Non-Markovian Continuous Quantum Measurement of Retarded Observables

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We reconsider the non-Markovian time-continuous measurement of a Heisenberg observable $\hat{x}$ and show for the first time that it can be realized by an infinite set of entangled von Neumann detectors. The concept of continuous readout is introduced and used to rederive the non-Markovian stochastic Schrödinger equation. We can prove that, contrary to recent doubts, the resulting non-Markovian quantum trajectories are true single system trajectories and correspond to the continuous measurement of a retarded functional of $\hat{x}$.

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Time-continuous measurement in quantum mechanics has long been an open theoretical issue because of the peculiarity of single quantum measurement itself. The Markovian theory emerged 20 years ago [1–3] from foundational considerations. The requests in quantum optics (and elsewhere) triggered another, partly independent, line of progress with expanding applications [4]. So far the Markovian theory of continuous measurement has become completely understood while the general non-Markovian one has remained an open issue even conceptionally.

Markovian time-continuous quantum measurement theory [2,3] includes the Markovian stochastic Schrödinger equation (MSSE) of the postmeasurement state vector ψ_t (cf. [1]), as correlated with the readout x_t of the detector system that measures a certain Heisenberg observable $\hat{x}_t$. A formal extension for the non-Markovian (even relativistic) case was published in Ref. [5]. This work calculated the asymptotic state $\psi_\infty[x]$, only as a function of the whole readout $\{x_t; t \in (-\infty, \infty)\}$, and determined correctly its probability distribution functional $p_\infty[x]$. It could not interpret intermediate conditional states because the concept of continuous readout was missing. This incomplete non-Markovian continuous measurement theory remained largely ignored; it has not been improved or advanced. Meanwhile, Strunz found non-Markovian quantum trajectories [6], and we invented their non-Markovian stochastic Schrödinger equation (NMSSE) [7,8]. This NMSSE and its modifications have been studied in subsequent works [9–12]. As in the Markovian case, one expected that the solutions of the NMSSE turn out to be realizable on a single copy of our quantum system via infinite many von Neumann detectors coupled to it. Such a realizability theorem holds for the solutions (quantum trajectories) of all diffusive MSSE [13]. Yet, Gambetta and Wiseman conjectured that the solution of the NMSSE cannot be observed on a single system [10]; I wrote cautiously [14]: these non-Markovian trajectories cannot be realized by any known way of monitoring [15].

The present work reaches the positive conclusion: the non-Markovian trajectories are measurable single system trajectories. A particular example can be the continuous measurement of a Heisenberg coordinate $\hat{x}_t$ with detectors of finite inertial time $1/\lambda$. Then the measured quantity becomes, e.g.,

$$\hat{z}_t = \lambda \int_0^t e^{-\lambda(t-\sigma)} \hat{x}_\sigma d\sigma. \quad (1)$$

Our work includes the more general case; see Eq. (22) later. We describe the detector system and prove that the NMSSE is indeed the equation of the continuously measured state. The proofs are based on the approach of Refs. [5–10], and an independent direct proof might be the subject of future research. The knowledge of the superoperator formalism is a request; it can be understood from [5,16] or learned from [17].

Stochastic unraveling.—Assume that a Heisenberg variable $\hat{x}_t$ of the system couples for times $t \geq 0$ to a harmonic reservoir variable whose equilibrium correlation function $\alpha(\tau - \sigma)$ will determine the reduced dynamics of the open system density operator $\hat{\rho}_t$:

$$\hat{\rho}_t = \mathcal{M}_t \hat{\rho}_0. \quad (2)$$

For simplicity, let $\alpha(\tau - \sigma)$ be real. Then the evolution superoperator $\mathcal{M}_t$ takes the following compact form [16]:

$$\mathcal{M}_t = \mathcal{T} \exp\left(-\frac{1}{2} \int_0^t d\tau \int_0^t d\sigma \hat{x}_{\tau,\Delta} \alpha(\tau - \sigma) \hat{x}_{\sigma,\Delta} \right). \quad (3)$$

Superoperator notation $\hat{x}_{\tau,\Delta} \hat{O}$ means $[\hat{x}_\tau, \hat{O}]$ for any operator $\hat{O}$ standing to the right of $\hat{x}_{\tau,\Delta}$, and $\mathcal{T}$ prescribes time ordering for all Heisenberg (super)operators standing to the right of $\mathcal{T}$.

