $K_{m_i} := [U, M_{m_i}]$ occurring in R_1 are of order $O(\Delta p \tau \|H\|/\hbar) + O(\tau^2 \|H\| \max\{\|H_\pm\|\}/\hbar^2)$, we can neglect terms containing products of two such commutators,

$$\begin{aligned} R_1 &= M_{m_N} K_{m_{N-1}} \cdots M_{m_1} U^{N-1} + 2 M_{m_N} M_{m_{N-1}} K_{m_{N-2}} \cdots M_{m_1} U^{N-1} + \cdots + (N-1) M_{m_N} \cdots K_{m_1} U^{N-1} \\ &= \frac{i \Delta p \tau}{\hbar} [H, \sigma_z] U_{m_N} \cdots U_{m_1} \sum_{n=1}^{N-1} \frac{n}{a_{m_{N-n}}} |M_{m_N}| \cdots |M_{m_1}| U^{N-1} \\ &\quad - \frac{\tau^2}{\hbar^2} \sum_{n=1}^{N-1} n [H, H_{m_{N-n}}] U_{m_N} \cdots U_{m_1} |M_{m_N}| \cdots |M_{m_{N-n}}| \cdots |M_{m_1}| U^{N-1}, \end{aligned} \tag{A2}$$

where $a_+ := -4\sqrt{p_0}$ and $a_- := 4\sqrt{1-p_0}$. Please note that in Eq. (A2) behind the sum sign the products of $|M|$'s and U's are meant to not contain $M_{m_{N-n}}$ and $U_{m_{N-n}}$, except if explicitly mentioned. In the last two lines we have commuted out the feedback $U_\pm$. The resulting error is of higher order and can be neglected.

Also in R_2 we only take into account the terms containing one commutator $K_{k,l} := [M_{m_k}, U_{m_l}]$, which is of order $O(\Delta p \tau \max\{\|H_\pm\|\}/\hbar)$,

$$\begin{aligned} R_2 &= U_{m_N} \cdots U_{m_2} K_{2,1} |M_{m_N}| \cdots |M_{m_3}| |M_{m_1}| U^N + \cdots + (U_{m_N} \cdots U_{m_2} K_{N,1} + U_{m_N} \cdots U_{m_3} U_{m_1} K_{N,2} + \cdots \\ &\quad + U_{m_N} U_{m_{N-2}} \cdots U_{m_1} K_{N,N-1}) |M_{m_{N-1}}| \cdots |M_{m_1}| U^N \\ &= -\frac{i \Delta p \tau}{\hbar} \sum_{k=2}^{N} \sum_{l=1}^{k-1} \frac{1}{a_{m_k}} [\sigma_z, H_{m_l}] U_{m_N} \cdots U_{m_{l+1}} U_{m_{l-1}} \cdots U_{m_1} |M_{m_N}| \cdots |M_{m_{k+1}}| |M_{m_{k-1}}| \cdots |M_{m_1}| U^{N-1} \\ &\quad - \frac{\tau^2}{\hbar^2} \sum_{k=2}^{N} \sum_{l=1}^{k-1} [H_{m_k}, H_{m_l}] U_{m_N} \cdots U_{m_{k+1}} U_{m_{k-1}} \cdots U_{m_{l+1}} U_{m_{l-1}} \cdots U_{m_1} |M_{m_N}| \cdots |M_{m_1}| U^{N-1}. \end{aligned} \tag{A3}$$

Let us briefly motivate the estimation of the order of magnitude of C_1 and C_2 given in Eqs. (29) and (31). First we observe that $R_1 = M_{m_n} \cdots M_{m_1} U^N C_1$ and $R_2 = U_+^{N_+} U_-^{N-N_+} |M_+|^{N_+} |M_-|^{N-N_+} U^N C_2$. A moment's thought shows that the order of magnitude of the summands contained in C_i is equal to the order of the commutators K_{m_i} and $K_{k,l}$ in R_i. Since there are approximately N^2 such summands in C_i, the norm of C_i can be estimated to be less than or equal to N^2 times the order of the commutators in R_i which leads to the claims (29) and (31).

