Non-Markovian open quantum systems: Input-output fields, memory, and monitoring

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Principles of monitoring non-Markovian open quantum systems are analyzed. We use a field representation of the environment [C. W. Gardiner and M. J. Collett, Phys. Rev. A **31**, 3761 (1985)] for the separation of its memory and detector part, respectively. We claim that the system-plus-memory compound becomes Markovian; the detector part is tractable by standard Markovian monitoring. Because of non-Markovianity, only the mixed state of the system can be predicted; the pure state of the system can be retrodicted. We present the corresponding non-Markovian stochastic Schrödinger equation.

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In their seminal paper [1], Gardiner and Collett used quantum white noise and the related Markovian quantum field to represent the dynamics of a quantum oscillator bath in the Markovian (memoryless) limit. This allowed the construction of exact stochastic differential equations to describe the influence of bath B on the embedded (i.e., open) quantum system S, the reaction of S on B, and the time-continuous monitoring of S. The theory became standard in quantum optics [2] and in many fields where a quantum system is open to natural or designed environmental influence [3]. If the memory of B cannot be ignored for S, then Markovian tools become jeopardized. In the non-Markovian (NM) case, S is coherently interacting with a finite part of B over a finite time. From different theoretical efforts [4–10], we distill a central question: How can we divide environment B into the *memory* M and *detector* D? Part M is continuously entangled with S but the compound S + M becomes a Markovian open system, as we shall argue. Part D contains information on S and can be continuously disentangled, i.e., monitored, without changing the dynamics of S.

As a matter of fact, the Markovian field representation [1] of B is capable of accounting for memory effects and leads to a natural separation between M and D. The local Markov field interacts with S in a finite range: this part makes the memory M. The output field carries away information on S, and it makes the detector D. Most features of the Markovian theory [1] of monitoring apply invariably to the composite system S + M.

Earlier, Jack, Collett, and Walls realized the role of a finite memory time in simulation [4] and in monitoring [5]. Those authors calculated, for the first time, the retrodiction of the mixed quantum state. Here we exploit the causality features of the standard Markovian bath [1–3] and, among other results, calculate the retrodicted pure state and the current mixed state.

We start with a brief summary of the standard Markovian field theory [1]. Then we identify M and D and outline how S + M becomes a Markovian open system. Its Markovian master equation is derived, followed by the derivation and discussion of the stochastic Schrödinger equation (SSE) of monitoring S.

Markovian field, non-Markovian coupling. The composite S + B dynamics is based on the total Hamiltonian,

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{SB}, \tag{1}$$

where $\hat{H}_S$ is the Hamiltonian of S, the bath Hamiltonian is $\hat{H}_B = \int \omega \hat{b}^\dagger_\omega \hat{b}_\omega d\omega$, and $\hat{H}_{SB} = i\hat{s}\int \kappa_\omega \hat{b}^\dagger_\omega d\omega + \text{H.c.}$ is their interaction, where $\hat{s}$ is an S operator that couples to the B modes. Here $\hat{b}_\omega$ are boson annihilation operators for the ω-frequency modes of B, satisfying $[\hat{b}_\omega,\hat{b}^\dagger_{\omega'}] = \delta(\omega-\omega')$. B can be called Markovian because of the flat spectrum. Memory effects are fully encoded in the coupling κ_ω. If the coupling is frequency independent, $\kappa_\omega = \text{const}$, then S is a Markovian open system; otherwise, it has a memory. We are interested in the latter case, i.e., in NM open systems. We assume that S and B are initially uncorrelated. Let, for simplicity, the initial B state be the vacuum $|0\rangle$ defined by $\hat{b}_\omega|0\rangle = 0$ for all ω.

