

## LETTER TO THE EDITOR

## Coupled Ito equations of continuous quantum state measurement and estimation

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Online at [stacks.iop.org/JPhysA/39/L575](http://stacks.iop.org/JPhysA/39/L575)**Abstract**

We discuss a nonlinear stochastic master equation that governs the time evolution of the estimated quantum state. Its differential evolution corresponds to the infinitesimal updates that depend on the time-continuous measurement of the true quantum state. The new stochastic master equation couples to the two standard stochastic differential equations of time-continuous quantum measurement. For the first time, we can prove that the calculated estimate almost always converges to the true state, also at low-efficiency measurements. We show that our single-state theory can be adapted to weak continuous ensemble measurements as well.

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**Introduction**

For seven decades after the completion of quantum theory, sequential measurements on a single-quantum system were not technically accessible. Advancements of experimental technology have finally allowed multiple measurements under full control. Time-continuous measurements on a single system have also become feasible. Their theory concluded to a flexible Ito-stochastic calculus. It describes the time evolution of the measured state  $\hat{\rho}_t$  under the continuous measurement of a certain variable  $\hat{q}$  as well as the evolution of the time-dependent measurement signal  $q_t$ . Recent perspectives of feedback control shed new light on the problem. The time-continuous, or real-time, state determination became an immediate theoretical task. Doherty *et al* [1] worked out a theory for a specific case (cf an application for feedback induced cooling [2]). A related different task has been discussed independently [3]. Our present proposal, sketched already in [4], extends the results of [1] for the whole class of systems under (time-)continuous measurement, and we prove the general convergence of the real-time estimate to the true state. Our concept will be slightly different from that of [1]. We do not think of integrating the stochastic differential equation of continuous measurement.

Rather, we emphasize that the estimated state  $\hat{\rho}_t^e$  satisfies a further Ito-stochastic equation driven by the (noisy) measurement signal  $q_t$ . This allows us to prove that the Hilbert–Schmidt distance between the unknown state  $\hat{\rho}_t$  and the calculated real-time estimate  $\hat{\rho}_t^e$  is decreasing until  $\hat{\rho}_t$  and  $\hat{\rho}_t^e$  will coincide:

$$\lim_{t \rightarrow \infty} \|\hat{\rho}_t^e - \hat{\rho}_t\| = 0. \quad (1)$$

This convergence implies for example the possibility of monitoring quantum oscillations in real time (cf [5]). The structure of our letter is as follows. We first introduce the elementary measurement-update cycle and the weak-measurement limit. Then we present the Ito equations of our proposal, and we provide the proof of the above convergence theorem. We also discuss and verify the theory for non-maximum efficiency of the continuous measurement. Finally, an application to collective measurements is shown.

### Unsharp measurement and update of states

In all continuous measurement and/or estimation theories, unsharp measurements play a central role. Consider the standard Gaussian model of an unsharp measurement of a variable  $\hat{q}$  [6]. If  $\hat{\rho}$  is the *a priori* state the measurement outcome  $q$  will have the following probability distribution:

$$p(q) = \text{tr}[G_\sigma(q - \hat{q})\hat{\rho}] = \langle G_\sigma(q - \hat{q}) \rangle_{\hat{\rho}}, \quad (2)$$

where  $G_\sigma$  is the normalized Gauss function of spread  $\sigma$ . The following standard update yields our *a posteriori* state:

$$\hat{\rho} \longrightarrow \frac{1}{p(q)} G_\sigma^{1/2}(q - \hat{q}) \hat{\rho} G_\sigma^{1/2}(q - \hat{q}). \quad (3)$$

If the *a priori* state  $\hat{\rho}$  is unknown then also the *a posteriori* one remains unknown. We can, nonetheless, estimate the *a priori* state, say by a certain  $\hat{\rho}^e$ . Then we apply the *same* update to our *a priori* estimate  $\hat{\rho}^e$  as to the true state above:

$$\hat{\rho}^e \longrightarrow \frac{1}{p^e(q)} G_\sigma^{1/2}(q - \hat{q}) \hat{\rho}^e G_\sigma^{1/2}(q - \hat{q}). \quad (4)$$

