Quantum gloves: Quantum states that encode as much as possible chirality and nothing else

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Communicating a physical quantity cannot be done using information only—i.e., using abstract cbits and/or qubits. Rather one needs appropriate physical realizations of cbits and/or qubits. We illustrate this by considering the problem of communicating chirality. We discuss in detail the physical resources this necessitates and introduce the natural concept of *quantum gloves*—i.e., rotationally invariant quantum states that encode as much as possible the concept of chirality and nothing more.

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I. INTRODUCTION

A question which has attracted much attention over the past years is how to encode a physical quantity into a finite quantum system; see [1] for many early studies of this question. Examples include encoding the time of an event, the phase and amplitude of a coherent state, a direction in space [1–7], and a reference frame [8–10]. One can formalize this problem as follows: one party, Alice, has a classical description of this physical quantity. By this we mean that Alice has perfect knowledge of the physical quantity. She encodes the description into the quantum system and sends it, using an ideal quantum communication channel, to the second party, Bob. Bob then carries out a measurement. The result of the measurement provides Bob with some information about the physical quantity.

An often overlooked aspect of this problem is that since Alice wants to communicate a physical quantity, the nature of the quantum system she uses to encode the information and the properties of the communication channel play an essential role in this problem. Thus there can be essential differences according to whether the particles used are bosons or fermions, according to whether the degrees of freedom are the spin of the particles, their position, etc. Some discussions of this point can be found in [11,7,12] (the latter work is based on some of the results presented here, but focuses only on this aspect).

In the present work we consider a particularly simple situation related to the question considered in [8–10] of transmitting a reference frame. We suppose that Alice only wants to tell Bob the chirality of her reference frame—i.e., whether it is a left- or a right-handed reference frame. This question is apparently very simple since only a binary quantity must be communicated. But therein lies the interest of the problem: since there are no real technical difficulties, one can focus on the essential role of the physical properties of the communication channel. Thus, for instance, it is impossible to compare chiralities by exchanging only classical information—i.e., by sending only abstract 0's and 1's. It is quite intuitive why this is so: bits measure the quantity of information, but have *per se* no meaning, in particular no meaning about geometric and physical concepts. Hence, if our world is invariant under *left*$\leftrightarrow$*right*, then mere information is unable to distinguish between *left* and *right*. In the Appendix the relation of this problem to particle physics is briefly discussed.

Now, information is physical, as Landauer used to emphasize and as every physicist knows today. Hence we must consider classical bits physically realized in some system. For example, the bits 0 and 1 could be realized by right-handed and left-handed gloves, respectively. It is obvious that such physical bits can be used to send chirality information. But bits realized by black and white balls could not do the job. Furthermore, if the physical bits can be encoded in a quantum system, then the problem is even more interesting because of the phenomenon of entanglement which allows Alice to prepare states which have no classical analog. Indeed by exploiting this aspect of quantum systems we will show that it is possible to communicate chirality perfectly. In

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this context we introduce the natural concept of *quantum gloves*—i.e., rotationally invariant quantum states that encode as much as possible the concept of chirality and nothing more. Furthermore, we will show that quantum gloves can be realized in a very economical way, using very little resources.

The amount of resources required to communicate a physical quantity is central to our discussion. Understanding it increases our understanding of the physical quantities and how the uncertainty principle of quantum mechanics puts constraints on the precision with which they can be represented. In the case of chirality, since a single bit must be communicated, Holevo's bound tells us that in principle a single qubit suffices. However Holevo's bound can generally only be achieved asymptotically, using block coding. In the present case, since the system is finite, finding the best encoding is non trivial. We shall show that there are quantum gloves which consist of only a single qubit. However, a number of trade-offs between physical resources are still possible, such as the number of particles which make up the quantum glove, the volume in space it occupies, the number of qubits communicated, etc.

The question of communicating chirality in the quantum setting was already introduced in [13]. Unfortunately (as was made clear in the final version of [13]) the idea presented there does not work as it is based on the incorrect assumption that under parity a spin pointing up in the $\vec{n}$ direction $|\uparrow_{\vec{n}}\rangle$ is flipped into a spin pointing down in the $-\vec{n}$ direction $|\downarrow_{\vec{n}}\rangle$ for all $\vec{n}$. In fact there are no degrees of freedom which transform in this way under parity. In particular—see the discussion below—spin degrees of freedom are invariant under parity.

The paper is organized as follows. In Sec. II we discuss the problem of communicating chirality using only classical systems which is of interest in itself and sets the stage for the quantum problem. Then we present in Sec. III a first example of quantum gloves, discussing in detail the resources required to realize them. In Sec. IV we show that many different kinds of quantum gloves can be constructed, depending on the resources used. In Sec. V a unified approach is developed based on the properties of the *chirality operator*, the operator which one must measure to determine the chirality of one's reference frame. We summarize our results in the Conclusion.

II. CLASSICAL GLOVES

Before turning to the problem of quantum gloves, let us consider the simpler problem of classical gloves—i.e., classical systems that can encode chirality. One possibility is, of course, for Alice to send Bob an orthonormal frame, represented, for instance, by three orthogonal vectors labeled from 1 to 3. These vectors could, for instance, be realized by having Alice send Bob arrows, labeled from 1 to 3.