We switch for an abstract field representation [1–3]. The bath field $\hat{b}(z)$ is defined by

$$\hat{b}(z) = \frac{1}{\sqrt{2\pi}}\int \hat{b}_\omega e^{-i\omega z} d\omega, \tag{2}$$

where z is a real one-dimensional spatial coordinate. For convenience, we set the velocity of propagation to 1. The canonical commutation relationship is local:

$$[\hat{b}(z),\hat{b}^\dagger(z')] = \delta(z-z'), \tag{3}$$

hence the field can be measured *independently* at all locations. In particular, it can be measured in the coherent-state overcomplete basis parametrized by the complex field $\xi(z)$. The (non-normalized) Bargman coherent states

$$|\xi\rangle = \exp\left(\int \xi(z)\hat{b}^\dagger(z)dz\right)|0\rangle \tag{4}$$

form an overcomplete basis: $\mathbf{M}|\xi\rangle\langle\xi^*| = \hat{1}$. Here $\mathbf{M}$ stands for the integral (mean) over ξ, with the normalized measure according to the standard complex white-noise statistics, specified by

$$\mathbf{M}\xi(z) = 0, \quad \mathbf{M}\xi(z)\xi(z') = 0, \quad \mathbf{M}\xi(z)\xi^*(z') = \delta(z-z'). \tag{5}$$

If we perform the measurement, the state of B collapses on $|\xi\rangle$ randomly, and the complex field $\xi(z)$ becomes the random readout. But its statistics depends on the premeasurement state. In the vacuum state $|0\rangle$, the readouts $\xi(z)$ follow the statistics, (5). This statistics gets modified by the B-S interaction. Typically, the mean becomes nonvanishing; cf. (11) or (21).

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Both the bath and the interaction Hamiltonians can be written in terms of the fields:

$$\hat{H}_B = \frac{i}{2}\int \hat{b}^\dagger(z)\partial_z\hat{b}(z)dz + \text{H.c.}, \tag{6}$$

$$\hat{H}_{SB} = i\hat{s}\int \hat{b}^\dagger(z)\kappa(z)dz + \text{H.c.}, \tag{7}$$

where $\kappa(z)$ is the Fourier transform of κ_ω.

The underlying picture [1–3] is that all B modes are spatial excitations along a single direction z. The coupling $\kappa(z)$ is supposed to vanish outside the *interaction range*, say, $z \in [0,T]$, where T is the memory time. Memory effects are fully confined here. (If $\kappa(z)$ decays only asymptotically to 0, a finite memory time can still be a robust approximation [4,5].) If κ were a δ function, $\kappa(z) \propto \delta(z)$, the interaction range would reduce to a single point $z=0$, memory effects would be absent, and S would be a Markovian open system.

Heisenberg picture. The solution of the Heisenberg field equation reads [1–3]

$$\hat{b}(z,t) = \hat{b}(z+t) + \int_0^t \hat{s}(t-\tau)\kappa(z+\tau)d\tau. \tag{8}$$

The first term $\hat{b}(z+t)$ on the right-hand side (r.h.s.) corresponds to free dispersionless propagation along the line, from right to left (cf. Fig. 1). The free field plays the role of the "conveyor belt" that carries information or perturbations one way: from right to left; never the opposite! As usual, the free field will later be identified as field (2) in the interaction picture:

$$\hat{b}_t(z) = \hat{b}(z+t). \tag{9}$$

The second term on the r.h.s. of (8) represents the interaction with S, localized inside the interaction range $z \in [0,T]$. In the *input range* $z \geqslant T$ the vacuum field is freely propagating. In the *output range* $z \leqslant 0$ the field is freely propagating and carrying *away* the perturbations emerged in the interaction range. For $t > 0$, the input field does not depend on whatever happens at $z < T$, and the dynamics of S remains undisturbed whatever happens at $z \leqslant 0$ to the output field. Most importantly, we can continuously observe the output field without altering the dynamics of S. Accordingly, the memory M will consist of the local field inside the interaction range and the detector D will consist of the output field. We emphasize that the coupling of M to the rest of B is Markovian, hence S + M becomes a Markovian open system (Fig. 2). The full armory of Markovian continuous measurement theories [1–3], including the Ito formalism, could be deployed—with some peculiarities, however.