Please note that the normalization factor is different than in equation (3). The normalizing function  $p^e(q)$  has, although we employ similar notation, no role as a probability distribution. We expect that by using weak measurements, i.e. when the unsharpness  $\sigma$  is very large, iterating the updates (2)–(4) brings the estimate and the true state closer to each other. It therefore makes sense to repeat the above measurement-update cycle many times at high frequency  $\nu$  in order to make the real-time estimate  $\hat{\rho}_t^e$  converge to the true state  $\hat{\rho}_t$ , as claimed in equation (1). Below we prove this remarkable convergence in the weak-measurement continuous-time limit [8], where both the unsharpness  $\sigma$  and the repetition frequency  $\nu$  of the measurement-update cycle tend to infinity while the ratio  $\nu/\sigma^2$  remains constant:

$$\sigma, \nu \longrightarrow \infty, \quad \frac{\nu}{\sigma^2} = \gamma. \quad (5)$$

The quantity  $\gamma$  is called the strength of the continuous measurement. In this limit, equations (2)–(4) result in three Ito-stochastic differential equations, respectively, for the time-dependent outcome (signal)  $q_t$  of the measurement, for the true state  $\hat{\rho}_t$  and for the estimate  $\hat{\rho}_t^e$ , which constitute the theory of continuous measurement and estimation.

### The three coupled Ito equations

Let us first postulate the heuristic theory. Consider an observable  $\hat{q}$  which we measure continuously. The signal is governed by a simple stochastic equation  $q_t = \langle \hat{q}_t \rangle_{\hat{\rho}_t} + \gamma^{-1/2} w_t$ , where  $\langle \hat{q}_t \rangle_{\hat{\rho}_t}$  stands for the mean value of  $\hat{q}$  in the current quantum state  $\hat{\rho}_t$ .  $w_t$  is the standard white-noise defined by  $E[w_t] = 0$  and  $E[w_t w_s] = \delta(t - s)$ , where  $E$  stands for the stochastic mean. This form of the observed value  $q_t$  is plausible: it fluctuates around the quantum mean value and the magnitude of the fluctuations decreases with the strength of the continuous measurement. Due to its nonlinearity, however, the naive equation must be replaced by the mathematically precise Ito equation

$$dQ = \langle \hat{q} \rangle_{\hat{\rho}} dt + \gamma^{-1/2} dW, \quad (6)$$

where  $Q_t, W_t$  are the time integrals of  $q_t$  and  $w_t$ , respectively. From now on, we call  $Q_t$  the integrated signal of the continuous measurement. For notational convenience of equation (6) and of further equations, the lower indices  $t$  are systematically ignored.

The second Ito equation governs the state  $\hat{\rho}_t$  under the continuous measurement of  $\hat{q}$ . For Markovian mechanisms, like ours, the Ito increment  $d\hat{\rho}_t$  of the state will only depend on the current state  $\hat{\rho}_t$  and on the current Ito increment  $dQ_t$  of the (integrated) signal:<sup>4</sup>

$$d\hat{\rho} = -i[\hat{H}, \hat{\rho}] dt - \frac{\gamma}{8} [\hat{q}, [\hat{q}, \hat{\rho}]] dt + \frac{\gamma}{2} \{\hat{q} - \langle \hat{q} \rangle_{\hat{\rho}}, \hat{\rho}\} (dQ - \langle \hat{q} \rangle_{\hat{\rho}} dt), \quad (7)$$

where  $\hat{H}$  is the Hamiltonian. Obviously, at any time  $t$ ,  $\hat{\rho}_t$  is conditioned on the previous measurement outcomes through  $\{Q_s; s \leq t\}$ .