On the other hand, it is impossible to communicate chirality using axial vectors only. An axial vector can, for instance, be realized physically by a rotating disk. The axial vector is the angular momentum of the disk. However, if the disk is completely symmetric (and therefore contains no other directional information than its angular momentum), then under inversion around its center, the spinning disk stays invariant. This means that it is impossible to encode chirality in one or many axial vectors—i.e., in one or many spinning disks—since under parity the spinning disks stay invariant.

However, it is interesting that one can encode chirality using one axial vector and one normal vector. Suppose Alice prepares the axial vector and the normal vector both pointing in the same direction and sends them to Bob. For instance, this could be realized by a spinning disk with UP written on one face and DOWN on the other. Then the axial vector is the angular momentum of the disk, and the vector is aligned with the axis of the disk and goes from the DOWN face to the UP face. This disk is no longer invariant under inversion and can be used to encode the chirality of Alice's reference frame. Indeed, if Bob has opposite chirality, he will find that the angular momentum and the vector pointing from DOWN to UP are opposite.

An alternative way of presenting the same thing is to suppose that Alice prepares a spinning disk of angular momentum $\vec{j}=(j_x,j_y,j_z)$ and suppose Bob uses a reference frame inverted about the origin. Then Bob will say that the angular momentum of the spinning disk has exactly the same components (j_x,j_y,j_z). But if Alice prepares a vector with components $\vec{v}=(v_x,v_y,v_z)$, then Bob will describe this vector as having components $(-v_x,-v_y,-v_z)$. The sign of the scalar product $\vec{v}\cdot\vec{j}$ can thus encode the chirality of the reference frame.

Note that all these methods are rather uneconomical and are far from what we call a perfect glove. Indeed, by sending Bob three vectors (her reference frame), Alice provides him with enough information to align his reference frame with hers; i.e.; an infinite amount of supplementary information is transmitted in addition to the chirality. On the other hand, in the example in which Alice sends Bob a marked spinning disk less information is conveyed. Indeed a single direction is transmitted. This could be used to align the z axis of Alice and Bob's reference frames, but the relative rotation around the z axis would be undefined. In addition information could also be encoded in the angle between the vector $\vec{v}$ and axial vector $\vec{a}$. We do not know whether classical methods more economical than this are possible.

III. QUANTUM GLOVES

A. Setting the problem

We now turn to the main subject of this article: namely, the problem of describing the chirality of a reference frame using quantum particles. We begin by describing precisely the setup. To this end let us consider the task of Alice and Bob from the point of view of an external observer. From the point of view of the external observer there are in fact four different situations according to whether he has the same chirality as Alice and/or Bob. In general he describes what happens as follows.

First Alice prepares a quantum state $|G\rangle$ which encodes the chirality of her reference frame. If Alice has the same chirality as the external observer, she will prepare the state

$|G^+\rangle$, whereas if she has the opposite chirality, she prepares the state $|G^-\rangle = P|G^+\rangle$ where P is the parity operator. Obviously, in order for Alice to perfectly encode her chirality in the quantum state the states $|G^\pm\rangle$ must be orthogonal:

$$\langle G^-|G^+\rangle = 0.$$

Alice then sends the quantum glove to Bob who measures a chirality operator of which $|G^\pm\rangle$ are two eigenstates with different eigenvalues. More precisely, if Bob has the same chirality as the external observer, he will measure the operator χ, whereas if he has the opposite chirality as the external observer, he will measure $P\chi P$.

Thus we can describe the quantum gloves in two ways. First we can consider the quantum state prepared by Alice $|G^\pm\rangle$ and how these transform one into the other under parity. The second, more abstract, approach is to consider the chirality operator measured by Bob χ and how it transforms under parity $\chi \to P\chi P$. We will use both approaches below.

Note that throughout this article we take the parity operator P to be the unitary operator that realizes inversion around the origin. It acts on vectors as $P\vec{v} = -\vec{v}$. The parity operator leaves axial vectors unchanged (it leaves spinning disks unchanged); hence, it also leaves spin degrees of freedom (for instance, the spin of an electron) unchanged. Acting with the parity operator twice always yields the identity: $P^2 = I$.

B. Summary of results

The nature of the physical resources used to encode chirality plays an essential role. Indeed Alice cannot use spin degrees of freedom alone to solve this problem since they are axial vectors. On the other hand, Alice can use the relative positions of particles. Indeed the relative position of two distinguishable particles—say, a proton and an electron—can describe a vector. This is the vector going from the proton to the electron. Under parity (inversion around the position of the proton) the vector will flip to the opposite vector. Thus using the relative position of four distinguishable particles one can describe a reference frame (one at the origin, the other three along the three axes).

It should therefore come as no surprise that one can construct quantum gloves using the relative position of four particles. What is more interesting is that it can be done perfectly (by this we mean that Bob will be certain of the chirality of Alice's reference frame) using only a small Hilbert space (effectively Alice only sends a single qubit) in a way which conveys no information about the orientation of Alice's reference frame: the quantum gloves are invariant under rotation.