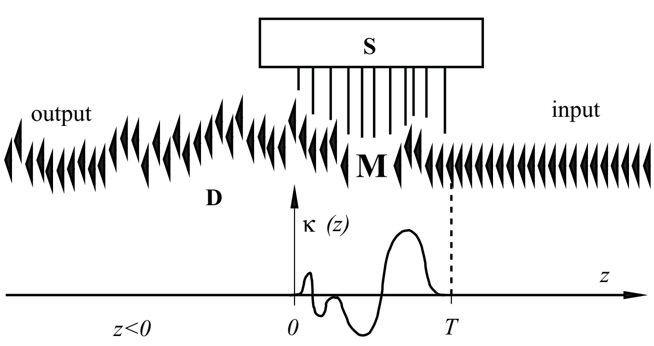

FIG. 1. The bath field $\hat{b}(z,t)$, when free, is propagating from right to left without dispersion at velocity 1. The unperturbed input field from range $z \geqslant T$ propagates through the interaction range $z \in [0,T]$ of nonzero coupling $\kappa(z)$; gets modified by system S, and gets entangled by system S; then leaves to freely propagate away to left infinity as the output field. The interaction range makes the memory M and the output range $z \leqslant 0$ makes the detector D, which can be continuously read out (monitored).

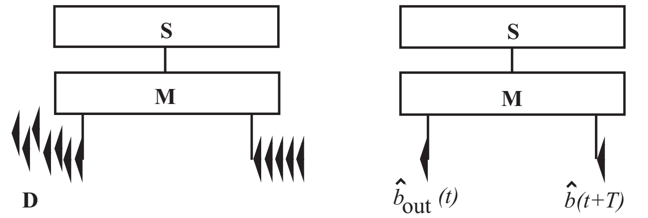

FIG. 2. Memory subsystem M is formed from the local field oscillators of the interaction range. S and M constitute a Markovian open system. It is pumped by the free Markovian input field and it creates the Markovian output field D that can be monitored (left). S + M are in fact pumped by the standard quantum white noise $\hat{b}(t+T)$ and monitored through the modified quantum white noise $\hat{b}_{\text{out}}(t)$ just like Markovian open quantum systems, apart from the delay T of readout with regard to the pump (right).

As we said, the D part of the field is the output field $\hat{b}(z \geqslant 0,t)$. The earliest location of monitoring is $z=0$ and it is common to introduce the notation $\hat{b}_{\text{out}}(t) = \hat{b}(0,t)$ and common to call it the output field:

$$\hat{b}_{\text{out}}(t) = \hat{b}(t) + \int_0^t \hat{s}(t-\tau)\kappa(\tau)d\tau. \tag{10}$$

This is the famous input-output relationship, which works for the NM case as well. The equation expresses the variable $\hat{b}_{\text{out}}(t)$, which one can continuously monitor without affecting the dynamics of S. Since $\kappa(\tau)$ vanishes for $\tau < 0$, the measured signal reflects a delayed and coarse-grained average of the S-variable $\hat{s}$.

In particular, if we read out $\hat{b}_{\text{out}}(t)$ in ideal heterodyne measurement—which corresponds to the measurement in the coherent state basis, (4)—the resulting signal $b_{\text{out}}(t)$ contains the standard complex white noise (5),

$$b_{\text{out}}(t) = \xi(t) + \int_0^t \langle\hat{s}(t-\tau)\rangle\kappa(\tau)d\tau, \tag{11}$$

where $\langle\hat{s}(t)\rangle$ is the quantum expectation value of the Heisenberg operator.