The third Ito equation governs the evolution of our estimate  $\hat{\rho}_t^e$ :

$$d\hat{\rho}^e = -i[\hat{H}, \hat{\rho}^e] dt - \frac{\gamma}{8} [\hat{q}, [\hat{q}, \hat{\rho}^e]] dt + \frac{\gamma}{2} \{\hat{q} - \langle \hat{q} \rangle_{\hat{\rho}^e}, \hat{\rho}^e\} (dQ - \langle \hat{q} \rangle_{\hat{\rho}^e} dt). \quad (8)$$

The structure of this equation coincides with the structure of equation (7), but  $dQ$  here depends on the true state  $\hat{\rho}$ , cf equation (6).

Equations (6)–(8) constitute the theory of continuous measurement *and* estimation. The second and third can also be called the stochastic master equations (SMEs) of measurement and estimation, respectively. The first two constitute the theory of continuous measurement and they were shortly derived from unsharp measurements (2)–(4) in the weak-measurement continuous-time limit, cf [7]. The proof relies on the approximation

$$p(q) \approx G_{\sigma}(q - \langle \hat{q} \rangle_{\hat{\rho}}) \quad (9)$$

valid for large  $\sigma$ . We can easily confirm the novel SME (8) from equation (4), without adapting the (otherwise simple) derivation [7] of (6), (7) from equations (2), (3). By change of variables, we are going to show that (3) and (4) become asymptotically identical. Let us consider expression (4) of the updated estimate and calculate the normalizing denominator for large  $\sigma$ :

$$p^e(q) \approx G_{\sigma}(q - \langle \hat{q} \rangle_{\hat{\rho}^e}). \quad (10)$$

Let us introduce new variables  $\hat{q}^e = \hat{q} - \langle \hat{q} \rangle_{\hat{\rho}^e} + \langle \hat{q} \rangle_{\hat{\rho}}$  and, of course,  $q^e = q - \langle \hat{q} \rangle_{\hat{\rho}^e} + \langle \hat{q} \rangle_{\hat{\rho}}$ . Please observe that  $p^e(q) = p(q^e)$  and re-write (4) into this form

$$\hat{\rho}^e \rightarrow \frac{1}{p(q^e)} G_{\sigma}^{1/2}(q^e - \hat{q}^e) \hat{\rho}^e G_{\sigma}^{1/2}(q^e - \hat{q}^e). \quad (11)$$

<sup>4</sup> The evolution equation for the measured state  $\hat{\rho}_t$  can be equivalently written as an autonomous equation [7, 9] driven by the standard Wiener-process  $W_t$ :  $d\hat{\rho} = -i[\hat{H}, \hat{\rho}] - \frac{1}{8} g^2 [\hat{q}, [\hat{q}, \hat{\rho}]] dt + \frac{1}{2} g \{\hat{q} - \langle \hat{q} \rangle_{\hat{\rho}}, \hat{\rho}\} dW$ . This form will be convenient for analytic calculations.

This is exactly the same equation as equation (3) updating the true state  $\hat{\rho}$ . Therefore the SME for  $\hat{\rho}^e$  will, in the new variables  $\hat{q}^e, q^e$ , coincide with the SME (7) of  $\hat{\rho}$ :

$$d\hat{\rho}^e = -i[\hat{H}, \hat{\rho}^e] dt - \frac{\gamma}{8}[\hat{q}^e, [\hat{q}^e, \hat{\rho}^e]] dt + \frac{\gamma}{2}\{\hat{q}^e - \langle \hat{q}^e \rangle_{\hat{\rho}^e}, \hat{\rho}^e\}(dQ^e - \langle \hat{q}^e \rangle_{\hat{\rho}^e} dt). \quad (12)$$

If we restore the original variables  $\hat{q} = \hat{q}^e + \langle \hat{q} \rangle_{\hat{\rho}^e} - \langle \hat{q} \rangle_{\hat{\rho}}$  and  $dQ = dQ^e + \langle \hat{q} \rangle_{\hat{\rho}^e} dt - \langle \hat{q} \rangle_{\hat{\rho}} dt$ , we obtain equation (8).

### Proof of convergence

For long times, both the actual state of the system,  $\hat{\rho}_t$ , and the estimated state,  $\hat{\rho}_t^e$ , become pure states. Therefore, it will be sufficient to prove that the fidelity  $\text{tr}[\hat{\rho}_t \hat{\rho}_t^e]$  tends to 1 for large  $t$ . We detail the proofs below.