Another surprising aspect is that with minor modifications these perfect quantum gloves can be realized with only two kinds of particles: we need one reference particle (say, a proton) to indicate the origin of the coordinate system and three other indistinguishable particles (say, electrons). The positions of the three indistinguishable particles with respect to each other and with respect to the reference particle encode the chirality of the reference frame. The restriction that one of the particles be different from the others can in fact also be dropped, although we do not know whether perfect gloves are possible in this case.

In Sec. IV we will further generalize these constructions and show that perfect quantum gloves can be realized with only three particles. One, the proton, indicates the origin of the coordinate system, and the other two can be indistinguishable. But in this case Alice must send more than a single qubit to Bob. The extra information can be used to convey some information about the orientation of her reference frame in addition to its chirality. We also show that perfect quantum gloves can be realized by using the relative position of two particles and spin degrees of freedom and that imperfect quantum gloves can be realized using the relative position of two particles only.

Note that in this paper we are not interested by the question of practical realization. Indeed all the examples we study have the status of "gedanken experimenten:" they illustrate points of principle without concern for the issue of feasability. Some of these states, and in particular those involving only the relative positions of two particles, may be realizable as excited atomic states. We leave to future research the question of finding how to implement quantum gloves in practice.

C. First example

We now describe how to construct quantum gloves involving four particles. We take as variables the position of the reference particle $\vec{x}_0$ and vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ going from the position of the reference particle to the positions of particles 1, 2, 3. We write $\vec{x}_i = r_i \vec{n}_{\Omega_i}$ where $r_i = |\vec{x}_i|$ and $\vec{n}_\Omega$ is a unit vector pointing in direction Ω. We can decompose any wave function of the four particles into a superposition of factorized wave functions of the form

$$\varphi(\vec{x}_0) f(r_1, r_2, r_3) Y_{l_1 m_1}(\Omega_1) Y_{l_2 m_2}(\Omega_2) Y_{l_3 m_3}(\Omega_3), \tag{1}$$

where Y_{lm} are the spherical harmonics. The dependence on $\vec{x}_0$ plays no role in what follows. Momentarily we also drop the dependence on the radial variables r_i. We will come back to them below.

The parity operator P realizes the reflection about the position of the reference particle. Thus $P\vec{n} = -\vec{n}$. It transforms spherical harmonics according to

$$PY_{lm} = (-1)^l Y_{lm}. \tag{2}$$

Thus the product of three spherical harmonics has parity

$$PY_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3} = (-1)^{l_1 + l_2 + l_3} Y_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3}. \tag{3}$$

Let us now consider the following two states.

(i) All three particles in S waves:

$$|S^3\rangle = Y_{00} Y_{00} Y_{00}. \tag{4}$$

(ii) All three particles in P waves, in a completely antisymmetric state (known as the Aharonov state):

$$|A\rangle = (Y_{11} Y_{10} Y_{1-1} + Y_{10} Y_{1-1} Y_{11} + Y_{1-1} Y_{11} Y_{1-0} - Y_{11} Y_{1-1} Y_{10} - Y_{1-1} Y_{10} Y_{11} - Y_{10} Y_{11} Y_{1-1})/\sqrt{6}.$$

Both states have zero total angular momentum. This implies that they are invariant under simultaneous rotations of all three particles. Under parity they transform as $P|S^3\rangle = |S^3\rangle$ and $P|A\rangle = -|A\rangle$.

The two quantum gloves are defined as

$$|G^{\pm}\rangle = \frac{|S^3\rangle \pm |A\rangle}{\sqrt{2}}. \tag{5}$$

These states are orthogonal, they are invariant under rotation, and under parity they transform as

$$P|G^+\rangle = |G^-\rangle, \quad P|G^-\rangle = |G^+\rangle. \tag{6}$$

This means that if Alice and Bob have the same chirality and try to construct state G^+, then they will construct the same state, independently of the orientation of their reference frames. On the other hand, if they have opposite chirality, then they will construct opposite states. By sending each other one of these states they can unambiguously learn whether they have the same or opposite chirality: Alice prepares state G^+ and sends it to Bob. Bob measures in the G^+, G^- basis. If he finds G^+, he concludes that they both have the same chirality. If he finds G^-, he concludes that they have opposite chirality. Because the states are invariant under rotation, this will work independently of the alignment of their reference frames and independently of whether the state was rotated during transmission.

D. Resources used

An essential question is to quantify the resources used by Alice to describe her chirality to Bob and, in particular, the amount of communication used. This question is far from trivial as we shall see. But before going into details we note that there are two kinds of communication involved. The first is the communication of the quantum glove $|G\rangle$. The second is the communication involved in setting up the protocol. We consider both aspects below.

1. Setting up the protocol

In problems of the type envisaged here, where two parties want to communicate some information, the resources required to set up the protocol are generally not taken into account. There are several good reasons for this.

First of all the amount of communication involved in setting up the protocol depends on whether the parties have a common language, a common vocabulary, etc. (This difficulty is analoguous to Kolmogorov complexity in computer science which is the minimum length of the program which will output a given number. The Kolmogorov complexity is defined up to a constant only, since it depends on the specific computer—i.e., on the specific Turing machine—used.) Thus the amount of communication involved in setting up the protocol is ill defined.