Markovian master equation. We construct the formal Markovian reduced dynamics of S + M in the Schrödinger picture. The Hamiltonian of M and the interaction are just $\hat{H}_B$ and $\hat{H}_{SB}$, respectively, restricted for the interaction range:

$$\hat{H}_M = \frac{i}{2}\int_0^T \hat{b}^\dagger(z)\partial_z\hat{b}(z)dz + \text{H.c.}, \tag{12}$$

$$\hat{H}_{SM} = i\hat{s}\int_0^T \hat{b}^\dagger(z)\kappa(z)dz + \text{H.c.} \tag{13}$$

We are not ready yet. The outer input field $\hat{b}(z > T)$, which we cut off, will be replaced by the time-dependent vacuum white noise $\hat{b}(T+t)$, which is *external* with regard to (w.r.t.) M since we take $t>0$. This noise couples to $\hat{b}(T)$ of the upper edge $z=T$ of M and pumps M via the following Hamiltonian:

$$\hat{H}_{Mt} = i\hat{b}^\dagger(T)\hat{b}(T+t) + \text{H.c.} \tag{14}$$

(This choice can be confirmed in the Heisenberg picture: the field equation $d\hat{b}(z)/dt = i[\hat{H}_M + \hat{H}_{Mt}, \hat{b}(z)]$ yields the correct solution, (9), for $z\in[0,T]$.) As for the output field, we trace out the modes for $z<0$, while we must retain $\hat{b}(z=0)=\hat{b}_{\text{out}}$ if monitoring is included. The total Hamiltonian is $\hat{H}_S+\hat{H}_M+\hat{H}_{Mt}+\hat{H}_{SM}$. We can directly write down the corresponding master equation for the density matrix of S + M:

$$\frac{\hat{\rho}_{SM}}{dt} = -i[\hat{H}_S+\hat{H}_M+\hat{H}_{SM},\hat{\rho}_{SM}] + \hat{b}(T)\hat{\rho}_{SM}\hat{b}^\dagger(T) - \frac{1}{2}[\hat{b}^\dagger(T)\hat{b}(T)\hat{\rho}_{SM} + \text{H.c.}]. \tag{15}$$

The non-Hamiltonian term on the r.h.s. is the typical second-order contribution of the white noise $\hat{H}_{Mt}$.

We have thus transformed the original NM open system S into a standard Markovian open system $S+M$ which is pumped by the vacuum white noise $\hat{b}(T+t)$ and can be monitored through $\hat{b}_{\text{out}}$ (cf. Fig. 2). In principle, the Markovian master equation, (15), would be a possible starting point to include monitoring. Unfortunately, the obtained equation is formal; its application would require further specifications on $\hat{H}_M$ regarding the boundary conditions. Rather, we choose an alternative tool.

Stochastic Schrödinger equation. We are interested in the dynamics of the monitored quantum state. NM SSEs [11–13] are traditionally used to describe open system dynamics, whereas their role in monitoring either was ignored [14–19] or was stronly suggested for investigation [4], then it led to difficulties [6–9]. The difficulties, related to the causal relationship between S and D, become transparent in our new treatment.

We work in the interaction picture: according to (9), we replace $\hat{b}(z)$ with $\hat{b}_t(z)=\hat{b}(z+t)$ and we replace $\hat{s}$ with $\hat{s}_t$. Interaction (13) becomes the functional of the standard vacuum white noise $\hat{b}(t)$:

$$\hat{H}_t = i\hat{s}_t\int_0^T \hat{b}^\dagger(t+\tau)\kappa(\tau)d\tau + \text{H.c.} \tag{16}$$

In the interaction picture the separate pump Hamiltonian, (14), is not needed. To construct the Schrödinger dynamics of S + M, let $|\Psi_S(0)\rangle$ stand for the initial state of S and $|0\rangle$ for the initial vacuum state of M. We choose an uncorrelated composite initial state:

$$|\Psi_{SM}(0)\rangle = |\Psi_S(0)\rangle|0\rangle. \tag{17}$$

Using (16), we get the following Schrödinger equation:

$$\frac{d|\Psi_{SM}(t)\rangle}{dt} = \left(\hat{s}_t\int_0^T \hat{b}^\dagger(t+\tau)\kappa(\tau)d\tau - \text{H.c.}\right)|\Psi_{SM}(t)\rangle. \tag{18}$$

Observe that the r.h.s. depends on the field $\hat{b}(t+\tau)$ for $\tau\in[0,T]$, i.e., for times later than t itself.