First we note that all three equations (6)–(8) are invariant under the trivial shifts  $\hat{q} \rightarrow \hat{q} + r$ ,  $dQ \rightarrow dQ + r dt$ , for arbitrary real constant  $r$ . In all time-local calculations we can, e.g., make  $\langle \hat{q} \rangle_{\hat{\rho}}$  zero by choosing  $r = -\langle \hat{q} \rangle_{\hat{\rho}}$  and we can restore the true result at the end of the calculation if we make a second shift by  $-r$ . This allows quicker calculations and we shall refer to this as shift invariance.

For long times, the solutions  $\hat{\rho}_t$  of the SME (6) are known to turn into pure states [1, 10]. To prove this, we show that the increment of  $E[\text{tr}[\hat{\rho}_t^2]]$  is non-negative:

$$dE[\text{tr}[\hat{\rho}^2]] = E[\text{tr}[2\hat{\rho} d\hat{\rho} + d\hat{\rho} d\hat{\rho}]] \geq 0. \quad (13)$$

Using the Ito equations (6), (7) in the shifted coordinate system, where  $\langle \hat{q} \rangle_{\hat{\rho}} = 0$ , the lhs can be written as the trace of a non-negative matrix:

$$\gamma^{-1} \frac{d}{dt} E[\text{tr}[\hat{\rho}^2]] = \text{tr}[\hat{\rho} \hat{q} \hat{\rho} \hat{q}] \equiv \text{tr}[(\hat{\rho}^{1/2} \hat{q} \hat{\rho}^{1/2})^2], \quad (14)$$

which is greater than zero and vanishes if  $\hat{\rho}_t$  is already a pure state. In certain marginal cases the growth of purity may get stalled, we discuss this problem later. However, in generic physical situations the rhs of equation (14) becomes zero only if  $\hat{\rho}_t$  turns into a pure state. The presented proof implies the longtime purity of  $\hat{\rho}_t^e$  as well since, in suitable variables, the SMEs of  $\hat{\rho}_t$  and  $\hat{\rho}_t^e$  are identical, cf equations (7) and (12), respectively.

Finally, we prove that in the generic case,  $\text{tr}[\hat{\rho}_t \hat{\rho}_t^e] \rightarrow 1$ , when  $t \rightarrow \infty$ . The task is to show that

$$dE[\text{tr}[\hat{\rho} \hat{\rho}^e]] = E[\text{tr}[d\hat{\rho} \hat{\rho}^e + \hat{\rho} d\hat{\rho}^e + d\hat{\rho} d\hat{\rho}^e]] \geq 0, \quad (15)$$

with equality if and only if  $\hat{\rho}^e = \hat{\rho}$  for all typical, physically interesting situations. Using shift invariance, we can set  $\langle \hat{q} \rangle_{\hat{\rho}} + \langle \hat{q} \rangle_{\hat{\rho}^e} = 0$ . Substituting equations (7), (8) yields the following result:

$$\begin{aligned} \gamma^{-1} \frac{d}{dt} E[\text{tr}[\hat{\rho} \hat{\rho}^e]] &= \langle \hat{q} \rangle_{\hat{\rho}}^2 \text{tr}[\hat{\rho} \hat{\rho}^e] + \text{tr}[\hat{\rho} \hat{q} \hat{\rho}^e \hat{q}] + \langle \hat{q} \rangle_{\hat{\rho}} \text{tr}[\hat{q} \{\hat{\rho}, \hat{\rho}^e\}] \\ &\equiv \text{tr}[\hat{\rho}^{1/2} (\hat{q} + \langle \hat{q} \rangle_{\hat{\rho}}) \hat{\rho}^e (\hat{q} + \langle \hat{q} \rangle_{\hat{\rho}}) \hat{\rho}^{1/2}]. \end{aligned} \quad (16)$$

The rhs is the trace of a non-negative matrix. This assures that the fidelity is monotonously increasing until  $\hat{\rho}^e = \hat{\rho}$  is reached asymptotically. We shall emphasize that the convergence may cease for marginal cases, see our discussion below. Nevertheless, in generic physical applications convergence will always be achieved. This claim is supported by numerical simulations, and also by the fact that in the one-qubit case, if  $[\hat{H}, \hat{q}] \neq 0$ , one can exactly prove that the rhs of equation (16) can vanish only if the true and the estimated states coincide,  $\hat{\rho}^e = \hat{\rho}$ .