Second, setting up the protocol can be separated from the actual communication task, both in time and in terms of the resources used. Indeed one can suppose that the protocol was discussed by the parties long before they actually communicate their information, or that the protocol was given to Alice and Bob by a third party.

In the present case there is at third important aspect which arises because setting up the protocol can be done using a classical channel that transmits information only—i.e., that uses only black and white balls. That is the protocol is independent of any physical realization of the communication channel.

This last remark provides an operational way of separating the cost of setting up the protocol from the cost of communicating chirality. Namely, we can suppose that Alice and Bob have unlimited access to an information-only classical channel which they use to set up the protocol in as much detail as they wish. But because this is an information-only channel, they cannot use it to communicate chirality. For this they must use the physical channel. They will try to do this as sparingly as possible, and we will quantify how much physical communication is used.

(There is another reason why the cost of setting up the protocol is often neglected. This concerns situations where one is concerned with "channel capacities." Then one wants to use the channel a very large number of times, and the cost of setting up the protocol is negligible in this limit. However, this does not apply in the present case because it does not make sense to communicate the chirality of many reference frames—more precisely, Alice can use a classical information-only channel to tell Bob that "the chirality of her second reference frame is opposite to the chirality of the first." It is only the communication of the chirality of the first reference frame which requires a physical communication channel. For this reason this argument does not apply in the present case.)

2. Computing resources

The above discussion shows that it is a well defined task to count only the physical resources involved in transmitting the glove $|G\rangle$. However, the amount of resources used in transmitting the glove is itself a complicated quantity and cannot be reduced to a single number. We devote the following paragraphs to discussing this question.

As an illustration let us compare the protocol described in Sec. III C with a protocol in which Alice uses four particles to encode her classical reference frame: one particle is at the origin, one particle very far along the $+x$ direction, and one particle very far along the $+y$ direction, and one particle very far along the $+z$ direction. This "classical reference frame" and the one obtained by reflection about the origin are different (the corresponding quantum states are orthogonal). Thus they could be used to encode chirality. Since only two reference frames are used, this method would also seem to require only one qubit of communication. However, it is clearly much less economical than the first method. What is the precise origin of the difference?

A first important point is to consider the possibility that during transmission from Alice to Bob the quantum glove undergoes a random rotation. Because of this random rotation, Alice cannot send Bob any information about the relative orientation of their reference frames. But she can still tell him about the relative chirality of their reference frames (since parity and rotations commute). By carrying out this random rotation, one sees an essential difference between the quantum gloves $|G^{\pm}\rangle$ and the classical reference frame. Indeed the quantum gloves are invariant under rotation; hence, the entropy, when the states are randomly rotated, stays 1 qubit. On the other hand, the entropy of the classical reference frame, when randomly rotated, becomes infinite: the classical reference frame not only encodes the chirality, but also an infinite amount of additional information about relative orientation.

[Note that the position $\vec{x}_0$ of the quantum glove could in principle also be used to transmit information. This is obviously irrelevant to the present problem. We can take, for instance, the position to be in a pure state $\psi(\vec{x}_0)$ on which both Alice and Bob agree, in which case no information can be transmitted in this way.]

There is another interesting way to compare the resources used by different quantum gloves. This is the volume they occupy in space. Indeed, if we suppose that the particles use hydrogenlike orbitals, then the lower the angular momentum of the particles, the closer they can lie to the origin. Thus L_{max}, the largest angular momentum of the particle, measures how much space they occupy. (For the quantum gloves described above $L_{max}=1$ whereas for the classical reference frame $L_{max}=\infty$.) Finally the number and type of particles used to realize the quantum gloves is another type of resource which can be compared (indeed we shall describe below quantum gloves using fewer than four particles).

This discussion shows that there is not a unique parameter which quantifies how much resources are used to encode the chirality of a reference frame. This is because encoding chirality cannot be done without reference to the physical system that is used. Thus, whereas the resources used in many quantum communication tasks can be quantified in terms of universal units such as bits, qubits, ebits, in the case of physical quantities protocols will be inequivalent when different physical systems are used to encode the same physical quantity.

IV. MANY OTHER QUANTUM GLOVES

A. Quantum gloves made from indistinguishable particles

We now go back to describing different kinds of quantum gloves. As a first extension of the above protocol, let us note that it required the four particles sent by Alice to be distinguishable. However, it can easily be extended to the case where some or all of the particles are indistinguishable. But now one needs to take care that the global wave function is symmetric or antisymmetric according to whether one is dealing with bosons or fermions.

As illustration we take the reference particle to be distinguishable from the other three particles which are taken to be indistinguishable fermions. Define $f^{symm(anti)}(r_1,r_2,r_3)$ to be symmetric (antisymmetric) functions of the radial coordinates r_1, r_2, r_3, respectively. Then the global wave function of the quantum gloves can be taken to be

$$|G^{\pm}\rangle = \varphi(\vec{x}_0)(f^{anti}|S^3\rangle \pm f^{symm}|A\rangle)/\sqrt{2}. \tag{7}$$

The case of three bosons is similar except that f^{symm} and f^{anti} are interchanged. Note that because the radial wave functions must be symmetric and antisymmetric, the particles will occupy a larger volume in space than in the case of distinguishable particles. Thus one has relaxed one condition (that the particles be distinguishable), but one has had to use more of another resource (space) in order to make these quantum gloves.