APPENDIX B

The state change due to an N series in the nonselective regime reads

$$\begin{aligned} \rho(t+\Delta t) &= \sum \Omega_{(m_i)} \rho \Omega^\dagger_{(m_i)} = \sum_{m_1 \cdots m_N} U_{m_N} \cdots U_{m_1} |M_{m_N}| \cdots |M_{m_1}| U(\Delta t) \rho U^\dagger(\Delta t) |M_{m_1}| \cdots |M_{m_N}| U^\dagger_{m_1} \cdots U^\dagger_{m_N} + \tilde{R}_1 + \tilde{R}_2 \\ &\quad + O(R_1^2) + O(R_2^2) + O(R_1 R_2) \end{aligned} \tag{B1}$$

with

$$\tilde{R}_i := \sum_{m_1 \cdots m_N} \{ R_i \rho U^\dagger(\Delta t) |M_{m_1}| \cdots |M_{m_N}| U^\dagger_{m_1} \cdots U^\dagger_{m_N} + \text{H.c.} \}, \quad i = 1,2. \tag{B2}$$

In order to carry out the summation in Eq. (B1), its terms can be expressed by means of binomial distributions. The latter can be approximated by integrals over Gaussians. For the first term in Eq. (B1) this recipe has already been demonstrated in Secs. III and IV. $\tilde{R}_1$ can be written in binomial form by observing that

$$\sum_{m_1\cdots m_N}\sum_{n=1}^{N-1} nb_{m_{N-n}}U_{m_N}\cdots U_{m_1}|M_{m_N}|\cdots|M_{m_{N-n+1}}||M_{m_{N-n-1}}|\cdots|M_{m_1}|\rho|M_{m_1}|\cdots|M_{m_N}|U^\dagger_{m_1}\cdots U^\dagger_{m_N}$$
$$=\frac{(N-1)(N-2)}{2}\left(\sum_m b_m U_m^L|M_m|^R(U_m^\dagger)^R\right)\sum_{N_+=0}^{N-1}\binom{N-1}{N_+}U_+^{N_+}U_-^{N-N_+-1}$$
$$\times|M_+|^{N_+}|M_-|_-^{N-N_+-1}\rho|M_+|^{N_+}|M_-|_-^{N-N_+-1}(U_+^\dagger)^{N_+}(U_-^\dagger)^{N-N_+-1}, \tag{B3}$$

where we have again used the notation that operators with uppercase L and R act from the left and from the right, respectively. A similar formula is obtained when instead of $|M_{m_{N-n}}|$ in the first line $U_{m_{N-n+1}}$ is missing. Then only U_m^L has to be replaced by $|M_m|^L$.

$\tilde{R}_2$ can be simplified employing

$$\sum_{m_1\cdots m_N}\sum_{k=2}^{N}\sum_{l=1}^{k-1} C_{m_k,m_l}U_{m_N}\cdots U_{m_{k+1}}U_{m_{k-1}}\cdots U_{m_1}|M_{m_N}|\cdots|M_{m_{l+1}}||M_{m_{l-1}}|\cdots|M_{m_1}|\rho|M_{m_1}|\cdots|M_{m_N}|U^\dagger_{m_1}\cdots U^\dagger_{m_N}$$
$$=\frac{(N-1)(N-2)}{2}\left(\sum_{m,\acute{m}=+,-} C_{m,\acute{m}}U_m^L|M_{\acute{m}}|^L(U_m^\dagger)^R|M_m|^R(U_{\acute{m}}^\dagger)^R|M_{\acute{m}}|^R\right)$$
$$\times\sum_{N_+=0}^{N-2}\binom{N-2}{N_+}U_+^{N_+}U_-^{N-N_+-2}|M_+|^{N_+}|M_-|^{N-N_+-2}\rho|M_+|^{N_+}|M_-|^{N-N_+-2}(U_+^\dagger)^{N_+}(U_-^\dagger)^{N-N_+-2}. \tag{B4}$$

Formulas (B3) and (B4) neglect commutators between the operators they contain. In our case, corrections containing these commutators would be of higher order and therefore too small.

Applying formulas (B3), (B4) to $\tilde{R}_1,\tilde{R}_2$, respectively, and expressing N_+/N in terms of variable s according to Eq. (38), we obtain

$$\tilde{R}_1=\frac{-i(N-3)\Delta p\Delta t}{2\hbar}\left\{[H,\sigma_z]\sum_m \frac{U_m}{a_m}\tilde{D}(\rho)|M_m|U_m^\dagger+\text{H.c.}\right\}-\frac{(N-3)\tau\Delta t}{2\hbar^2}\sum_m\{[H,H_m]|M_m|\tilde{D}(\rho)|M_m|U_m^\dagger+\text{H.c.}\}, \tag{B5}$$