### Extension for low-efficiency measurements

The theory of continuous measurement (6), (7) corresponds to perfectly efficient continuous measurements, i.e., the signal-to-noise ratio reaches the quantum mechanically possible maximum value. Real continuous measurements are producing and/or are accompanied by an excess noise. Therefore, they cannot preserve or reach the purity of continuously measured states although they limit their mixedness. Wiseman and Milburn [11] have already incorporated the efficiency parameter  $\eta \in [0, 1]$  into the theory (6), (7), and we do it for the novel SME (8) as well:

$$dQ = \langle \hat{q} \rangle_{\hat{\rho}} dt + (\eta\gamma)^{-1/2} dW \quad (17)$$

$$d\hat{\rho} = -i[\hat{H}, \hat{\rho}] dt - \frac{\gamma}{8} [\hat{q}, [\hat{q}, \hat{\rho}]] dt + \frac{\eta\gamma}{2} \{\hat{q} - \langle \hat{q} \rangle_{\hat{\rho}}, \hat{\rho}\} (dQ - \langle \hat{q} \rangle_{\hat{\rho}} dt), \quad (18)$$

$$d\hat{\rho}^e = -i[\hat{H}, \hat{\rho}^e] dt - \frac{\gamma}{8} [\hat{q}, [\hat{q}, \hat{\rho}^e]] dt + \frac{\eta\gamma}{2} \{\hat{q} - \langle \hat{q} \rangle_{\hat{\rho}^e}, \hat{\rho}^e\} (dQ - \langle \hat{q} \rangle_{\hat{\rho}^e} dt). \quad (19)$$

Our SME (19) of estimation works for lower efficiencies  $\eta < 1$  as well. We expect that the convergence (1) of the estimate  $\hat{\rho}_t^e$  and the true state  $\hat{\rho}_t$  will slow down if  $\eta \ll 1$ , still it exists for all nonzero efficiencies  $\eta$ . The former proof cannot be applied directly because it relied upon the longtime purity of both  $\hat{\rho}_t$  and  $\hat{\rho}_t^e$ . Yet, we can reduce the proof of the case  $\eta < 1$  to the former proof of the case  $\eta = 1$ . Let us outline the steps.

In the two SMEs (18), (19), let us separate the excess noise from that which is necessary for a given measurement efficiency  $\eta$ :

$$d\hat{\rho} = -i[\hat{H}, \hat{\rho}] dt - \frac{(1-\eta)\gamma}{8} [\hat{q}, [\hat{q}, \hat{\rho}]] dt - \frac{\eta\gamma}{8} [\hat{q}, [\hat{q}, \hat{\rho}]] dt + \frac{\eta\gamma}{2} \{\hat{q} - \langle \hat{q} \rangle_{\hat{\rho}}, \hat{\rho}\} (dQ - \langle \hat{q} \rangle_{\hat{\rho}} dt), \quad (20)$$

and similarly for (19). It is known that the noise term proportional to  $1 - \eta$ , like any further noise terms, can always be reproduced formally by an interaction Hamiltonian with a ‘heat bath’. Accordingly, we can transform the original SMEs of continuous measurement and estimation at measurement strength  $\gamma$  and efficiency  $\eta < 1$  into the theory of continuous measurement and estimation of the system+bath at measurement strength  $\eta\gamma$  and efficiency  $\eta = 1$ . Vice versa, if we trace over the bath, these SMEs reduce to the SMEs of the original system. According to our earlier theorem, valid for  $\eta = 1$ , the true and the estimated states of the system+bath converge to each other. This convergence implies, via tracing over the bath, that also  $\hat{\rho}_t$  and  $\hat{\rho}_t^e$  converge to each other whereas both may remain mixed forever.