We do not know whether it is possible to realize quantum gloves with only four indistinguishable particles (in the example above the reference particle $\vec{x}_0$ is different from the other three). On the other hand, if we take more than four particles, they can be all identical. One possibility is to take one particle as particle 1, two particles close together as particle 2, three particles close together as particle 3, etc. We do not know, however, what is the optimal way of doing this.

B. Quantum gloves made from three particles

We now show how Alice can encode the chirality of her reference frame in the relative position of three particles, one of which (the proton) is distinguishable from the other two (the electrons). We take as variables the position of the reference particle $\vec{x}_0$ and vectors $\vec{x}_1=r_1\vec{n}_{\Omega_1}$ and $\vec{x}_2=r_2\vec{n}_{\Omega_2}$ going from the position of the reference particle to the positions of particles 1 and 2. We can decompose wave functions of the three particles into factorized wave functions of the form

$$\varphi(\vec{x}_0)f(r_1,r_2)Y_{l_1m_1}(\Omega_1)Y_{l_2m_2}(\Omega_2). \tag{8}$$

From now on we drop the dependence on $\vec{x}_0$ and on r_1, r_2.

The rules of addition of angular momentum imply that all states with zero total angular momentum are combinations of states of the form of Eq. (8) with $l_1=l_2$. Hence they always have parity $P=+1$ and cannot be used to encode the chirality of a reference frame. However, there exist spaces with total angular momentum $L_{tot}\neq 0$ of opposite parity. For simplicity we consider the case $L_{tot}=1$. Thus, for instance, the states

$$|\alpha_{11}\rangle = \frac{Y_{00}Y_{11}+Y_{11}Y_{00}}{\sqrt{2}},$$

$$|\alpha_{10}\rangle = \frac{Y_{00}Y_{10}+Y_{10}Y_{00}}{\sqrt{2}},$$

$$|\alpha_{1-1}\rangle = \frac{Y_{00}Y_{1-1}+Y_{1-1}Y_{00}}{\sqrt{2}}$$

form a basis of a symmetric irreducible representation of the rotation group with total angular momentum $L_{tot}=1$ and parity $P=-1$. On the other hand, the states

$$|\beta_{11}\rangle = \frac{Y_{10}Y_{11}-Y_{11}Y_{10}}{\sqrt{2}},$$

$$|\beta_{10}\rangle = \frac{Y_{1-1}Y_{11}-Y_{11}Y_{1-1}}{\sqrt{2}},$$

$$|\beta_{1-1}\rangle = \frac{Y_{1-1}Y_{10}-Y_{10}Y_{1-1}}{\sqrt{2}}$$

form a basis of an antisymmetric irreducible representation of the rotation group with total angular momentum $L_{tot}=1$ and parity $P=+1$.

The quantum gloves consist of two spaces of dimension 3, each of which constitutes an irreducible representation of the rotation group with total angular momentum $L_{tot}=1$. A basis of this spaces is

$$|G^{\pm}_{11}\rangle = \frac{\alpha_{11} \pm \beta_{11}}{\sqrt{2}},$$

$$|G^{\pm}_{10}\rangle = \frac{\alpha_{10} \pm \beta_{10}}{\sqrt{2}},$$

$$|G^{\pm}_{1-1}\rangle = \frac{\alpha_{1-1} \pm \beta_{1-1}}{\sqrt{2}},$$

where $|G^{\pm}_{LM}\rangle$ is a quantum glove state with total angular momentum L and angular momentum along the z axis equal to M. We denote by $\Pi_{G^{\pm}}$ the projectors onto these spaces. Thus

$$\Pi_{G^+} = |G^+_{11}\rangle\langle G^+_{11}| + |G^+_{10}\rangle\langle G^+_{10}| + |G^+_{1-1}\rangle\langle G^+_{1-1}| \qquad (9)$$

and similarly for Π_{G^-}. It is immediate to check that $\langle G^+_{1M}|G^-_{1M'}\rangle=0$ for all M, M'. This implies that the projectors Π_{G^+} and Π_{G^-} are orthogonal. Furthermore, we note that the projectors $\Pi_{G^{\pm}}$ are invariant under simultaneous rotations of both particles 1 and 2 around the reference particle. This follows from the fact that they project onto the spaces spanned by all the vectors of an irreducible representation of the rotation group. Finally we note that under parity these projectors transform as