where $\tilde{D}(\rho)=\int_{-\infty}^{\infty}M_s\rho M_s^\dagger ds$ with M_s as given by Eq. (42) with Δt replaced by $(N-1)\tau$. $\tilde{R}_2$ now reads

$$\tilde{R}_2=-\frac{i(N-3)\Delta p\Delta t}{2\hbar}\sum_{m,\acute{m}}\left\{\frac{[\sigma_z,H_{\acute{m}}]}{a_m}U_m|M_{\acute{m}}|\tilde{D}(\rho)|M_{\acute{m}}|U_{\acute{m}}^\dagger|M_m|U_m^\dagger+\text{H.c.}\right\}$$
$$-\frac{(N-3)\tau\Delta t}{2\hbar^2}\sum_{m\neq\acute{m}}\{[H_m,H_{\acute{m}}]|M_m||M_{\acute{m}}|\tilde{D}(\rho)|M_{\acute{m}}|U_{\acute{m}}^\dagger|M_m|U_m^\dagger+\text{H.c.}\}. \tag{B6}$$

In $\tilde{R}_2$, when inserting M_s from Eq. (42) in $\tilde{D}(\rho)$, Δt has to be replaced by $(N-2)\tau$. The second sum in Eq. (B6) vanishes since the summand with $m=+,\acute{m}=-$ and the summand with $m=-,\acute{m}=+$ add to zero. Inserting the lowest order of $\tilde{D}(\rho)$, namely $\tilde{D}(\rho)\approx\rho$, it is easy to find the contributions of $\tilde{R}_1$ and $\tilde{R}_2$ to the change of state in the last two lines of Eq. (C6).

APPENDIX C

Having calculated $\tilde{R}_1$ and $\tilde{R}_2$, we want to sketch how to process the main contribution to the state change, which is represented by the first term in Eq. (B1). As mentioned above, this part of the operation can be written by means of a binomial distribution and can then be expressed with operation elements whose modulus $|M(N_+,N)|$ is the square root of Gaussians, cf. Eq. (36). In contrast to Secs. III and IV, we now take into account the full q-number denominators of the modulus $|M(N_+,N)|$ in Eq. (36). Expressing the operation elements in terms of variable s [cf. Eq. (38)], we obtain for their modulus

$$|M_s|=\frac{1}{\sqrt[4]{2\pi/(\hat{\gamma}\Delta t)}}\exp\left\{-\hat{\gamma}\frac{(\sigma_z-s)^2}{4}\Delta t\right\}, \tag{C1}$$

with

$$\hat{\gamma}:=\frac{(\Delta p)^2}{4E_+E_-\tau}. \tag{C2}$$

Expanding the unitary part of the operation up to order Δt^2 leads to the state change (without the contribution from R_1 and R_2)

$$\rho(t+\Delta t)=D(\rho)-\frac{i\Delta t}{\hbar}[H+H_{AV},D(\rho)]-\frac{\Delta t^2}{2\hbar^2}\{(H+H_{AV})^2,\rho\}+\frac{\Delta t^2}{\hbar^2}(H+H_{AV})\rho(H+H_{AV}) \tag{C3}$$

with

$$D(\rho)=\int_{-\infty}^{\infty}\exp\left\{\frac{i}{2\hbar}(\Delta H^L-\Delta H^R)s\Delta p\Delta t\right\}\exp\left\{-\frac{\hat{\gamma}^L}{4}(\sigma_z^L-s)^2\Delta t\right\}\exp\left\{-\frac{\hat{\gamma}^R}{4}(\sigma_z^R-s)^2\Delta t\right\}ds\frac{\sqrt[4]{\hat{\gamma}^L\hat{\gamma}^R}}{\sqrt{2\pi/\Delta t}}\rho. \tag{C4}$$

We note that in Eq. (C3), H is meant to act in operator products directly on ρ. This is due to the order of operators in the operation elements [cf. Eq. (43)]. The integral $D(\rho)$ has a closed-form solution which can be expanded in powers of Δt and Δp. γ without a hat is given by Eq. (40),

$$\begin{aligned}D(\rho)=&\Bigg(1-\Delta t\Bigg\{\frac{\gamma}{8}(\sigma_z^L-\sigma_z^R)^2\left[1-\frac{1}{2}\left(\frac{\Delta p(p_0-1/2)}{p_0\tilde{p}_0}\right)^2(1-\sqrt{p_0\tilde{p}_0}[p_0\tilde{p}_0+3-2^{-1/4}])\right]+\frac{\Delta p^2}{8\gamma\hbar^2}(\Delta H^L-\Delta H^R)^2\\&+i\frac{\Delta p}{4\hbar}(\Delta H^L-\Delta H^R)(\sigma_z^L+\sigma_z^R)\Bigg\}+\Delta t^2\frac{\gamma^2}{32}(\sigma_z^L-\sigma_z^R)^2+O(\Delta p\Delta t^2)+O(\Delta p^2\Delta t^2)+O(\Delta t^3)\Bigg)\rho.\end{aligned} \tag{C5}$$