### Application to ensembles

The weak-measurement paradigm also plays a role in applications where a large ensemble of the unknown state  $\hat{\rho}$  is accessible to the experiment, cf e.g. [3]. Crucial in this context is the approximation that the collective state of the ensemble preserves the uncorrelated form  $\hat{\rho}_t^c = \hat{\rho}_t^{\otimes N}$  if  $N$  is very large and the strength  $\gamma^c$  of the collective measurement is very small. In particular, we consider the same observable  $\hat{q}$  on each component and we measure their *sum*  $\hat{q}^c$  in a collective continuous measurement of strength  $\gamma^c$ . For the sake of simplicity only, we assume  $\eta = 1$  and apply the theory (6)–(8):

$$dQ^c = \langle \hat{q}^c \rangle_{\hat{\rho}^c} dt + (\gamma^c)^{-1/2} dW, \quad (21)$$

$$d\hat{\rho}^c = -i[\hat{H}^c, \hat{\rho}^c] dt - \frac{\gamma^c}{8} [\hat{q}^c, [\hat{q}^c, \hat{\rho}^c]] dt + \frac{\gamma^c}{2} \{\hat{q}^c - \langle \hat{q}^c \rangle_{\hat{\rho}^c}, \hat{\rho}^c\} (dQ^c - \langle \hat{q}^c \rangle_{\hat{\rho}^c} dt), \quad (22)$$

$$d\hat{\rho}^{ce} = -i[\hat{H}^c, \hat{\rho}^{ce}] dt - \frac{\gamma^c}{8} [\hat{q}^c, [\hat{q}^c, \hat{\rho}^{ce}]] dt + \frac{\gamma^c}{2} \{\hat{q}^c - \langle \hat{q}^c \rangle_{\hat{\rho}^{ce}}, \hat{\rho}^{ce}\} (dQ^c - \langle \hat{q}^c \rangle_{\hat{\rho}^{ce}} dt), \quad (23)$$

where  $\hat{H}^c$  is the collective Hamiltonian, i.e., the sum of the same  $\hat{H}$  for all  $N$  components. We extend the approximation  $\hat{\rho}^c = \hat{\rho}^{\otimes N}$  for the estimate  $\hat{\rho}^{ce} = (\hat{\rho}^e)^{\otimes N}$ . Substituting these forms, we obtain closed equations of the ensemble-continuous measurement and single-system estimation:

$$dQ^c = N \langle \hat{q} \rangle_{\hat{\rho}} dt + (\gamma^c)^{-1/2} dW, \quad (24)$$

$$d\hat{\rho} = -i[\hat{H}, \hat{\rho}] dt - \frac{\gamma^c}{8} [\hat{q}, [\hat{q}, \hat{\rho}]] dt + \frac{\gamma^c}{2} \{\hat{q} - \langle \hat{q} \rangle_{\hat{\rho}}, \hat{\rho}\} (dQ^c - N \langle \hat{q} \rangle_{\hat{\rho}} dt), \quad (25)$$

$$d\hat{\rho}^e = -i[\hat{H}, \hat{\rho}^e] dt - \frac{\gamma^c}{8} [\hat{q}, [\hat{q}, \hat{\rho}^e]] dt + \frac{\gamma^c}{2} \{\hat{q} - \langle \hat{q} \rangle_{\hat{\rho}^e}, \hat{\rho}^e\} (dQ^c - N \langle \hat{q} \rangle_{\hat{\rho}^e} dt). \quad (26)$$

These equations are identical to equations (6)–(8) of continuous measurement and estimation on a single system, apart from two things. First, the strength  $\gamma^c$  of the collective measurement survives as the strength of the single-state measurement. Second, the SMEs are governed by the collective signal  $Q_t^c$ , as they should be. This latter fact leads usually to a faster convergence of  $\hat{\rho}_t$  and  $\hat{\rho}_t^e$  than the single-state method, as it is plausible and could be verified from a detailed analysis.