$$P\Pi_{G^+}P = \Pi_{G^-}, \quad P\Pi_{G^-}P = \Pi_{G^+}. \qquad (10)$$

This means that if Alice and Bob have the same chirality, then their definitions of $\Pi_{G^{\pm}}$ will coincide. But if they have opposite chirality, then what Alice calls Π_{G^+}, Bob will call Π_{G^-}, and similarly what Alice calls Π_{G^-}, Bob will call Π_{G^+}. The protocol is then similar to the previous case: Alice prepares a state in Π_{G^+} and sends it to Bob. Bob projects the state onto the $\Pi_{G^{\pm}}$ spaces. If he finds space G^+, he concludes they both have the same chirality, and if he finds space G^-, he concludes that they have opposite chirality. Note that because the spaces $\Pi_{G^{\pm}}$ are invariant under rotation, this will work even if the state Alice sends undergoes a random rotation during transmission. Conversely, if the state does not undergo any rotation, Alice can send Bob some information about the orientation of her reference frame. For instance, if she sends Bob the state $|G^{\pm}_{11}\rangle$ aligned with her z axis, then by measuring the state Bob can learn information about the orientation of Alice's reference frame. If one averages over rotations, then this protocol uses $1+\ln 3$ qubits of communication, whereas the protocol using four particles used only one qubit of communication. Once more, one sees how one resource is traded for another. The above protocol can be generalized to the case where particles 1 and 2 are identical exactly as in the case where four particles were sent.

C. Quantum gloves made from spins and relative positions

As we mentioned above it is also possible to encode the chirality of a reference frame using one vector and one axial vector. This can be done classically by having Alice prepare a spinning disk of angular momentum $\vec{j}$ and a vector $\vec{v}$. The sign of the scalar product $\vec{j}\cdot\vec{v}$ then encodes the chirality of the reference frame.

An interesting semiquantum implementation of this construction is for Alice to send Bob a photon propagating along direction $\vec{v}$ with right circular polarization. Bob then measures the photon in the right-left circular basis. The system sent in this case has both a classical degree of freedom (the direction of propagation) and a quantum degree of freedom (the spin of the photon). Hence, if one averages the system over the rotation group, one finds that its entropy becomes infinite: this system requires an infinite number of qubits.

But this construction also has a purely quantum implementation: by using two spin-$1/2$ degrees of freedom and one relative position it is possible to construct two states of total angular momentum zero (and therefore invariant under rotation) but of opposite parity:

$$|\alpha\rangle = |\text{singlet}\rangle Y_{00},$$

$$|\beta\rangle = \frac{|\uparrow\uparrow\rangle Y_{1-1} - |\text{triplet}\rangle Y_{10} + |\downarrow\downarrow\rangle Y_{11}}{\sqrt{3}}, \qquad (11)$$

where $|\text{singlet}\rangle=(|\uparrow\rangle|\downarrow\rangle-|\downarrow\rangle|\uparrow\rangle)/\sqrt{2}$ and $|\text{triplet}\rangle=(|\uparrow\rangle|\downarrow\rangle+|\downarrow\rangle|\uparrow\rangle)/\sqrt{2}$. We now use the fact that any wave function $|\psi_{spin}\rangle$ composed only of spin degrees of freedom is invariant under parity: $P|\psi_{spin}\rangle=|\psi_{spin}\rangle$. This implies that $P|\alpha\rangle=+|\alpha\rangle$ and $P|\beta\rangle=-|\beta\rangle$, and hence $|G^{\pm}\rangle=(|\alpha\rangle\pm|\beta\rangle)/\sqrt{2}$, constitute good quantum gloves.

With a single spin $1/2$ and the relative position of two particles one cannot construct a state of total angular momentum zero. However, one can construct two spaces of dimension 2, of total angular momentum $1/2$, and of opposite parity. Bases of these spaces are

$$|\alpha_{1/2+1/2}\rangle = Y_{00}|\uparrow\rangle,$$

$$|\alpha_{1/2-1/2}\rangle = Y_{00}|\downarrow\rangle \qquad (12)$$

and

$$|\beta_{1/2+1/2}\rangle = \frac{Y_{10}|\uparrow\rangle - \sqrt{2}Y_{11}|\downarrow\rangle}{\sqrt{3}},$$

$$|\beta_{1/2-1/2}\rangle = \frac{Y_{10}|\downarrow\rangle - \sqrt{2}Y_{1-1}|\uparrow\rangle}{\sqrt{3}}. \qquad (13)$$

D. Quantum gloves made from relative position of two particles

Finally let us show that one can construct approximate quantum gloves using the relative position of two particles only. That this should be the case can be seen from the example discussed above of the spinning disk with asymmetric upper and lower sides. Indeed consider an electron in orbit around a proton. The angular momentum of the electron defines the axial vector $\vec{a}$. However, the wave function of the electron need not be symmetric between the upper and lower sides of the plane of rotation, hence encoding a vector parallel to $\vec{a}$. The arguments given above show that this should allow Alice to encode the chirality of her reference frame. We now show how this can be done.