Collecting all terms up to order $O(\Delta t^2)$ and $O(\Delta t\Delta p^2)$ we obtain the following difference equation:

$$\begin{aligned}\Delta\rho=&\Delta t\Bigg\{-\frac{i}{\hbar}[H+H_{AV},\rho]-\frac{\gamma}{8}\left[1-\frac{1}{2}\left(\frac{\Delta p(p_0-1/2)}{p_0\tilde{p}_0}\right)^2(1-\sqrt{p_0\tilde{p}_0}[p_0\tilde{p}_0+3-2^{-1/4}])\right][\sigma_z,[\sigma_z,\rho]]-\frac{\Delta p^2}{8\gamma\hbar^2}[\Delta H,[\Delta H,\rho]]\\&-\frac{i\Delta p}{4\hbar}[\Delta H,\{\sigma_z,\rho\}]\Bigg\}+\Delta t^2\Bigg(-\frac{i\gamma}{8\hbar}([\sigma_z,[\sigma_z,[H,\rho]]]+[H_{AV},[\sigma_z,[\sigma_z,\rho]]])-\frac{1}{2\hbar^2}\{(H+H_{AV})^2,\rho\}\\&+\frac{1}{\hbar^2}(H+H_{AV})\rho(H+H_{AV})+\frac{\gamma^2}{32}[\sigma_z,[\sigma_z,\rho]]\Bigg)+\left(-\frac{i\Delta t^2\gamma}{8\hbar}+\frac{3i\Delta t\Delta p^2}{32\hbar p_0\tilde{p}_0}\right)\Bigg(\{[H,\sigma_z]\rho\sigma_z+\text{H.c.}\}-\frac{4i}{\hbar\Gamma}[[H,H_{AV}],\rho]\\&+\{[\sigma_z,\tilde{p}_0H_++p_0H_-]\rho\sigma_z+\text{H.c.}\}\Bigg)O(N\Delta p\Delta t^2)+O(\Delta p\Delta t^2)+O(\Delta p^2\Delta t^2)+O(\Delta t^3)\end{aligned} \tag{C6}$$

with $\tilde{p}_0:=1-p_0$. In the order terms Δt occurs in products with one of the three: $H/\hbar, H_\pm/\hbar$, or γ.

[1] A.C. Doherty, K. Jacobs, and G. Jungman, e-print quant-ph/0006013.

[2] H.M. Wiseman, Phys. Rev. A **51**, 2459 (1995).

[3] J. Audretsch, Th. Konrad, and A. Scherer, Phys. Rev. A **65**, 033814 (2002).

[4] J. Audretsch, Th. Konrad, and A. Scherer, Phys. Rev. A **63**, 052102 (2001).

[5] A.C. Doherty, S.M. Tan, A.S. Parkins, and D.F. Walls, Phys. Rev. A **60**, 2380 (1999).

[6] H.M. Wiseman, Phys. Rev. A **49**, 2133 (1995).

[7] A.N. Korotkov, Physica B **280**, 412 (2000).

[8] A.N. Korotkov. Phys. Rev. B **63**, 115403 (2001).

[9] A.O. Caldeira and A.J. Leggett. Physica A **121**, 587 (1983).

[10] L. Diósi, Physica A **199**, 517 (1993).

[11] A. Barchielli, L. Lanz, and G.M. Prosperi, Nuovo Cimento Soc. Ital. Fis., B **72**, 79 (1982).

[12] C.M. Caves and G.J. Milburn, Phys. Rev. A **36**, 5543 (1987).

[13] J. Audretsch, M.B. Mensky, and V. Namiot, Phys. Lett. A **237**, 1 (1997).

[14] M. Brune, S. Haroche, V. Lefevre, J.M. Raimond, and N. Zagury, Phys. Rev. Lett. **65**, 976 (1990).

[15] J. Audretsch and M.B. Mensky, e-print quant-ph/9808062.

[16] L. Diósi, Phys. Lett. A **129**, 419 (1988).