## Remarks

As we anticipated in the text, there are exceptions from convergence (1) and from longtime purity of  $\hat{\rho}_t$  and  $\hat{\rho}_t^e$ . For trivial dynamics  $[\hat{H}, \hat{q}] = 0$ , the estimate  $\hat{\rho}_t^e$  will get stuck in any eigenstate of  $\hat{q}$ , independently of  $\hat{\rho}_t$  which would converge to any other eigenstate as time goes by. However, these cases are of marginal importance. In real tasks the dynamics is nontrivial and  $[\hat{H}, \hat{q}]$  does not vanish. We conjecture the following condition as sufficient for the universal convergence. Consider the *Heisenberg* operator  $\hat{q}_t$  of the measured observable and determine the largest common eigenspace of all  $\hat{q}_t$  for  $t \geq 0$ . If this eigenspace is empty or one-dimensional then the convergence of  $\hat{\rho}_t$  and  $\hat{\rho}_t^e$  is always guaranteed. For instance, the position measurement of a particle yields a convergent state estimate in one dimension. The two-dimensional motion may be different. For a free particle, the simultaneous continuous measurement of both coordinates  $\hat{q}_x$  and  $\hat{q}_y$  is necessary otherwise the measured state may not become pure and the estimate may not converge to it. Interestingly, there is a chance of purity and convergence if we measure but one coordinate  $\hat{q}_x$ , provided a potential rotates  $\hat{q}_x$ 's Heisenberg-version in a proper non-trivial way.

In practice, the (integrated) signal  $Q_t$  is obtained from the experimental device doing the continuous measurement, so that  $Q_t$  does not need computational efforts. On the other hand, the real-time estimate  $\hat{\rho}_t^e$  must be online calculated from  $Q_t$  and one is interested in good algorithms. There are several options depending on the concrete task. In the case of optimum detection efficiency  $\eta = 1$ , we can use a pure state estimate from the beginning. Then the density matrix equation (8) is equivalent to the following stochastic Schrödinger equation for the state-vector estimate:

$$d\psi^e = -i\hat{H}\psi^e dt - \frac{\gamma}{8} (\hat{q} - \langle \hat{q} \rangle_{\psi^e})^2 \psi^e dt + \frac{\gamma}{2} (\hat{q} - \langle \hat{q} \rangle_{\psi^e}) \psi^e (dQ - \langle \hat{q} \rangle_{\psi^e} dt). \quad (27)$$

Of course, if we calculate  $d\hat{\rho}_i^e = d[\psi_i^e(\psi_i^e)^\dagger]$  from the above stochastic Schrödinger equation, we get back (8).

## Summary

To complete the standard theory of continuous measurement, we have constructed a third Ito-stochastic equation for the real-time state estimate, exploiting the measured signal. Our theory (17)–(19) applies to any system under time-continuous measurement. In this way, we have largely extended similar heuristic proposals [1, 2] which used Gaussian estimates  $\hat{\rho}_i^e$  and requested, in principle, the perfect efficiency (signal to noise) of the time-continuous measurement. We proved analytically that our novel SME for the state estimate yields the true state for any non-trivial dynamics at any nonzero efficiency of the measurement. The recent work [4] on real-time estimate has sketched the Ito equations in an alternative representation, without details of derivation and proof of the estimate's convergence to the true state. Our theory, due to the plain structure of the equations, can invariably be applied when several observables are measured simultaneously, like the canonical coordinate and momentum  $\hat{q}$ ,  $\hat{p}$  or the spatial coordinates  $\hat{q}_x$ ,  $\hat{q}_y$ ,  $\hat{q}_z$  of a particle, as well as two or more components of a Pauli spin. We also showed that the theory applies when the state estimate relies on the collective continuous measurement on a large number of copies. Our SME applies to the experimental realizations of single-state control, and we expect that it will contribute to a direct solution of state tomography from continuous measurement on ensembles, cf e.g. [3].

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