We could consider the reservoir as the detector of $\hat{x}_t$. Technically, it is more tractable if we consider standard von Neumann detectors; hence, we replace the reservoir by them. However, we require that their influence on the system be the same as the reservoir's. We assume, for simplicity, that the detectors are able to fully monitor the system's trajectory $\psi_t[x]$ for all time $t \geq 0$, in a function of the detection readout $\{x_\tau; \tau \in [0, t]\}$ whose probability distribution is denoted by $p_t[x]$. Then the stochastic mean of the trajectories will reproduce the open system evolution:

$$\hat{\rho}_t = \mathbf{M}\psi_t[x]\psi_t^\dagger[x], \tag{4}$$

for all $t \geq 0$, since the detector's influence is the same as the reservoir's. We say that the trajectories $\psi_t[x]$ *unravel* the open system dynamics (2).

In the Markovian special case $\alpha(\tau - \sigma) = g^2\delta(\tau - \sigma)$. Then the conditional state vector $\psi_t[x]$ satisfies the MSSE [1–3]. The NMSSE [7,8] became a candidate of being the equation of non-Markovian continuous measurement of $\hat{x}_t$. Here we use the simple real-noise version [10–12]. For the unnormalized state vector $\Psi_t[z]$, the NMSSE reads:

$$\frac{d\Psi_t[z]}{dt} = z_t\hat{x}_t\Psi_t[z] - 2\hat{x}_t\int_0^t \alpha(t-\tau)\frac{\delta\Psi_t[z]}{\delta z_\tau}d\tau, \tag{5}$$

where z_τ is a real random variable for $\tau \in [0, t]$. The true postmeasurement state is obtained via normalization $\psi_t[z] = \Psi_t[z]/\|\Psi_t[z]\|$. The probability distribution of z is the following:

$$p_t[z] = \tilde{G}_{[0,t]}[z]\|\Psi_t[z]\|^2, \tag{6}$$

where $\tilde{G}_{[0,t]}[z]$ is defined by (A3). With this statistics, the solutions $\psi_t[z]$ unravel the non-Markovian open system dynamics (2) and (3):

$$\hat{\rho}_t = \mathbf{M}\psi_t[z]\psi_t^\dagger[z]. \tag{7}$$

Although to calculate the analytic form (6) of $p_t[z]$ would be cumbersome, it follows from the method [8] that

$$\mathbf{M}\,z_t = 2\int_0^t \alpha(t-\sigma)\langle\hat{x}_\sigma\rangle_t d\sigma, \tag{8}$$

where $\langle\hat{x}_\sigma\rangle_t$ is $\hat{x}_\sigma$'s quantum expectation value at time t in the conditional state $\psi_t[z]$. This suggests that the NMSSE (5) measures the retarded functional of $\hat{x}_t$ rather than $\hat{x}_t$ itself. Compared to the Markovian case, there has been one serious issue left: Whether the trajectory $\psi_t[z]$ can, like the Markovian trajectories, be realized on a single system by sequential von Neumann measurements of which z_t is the readout? We answer in the positive and construct the corresponding von Neumann detectors.

Non-Markovian measurement device.—The construction will be very similar to the Markovian one [2,18] in that we replace the reservoir by a dense sequence of standard von Neumann detectors. To learn what happens, let us first consider a single von Neumann detector of initial density matrix $D_0(x;x')$ and couple it to our system at time τ in order to measure the current Heisenberg operator $\hat{x}_\tau$. Following von Neumann (last three pages in [19]), we choose $\delta(t-\tau)\hat{x}_\tau(-i\partial/\partial x)$ for the interaction Hamiltonian. We can write the initial composite state of the detector + system as $D_0(x;x')\hat{\rho}_0$. Fortunately, we can and shall restrict all forthcoming calculations on the elements $x = x'$ since we shall eventually collapse on (or trace over) the pointer coordinates. After the interaction, the total state becomes entangled at τ and the pointer x gets shifted by $\hat{x}_\tau$:

$$D_0(x;x)\hat{\rho}_0 \rightarrow D_0(x - \hat{x}_{\tau,L}; x - \hat{x}_{\tau,R})\hat{\rho}_0. \tag{9}$$