As in the case of Markovian open systems, we have to match the unitary evolution, (18), with the continuous readout of $\hat{b}(t)$. To this end, we project the M part of the composite state $|\Psi_{SM}(t)\rangle$ on the coherent-state basis, (4): $|\Psi_S[\xi^*]\rangle = \langle\xi^*|\Psi_{SM}(t)\rangle$. The Schrödinger equation, (18), reads

$$\frac{d|\Psi_S[\xi^*;t]\rangle}{dt} = \hat{s}_t\int_0^T d\tau\kappa(\tau)\xi^*(t+\tau)|\Psi_S[\xi^*;t]\rangle - \hat{s}_t^\dagger\int_0^T d\tau\kappa^*(\tau)\frac{\delta|\Psi_S[\xi^*;t]\rangle}{\delta\xi^*(t+\tau)}. \tag{19}$$

This equation is just the Schrödinger equation, (18), in a different representation. But it is more than that if we consider the monitoring and readout of $\hat{b}(t)$. Then $|\Psi_S[\xi^*]\rangle$ is the (non-normalized) conditional state vector of S, depending on the measured signal $b_{\text{out}}=\xi$. Since the signal is stochastic, we call (19) the NM SSE.

We have come to a landmark. The r.h.s. would contain the measured signal $\xi(t+\tau)$ at later times w.r.t. t; these data are not yet available at time t. We can still exploit the SSE in two ways. Either we propagate the *conditional mixed state* or we propagate the *retrodicted pure state*. In both cases, we prepare the initial state, (17), of S + M at time $t=0$, let it go, and start to read out the signal $b_{\text{out}}(t)=\xi(t)$. The field in M becomes entangled with S so we can never monitor the pure state $|\Psi_S\rangle$ of S. Nonetheless, at each time $t>0$ we propagate (calculate) the solution of (19) by using the latest readouts $\xi(t)=b_{\text{out}}(t)$ and by setting auxiliary values for $\xi(t+\tau)$ with $\tau\in(0,T)$. The latter data have not yet been measured; we acknowledge our ignorance by tracing out the corresponding field degrees of freedom. Accordingly, we derive the following conditional mixed state from the pure-state solution:

$$\hat{\rho}_S[b_{\text{out}},b^*_{\text{out}};t] = \mathbf{M}|\Psi_S[\xi^*;t]\rangle\langle\Psi_S[\xi;t]|_{\xi(\sigma<t)\equiv b_{\text{out}}(\sigma<t)}. \tag{20}$$

This mixed state (with a normalizing factor) is the true conditional state of S under monitoring. If we stick to the idea of a conditional pure state, we exploit the measured signal $b_{\text{out}}(t)$ differently. We use the SSE, (19), at time t to retrodict the state propagation at time $t-T$. Until time $t=T$, measured data are not sufficient to retrodict any pure state. From time $t=T$ on, we start to propagate the initial state $|\Psi_S(0)\rangle$, using the signal $b_{\text{out}}=\xi$ measured until time t. At each time $t>T$, we have $|\Psi_S[b^*_{\text{out}};t-T]\rangle$ as the solution of the SSE. And this (with a normalizing factor) is our retrodicted conditional pure state for S. The pure state $|\Psi_S[b^*_{\text{out}};t-T]\rangle$ looks like a mere mathematical construction, though it will appear—as if it were the true state—in expression (21) of the measured output signal.