Consider the following two orthogonal states:

$$|g_+\rangle = \frac{Y_{00}+Y_{01}}{\sqrt{2}},$$

$$|g_-\rangle = \frac{Y_{00}-Y_{01}}{\sqrt{2}}. \tag{14}$$

Under parity they transform as $P|g_+\rangle=|g_-\rangle$ and $P|g_-\rangle=|g_+\rangle$. Thus they seem good candidates for quantum gloves. However, these are not perfect quantum gloves because under rotation $|g_+\rangle$ does not stay orthogonal to $|g_-\rangle$. Thus, if Alice and Bob's reference frames are not aligned or if the quantum glove undergoes a random rotation during transmission, then they cannot learn with certainty whether they have the same chirality. More precisely one computes that

$$\rho_\pm = \int dR U_R |g_\pm\rangle\langle g_\pm| U_R^\dagger \tag{15}$$

$$= \tfrac{1}{6}(|Y_{11}\rangle\langle Y_{11}| + |Y_{10}\rangle\langle Y_{10}| + |Y_{1-1}\rangle\langle Y_{1-1}|) + \tfrac{1}{2}|Y_{00}\rangle Y_{00} \pm \tfrac{1}{4}(|Y_{00}\rangle\langle Y_{10}| + |Y_{10}\rangle\langle Y_{00}|), \tag{16}$$

where the integration in Eq. (15) is over all rotations R and U_R is the unitary transformation that realizes rotation R. Bob's task is thus to distinguish between the two density matrices $\rho_\pm$. Since these density matrices are nonorthogonal, he only has a finite chance of success. We will show below that this is a general feature and that it is impossible to make perfect rotationally invariant quantum gloves out of the relative position of two particles. We expect that by increasing the size of the Hilbert space—i.e., by having the gloves have large angular momentum—it is possible to make them better and better.

E. Quantum gloves and decoherence free subspaces

It is interesting to note that the quantum gloves constructed in the examples described above are very closely related to the decoherence free subspaces considered in [14–16] and recently realized experimentally in [17]. Indeed in these works the aim was to construct orthogonal states or subspaces that are invariant under rotation. The main difference is that for these applications it is indifferent whether the subspaces are realized using spin degrees of freedom or using relative position of particles. Thus, for instance, the states realized in [17] are states of the polarization (i.e., angular momentum) of four photons and therefore are good decoherence free spaces, but cannot serve as quantum gloves.

V. CHIRALITY OPERATOR

In what preceeded we focused on the properties of the quantum states $|G_\pm\rangle$ sent by Alice and supposed that Bob always measured the same operator χ. This approach is the one which would be adopted by the external observer if he has the same chirality as Bob. In this section we consider the opposite situation where the external observer has the same chirality as Alice. In this case Alice always prepares the same state $|G_+\rangle$. But Bob will measure either χ or $P\chi P$ according to his chirality.

We study here the properties of the chirality operator χ. This will provide us with a very general approach to the problem of quantum gloves and will allow us to classify many possible realizations of quantum gloves. We will suppose that the quantum gloves are perfect—i.e., that Bob can perfectly distinguish whether or not Alice has the same chirality as him. We will also suppose rotational invariance in the sense that we require that a quantum glove $|G_+\rangle$ and the rotated glove $R|G_+\rangle$ have exactly the same properties.

With these conditions the chirality operator must obey several conditions. First of all the quantum gloves $|G_+\rangle$ and $|G_-\rangle$ must be eigenstates of χ with different eigenvalues:

$$\chi|G^+\rangle = \gamma^+|G^+\rangle,$$

$$\chi|G^-\rangle = \gamma^-|G^-\rangle, \quad \gamma^+ \neq \gamma^-, \tag{17}$$

where

$$\langle G^-|G^+\rangle = 0, \tag{18}$$

$$P|G^+\rangle = |G^-\rangle, \quad P|G^-\rangle = |G^+\rangle. \tag{19}$$

Properties (17) [or equivalently (18)] are necessary in order to have perfect quantum gloves.

For simplicity we will suppose that the eigenvalues γ^+ and γ^- are opposite: $\gamma^-=-\gamma^+$ (although this is not essential for the next part of the argument based on rotational invariance). Then Eqs. (17) and (19) imply that

$$P\chi P = -\chi. \tag{20}$$

Note that, when Eq. (20) is obeyed χ may have a zero eigenvalue. The corresponding eigenspace cannot be used to encode chirality. In what follows we restrict our attention to the subspace on which χ is nonzero.

We now consider rotation invariance in the sense that we require that if $|G_+\rangle$ is a quantum glove, then the rotated glove $R|G_+\rangle$ have exactly the same properties. In particular this implies that if $|G^+\rangle$ is an eigenstate of χ, then $R|G^+\rangle$ is also an eigenstate of χ with the same eigenvalue:

$$\chi R|G^+\rangle = \gamma^+ R|G^+\rangle.$$

The same holds for $R|G_-\rangle$. These properties imply that the chirality operator is invariant under rotation:

$$R^\dagger \chi R = \chi \quad \forall \; R \in \mathrm{SU}_2. \tag{21}$$

Let us now consider the Hilbert space of the quantum gloves. For definiteness we shall suppose that it is realized by some spin degrees of freedom and the relative position of several particles. Let us denote by $\vec{L}$ the total angular momentum operator acting on this Hilbert space. Then Eq. (21)

implies that χ commutes with the generators of the rotation group $\vec{L}$. In particular χ commutes with the total angular momentum operator $L^2=L_x^2+L_y^2+L_z^2$.