In superoperator notations $\hat{x}_{\tau,L}\hat{O} = \hat{x}_\tau\hat{O}$ and $\hat{x}_{\tau,R}\hat{O} = \hat{O}\hat{x}_\tau$. It is the *readout* of the pointer x that turns the total state into the following conditional postmeasurement state, depending on the readout, of the system alone:

$$\hat{\rho}(x) = \frac{1}{p(x)}D_0(x - \hat{x}_{\tau,L}; x - \hat{x}_{\tau,R})\hat{\rho}_0. \tag{10}$$

The readout x has the probability distribution $p(x)$ whose expression follows from the normalization of the above conditional state:

$$p(x) = \mathrm{Tr}D_0(x - \hat{x}_{\tau,L}; x - \hat{x}_{\tau,R})\hat{\rho}_0. \tag{11}$$

Now, let us choose a fine discretization $\tau = n\epsilon$ of the time, $n = 0, \pm1, \pm2, \ldots$. We install an infinite sequence of von Neumann detectors; they could be numbered by the integers n, but we label them by the discretized times $\tau = n\epsilon$. The pointer coordinates of the detectors will be, respectively, denoted by x_τ. The detector of *label* $\tau = n\epsilon$ measures the Heisenberg operator $\hat{x}_\tau$ of the system via the mechanism (9)–(11) provided we switch the von Neumann interactions on. We do so for the non-negative labels; i.e., we choose the interaction Hamiltonian $\sum_{\tau\geq 0}\delta(t-\tau)\hat{x}_\tau(-i\partial/\partial x_\tau)$.

We depart from the Markovian construction and assume *initially correlated detectors*. Let their initial wave function be:

$$\phi_0[x] = \sqrt{\mathcal{N}}\exp\Big(-\epsilon^2\sum_{\tau,\sigma}x_\tau\alpha(\tau-\sigma)x_\sigma\Big), \tag{12}$$

where the summation extends for all discretized values of both τ and σ. The notation $[x]$ anticipates the continuous (or weak-measurement) limit [2,18] $\epsilon \rightarrow 0$ where the above wave function becomes the square root of the Gaussian functional (A1): i.e., $\phi_0[x] = \sqrt{G[x]}$. We carry out the explanation in the continuous limit. The total initial density matrix reads:

$$\hat{\rho}_0[x;x'] = \sqrt{G[x]}\hat{\rho}_0\sqrt{G[x']}. \tag{13}$$

As we switched on the detectors of labels $\tau \geq 0$ only, at time $t > 0$ each pointer coordinate x_τ with $\tau \in [0, t]$ will have been shifted by $\hat{x}_\tau$ and the following composite state emerges [cf. (9)]:

$$\hat{\rho}_t[x;x] = \mathcal{T}\sqrt{G[x - \theta_{[0,t]}\hat{x}_L]}\sqrt{G[x - \theta_{[0,t]}\hat{x}_R]}\hat{\rho}_0, \tag{14}$$

where $\theta_{[0,t]}$ denotes the characteristic function $\theta_{[0,t]}(\tau)$ of the period $[0, t]$. This can be written into the following compact form:

$$\hat{\rho}_t[x;x] = \mathcal{T}G[x - \theta_{[0,t]}\hat{x}_c]\mathcal{M}_t\hat{\rho}_0, \tag{15}$$

using the Eqs. (3) and (A1) and the superoperator notation $\hat{x}_c\hat{O} = \frac{1}{2}\{\hat{x}, \hat{O}\}$. This remarkable novel form guarantees explicitly that the reduced density matrix $\hat{\rho}_t$ of the system satisfies the open system evolution (2) and (3) as it should. Indeed, the tracing over the detectors' Hilbert space is equivalent with the functional integration of the diagonal elements (15) over all x_τ, which cancels the factor G and leaves us with (2).