The addition properties of angular momentum imply that the Hilbert space decomposes into a direct sum of spaces $H=H_0\oplus H_{1/2}\oplus H_1\oplus H_{3/2}\oplus\cdots$ where H_L is the space with total angular momentum L. We thus obtain that χ is block diagonal in this representation. From now on we focus on a specific subspace H_L of total angular momentum L.

The space of total angular momentum L can further be decomposed into the direct sum of a number of irreducible representations of SU_2. Some of these irreducible representations have positive parity, denote them H_{L_+}, whereas others have negative parity, denote them H_{L_-}. An arbitrary quantum glove of total angular momentum L can thus be written as $|G_L^+\rangle=\alpha|\psi_{L_+}\rangle+\beta|\psi_{L_-}\rangle$ where $|\psi_{L_+}\rangle\in H_{L_+}$ has total angular momentum L and positive parity and $|\psi_{L_-}\rangle\in H_{L_-}$ has total angular momentum L and negative parity. Then the other glove has the form $|G_L^-\rangle=P|G_L^+\rangle=\alpha|\psi_{L_+}\rangle-\beta|\psi_{L_-}\rangle$. Orthogonality of the right and left gloves then implies that $\alpha=\beta=1/\sqrt{2}$:

$$|G_\pm\rangle=\frac{1}{\sqrt{2}}(|\psi_{L+}\rangle\pm|\psi_{L-}\rangle). \tag{22}$$

This provides a systematic way of constructing all possible quantum gloves in a given Hilbert space. One simply decomposes the total Hilbert space into a direct sum of spaces of different total angular momentum and different parity. The quantum gloves are then arbitrary states of the form of Eq. (22). The states constructed in the previous sections are particular examples of such quantum gloves which use irreducible representations with small values of L. The approach based on the chirality operator shows how to generalize this to other values of L.

This also shows why one cannot construct rotationally invariant perfect quantum gloves using the relative position of two particles only; see Sec. IV D. In this case to each value of L corresponds a single irreducible representation of SU_2 and one cannot construct states of the form (22).

VI. CONCLUSION

Quantum information can be thought of independently of any implementation, similarly to classical information. This rather trivial remark implies that quantum information can only achieve tasks which are expressed in pure information theoretical terms, like cloning and factoring, but cannot perform physical tasks like aligning reference frames or defining temperature. Thus, for instance, quantum teleportation is an information concept and does not permit the teleportation of a physical object, including its mass and chirality. This underlines that *information is physical*, but *physics is more than information*.

Here we have considered the problem in which one party wants to transmit to another the chirality of his reference frame. This is a physical quantity and cannot be sent using information only. Thus the physical nature of the communication channel plays an essential role in understanding this problem. This aspect has often been overlooked or passed under silence. But because in the present case the quantity we want to transmit is so simple—it is only a dichotomic variable—we can focus on these aspects without being distracted by mathematical details.

We have shown that it is possible to construct rotationally invariant quantum states, called "quantum gloves," which can be used to perfectly encode the chirality of a reference frame. We have discussed how, in order to pose the problem correctly, one must separate the communication of the actual glove state from the communication involved in setting up the protocol. We have also seen that whereas spin degrees of freedom alone cannot make quantum gloves, relative positions of particles, or combinations of relative position and spin can make good quantum gloves. Furthermore, one can make trade-offs between resources used: number of qubits transmitted versus number of particles sent versus volume occupied in space, etc.

We hope that this work will stimulate further research on the problem of transmitting physical quantities through physical communication channels. Indeed such research gives us a deep insight into the meaning of these quantities and how they are related to the elementary properties of the physical systems used to encode them.

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APPENDIX: CHIRALITY AND PARTICLE PHYSICS

In the Introduction we noted that "if our world is invariant under *left*↔*right*, then mere information is unable to distinguish between *left* and *right*." But of course particle physics has tought us that our world is not invariant under *left*↔*right*. Indeed the Hamiltonian of elementary particle physics, describing the behavior of kaons, etc., is not invariant under *left*↔*right*. Thus one can prepare a quantum state of elementary particles $|\psi_0\rangle$ which is invariant under parity, $P|\psi_0\rangle=|\psi_0\rangle$, let it evolve, and the final state $e^{-iHt}|\psi_0\rangle$ is no longer invariant under parity. Thus the universe is in fact endowed with an absolute chirality. In this case Alice no longer needs to reveal to Bob some physical information. She only needs to measure her chirality with respect to the absolute chirality of the universe and tell the result to Bob. This can be done using information only—i.e., using only black and white balls. See [18] for a discussion.

In the present work we have supposed that Alice and Bob do not have access to a parity violating Hamiltonian and are restricted to manipulating some simple degrees of freedom such as spin, position, etc. In this case the chirality of their reference frame must be encoded in the quantum states they use. Thus the question we study here is, what are the physical degrees of freedom that allow one to encode chirality and what is the most economical way of doing so if one does not have access to a parity violating Hamiltonian. It would certainly be very interesting to revisit this problem in the light of the known properties of the particle physics Hamiltonian. For instance, some particles, such as pions, have an intrinsic parity, and this could presumably be exploited when constructing quantum gloves.

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