Continuous readout.—It is crucial to realize that the true time evolution of the system's conditional state depends on our chosen schedule of reading out the pointers x_τ. We can read out any x_τ at any time since all detectors are always available. Of course, we better read out the value x_τ at a *time* that is later than the *label* τ of the detector because the detector will only have coupled to the system at time τ. Hence, a natural schedule is that we read out x_τ immediately at time τ. Hence, until any given time $t > 0$ we would read out all pointers x_τ for the period $[0, t]$ and no others. To calculate the conditional postmeasurement state $\hat{\rho}_t[x]$ of the system at time t, we trace (integrate) the total density matrix (15) over all x_τ with $\tau \notin [0, t]$:

$$\hat{\rho}_t[x] = \frac{1}{p_t[x]} \int \hat{\rho}_t[x; x] \prod_{\tau \notin [0,t]} dx_\tau. \tag{16}$$

This postmeasurement density matrix $\hat{\rho}_t[x]$ of the system depends on the readouts x_τ of τ from $[0, t]$ only. By substituting (15), we obtain

$$\hat{\rho}_t[x] = \frac{1}{p_t[x]} \mathcal{T} G_{[0,t]}[x - \hat{x}_c]\mathcal{M}_t\hat{\rho}_0, \tag{17}$$

where $G_{[0,t]}[x]$ is the marginal distribution of $G[x]$, similar to (A3). This is our ultimate equation for the non-Markovian continuous measurement of the Heisenberg observable $\hat{x}_t$, completing the theory [5] (which only gave $\hat{\rho}_\infty[x]$). Recall that, as always, the denominator $p_t[x]$ assures $\mathrm{Tr}\hat{\rho}_t[x] = 1$ as well as it yields the probability distribution of the readouts.

In order to find the measurement process that corresponds to the NMSSE (5), we alter our readout schedule. Instead of the Heisenberg variables $\{x_\tau; \tau \in [0, t]\}$ we read out the following linear functional of them:

$$z_\tau = 2 \int_{-\infty}^{\infty} \alpha(\tau - \sigma)x_\sigma d\sigma, \tag{18}$$

which we also write as $z = 2\alpha x$. We reexpress the total density matrix (15) in the new pointer variables:

$$\hat{\rho}_t[z; z] = \mathcal{T}\tilde{G}[z - 2\alpha\theta_{[0,t]}\hat{x}_c]\mathcal{M}_t\hat{\rho}_0, \tag{19}$$

where we used the identity $G[x] = \text{Jacobian} \times \tilde{G}[z]$. Again, we suppose that we read out each pointer of label τ (i.e., z_τ) at *time* τ. Until time $t > 0$, this schedule implies that all pointers z_τ for the period $[0, t]$ are read out and the rest of them are not. The conditional state of the system is defined by

$$\hat{\rho}_t[z] = \frac{1}{p_t[z]} \int \hat{\rho}_t[z; z] \prod_{\tau \notin [0,t]} dz_\tau, \tag{20}$$

which transforms (19) into

$$\hat{\rho}_t[z] = \frac{1}{p_t[z]} \mathcal{T}\tilde{G}_{[0,t]}[z - 2\alpha\theta_{[0,t]}\hat{x}_c]\mathcal{M}_t\hat{\rho}_0, \tag{21}$$

where $\tilde{G}_{[0,t]}[z]$ is the marginal distribution (A3) of $\tilde{G}[z]$. This is our ultimate equation for the non-Markovian continuous measurement of the observable

$$\hat{z}_t = 2 \int_0^t \alpha(t - \sigma)\hat{x}_\sigma d\sigma, \tag{22}$$

which is a *retarded* functional of the Heisenberg variable $\hat{x}_\tau$. This interpretation of $\hat{\rho}_t[z]$ can shortly be inspected. Recall that at *time* t we read out the pointer of *label* t: i.e., z_t. The factor $\tilde{G}_{[0,t]}[z - 2\alpha\theta_{[0,t]}\hat{x}_c]$ in the expression (21) of the measured state shows that at time t the pointer z_t localizes around (i.e., measures) the observable (22). Equation (8) holds between the readout z_t in (21) and the retarded variable $\hat{z}_t$ (22); instead of the direct proof we are going to prove the complete equivalence of the NMSSE (5) with our construction summarized by Eq. (21).

Stochastic Schrödinger equation.—We are going to prove that the NMSSE (5) governs the evolution (21). Let us find $\hat{\rho}_t[z]$ in the form

$$\hat{\rho}_t[z] = \frac{1}{p_t[z]} \tilde{G}_{[0,t]}[z]\Psi_t[z]\Psi_t^\dagger[z], \tag{23}$$

where $\Psi_t[z]$ is the unnormalized conditional state vector of the system. Taking the trace of both sides, the norm condition yields exactly the $p_t[z]$ (6) that belongs to the NMSSE (5). Inserting (23) as well as $\hat{\rho}_0 = \psi_0\psi_0^\dagger$ into (21), it reduces to

$$\Psi_t[z]\Psi_t^\dagger[z] = \frac{1}{\tilde{G}_{[0,t]}[z]} \mathcal{T}\tilde{G}_{[0,t]}[z - 2\alpha\theta_{[0,t]}\hat{x}_c]\mathcal{M}_t\psi_0\psi_0^\dagger. \tag{24}$$

Substituting Eqs. (3) and (A3), the right-hand side factorizes and we can write equivalently:

$$\Psi_t[z] = \mathcal{T}\exp\left(\int_0^t z_\tau\hat{x}_\tau d\tau - \int_0^t d\tau \int_0^t d\sigma \hat{x}_\tau \alpha(\tau - \sigma)\hat{x}_\sigma\right)\psi_0. \tag{25}$$

This $\Psi_t[z]$ is the solution of the NMSSE (5), as can be seen by substitution. That completes our proof.

Summary.—We proved for the first time that both the formalism [5] of non-Markovian measurement theory and the NMSSE [7] are equivalent with using correlated von Neumann detectors in the weak-measurement continuous limit, i.e., with the continuous readout of the values of a given retarded functional of a Heisenberg variable on a single quantum system. Our merit is the constructive proof of existence of the underlying standard quantum mechanical measurement process. The results should be generalized in various directions. We can interpret complex reservoir correlation functions, too, if we include the mechanism of feedback [5]. We might retain the original reservoir as a detector [10], to extract information by measuring the reservoir but without altering the non-Markovian reduced dynamics of the monitored system. Then the measured retarded observable might be identified by a reservoir field. (Theories advocating non-Markovian stochastic *modification* of quantum theory [11,12,20] refuse the measurement interpretation of the stochastic field.) The concept of relativistically invariant continuous measurement [5] can be reconsidered for the intermediate states $\psi_t[x]$ as well. Our work might lead to efficient numeric simulation algorithms or, conversely, might make us understand why they do not exist.

Appendix.—Let x_τ be a random time-dependent real variable and consider the normalized Gaussian distribution functional of $\{x_\tau; \tau \in (-\infty, \infty)\}$:

$$G[x] = \mathcal{N} \exp\left(-2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\sigma x_\tau \alpha(\tau - \sigma) x_\sigma\right), \tag{A1}$$

$\alpha(\tau - \sigma)$ is a real positive definite kernel. We define its inverse by $\int_{-\infty}^{\infty} \alpha^{-1}(\tau - s)\alpha(s - \sigma) ds = \delta(\tau - \sigma)$. Introduce the normalized functional Fourier transform of $G[x]$, too:

$$\tilde{G}[z] = \tilde{\mathcal{N}} \exp\left(-\frac{1}{2} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\sigma z_\tau \alpha^{-1}(\tau - \sigma) z_\sigma\right). \tag{A2}$$

We need certain marginal distributions as well, e.g.,

$$\tilde{G}_{[0,t]}[z] = \int \tilde{G}[z] \prod_{\tau \notin [0,t]} dz_\tau, \tag{A3}$$

and similarly for $G_{[0,t]}[x]$. These marginal distributions are also Gaussian, e.g.,

$$\tilde{G}_{[0,t]}[z] = \tilde{\mathcal{N}}_{[0,t]} \exp\left(-\frac{1}{2} \int_0^t d\tau \int_0^t d\sigma z_\tau \alpha_{[0,t]}^{-1}(\tau, \sigma) z_\sigma\right), \tag{A4}$$

where the restricted new kernel $\alpha_{[0,t]}^{-1}(\tau, \sigma)$ is defined by $\int_0^t \alpha_{[0,t]}^{-1}(\tau, s)\alpha(s - \sigma) ds = \delta(\tau - \sigma)$ for all $\tau, \sigma \in [0, t]$.

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