So far we have not determined the statistics of the signal b_{out}. The candidate expression (11) does not resolve the selective evolution of S under monitoring. This selection is only given by the SSE, (19), together with its interpretation, (20). As we said before, the signal b_{out} would be the standard complex white noise, (5), of zero mean had we switched off the interaction. With the interaction on, the typical change is that the mean of b_{out} will be nonvanishing. Lessons from the

Markovian special case and the nonselective NM form, (11), suggest the following expression:

$$b_{\text{out}}(t) = \xi(t) + \int_0^t \langle \hat{s}_{t-\tau} \rangle_{t-\tau} \kappa(\tau) d\tau, \tag{21}$$

where $\langle \hat{s}_{t-\tau} \rangle_{t-\tau}$ is the quantum expectation value of $\hat{s}_{t-\tau}$ in the conditional mixed state $\hat{\rho}_S[b_{\text{out}}, b^*_{\text{out}}; t-\tau]$ or, alternatively, in the conditional pure state $|\Psi_S[b^*_{\text{out}}; t-\tau]\rangle$. We show later that the second choice is the right one.

Structured bath. NM open systems are often derived from Markovian coupling $\kappa_\omega = 1$ to an NM bath of nonflat spectral density $\alpha_\omega \geqslant 0$. A prototype of the NM SSE was obtained in 1997 [11]:

$$\frac{d|\Psi_S[a^*;t]\rangle}{dt} = \hat{s}_t a^*_t |\Psi_S[a^*;t]\rangle - \hat{s}^\dagger_t \int_0^t d\sigma \alpha(t-\sigma) \frac{\delta |\Psi_S[a^*;t]\rangle}{\delta a^*_\sigma}. \tag{22}$$

Here a^*_t must be a (Gaussian) complex colored noise of zero mean and of correlation

$$\mathbf{M} a_t a^*_\sigma = \alpha(t-\sigma), \tag{23}$$

where $\alpha(t)$ is the bath correlation function, i.e., the Fourier transform of α_ω. The interpretation of this equation drew permanent attention. Gambetta and Wiseman showed [6,8] that no monitoring process exists for $|\Psi_S[a^*;t]\rangle$ itself. If, however, the support of $\alpha(t)$ is finite (there is a finite memory time), then SSEs like (22) can predict the mixed conditional state at t and retrodict the pure conditional state at t minus the memory time [7,9]. Now we are in a position to unfold the causality structure of the 1997 SSE [11]: we rewrite it in the form of SSE (19).

The point is that the said NM bath with Markovian coupling can equivalently be substituted by the Markovian B with NM coupling satisfying $|\kappa_\omega|^2 = \alpha_\omega$. Precisely, if we solve

$$\alpha(t) = \int \kappa(t+\tau)\kappa^*(\tau) d\tau \tag{24}$$

for $\kappa(t)$ with condition $\kappa(\tau) = 0$ for $\tau < 0$ [20], then we can express a_t through the standard complex white noise, (5):

$$a_t = \int \xi(t+\tau)\kappa^*(\tau) d\tau. \tag{25}$$

By inserting this into (22), the resulting equation coincides with NM SSE (19). Therefore the discussion and resolution of the causality issue of monitoring (cf. the preceding paragraph) can be directly adapted to the old form of the SSE [21].

Let us verify the Girsanov transformation $\xi(t) \Rightarrow b_{\text{out}}(t)$ underlying our heuristic expression, (21), of the output signal. We exploit the Girsanov transformation $a_t \Rightarrow \tilde{a}_t$ accomplished by (16) in [12]:

$$\tilde{a}_t = a_t + \int_0^t \alpha(t-\tau) \langle \hat{s}_{t-\tau} \rangle_{t-\tau} d\tau, \tag{26}$$

where a_t and $\tilde{a}_t$ are related to $\xi(t)$ and $b_{\text{out}}(t)$, respectively, by convolution (25). Let us arrange all terms on one side, apply (25), and insert (24), yielding

$$\int \left(b_{\text{out}}(t+\sigma) - \xi(t+\sigma) - \int_0^t \langle \hat{s}_{t-\tau} \rangle_{t-\tau} \kappa(\tau+\sigma) d\tau \right) \times \kappa^*(\sigma) d\sigma = 0. \tag{27}$$

The removal of the outer convolution, legitimated at least when κ_ω is nowhere 0, yields our result (21). From [12] we know that $\langle \hat{s}_{t-\tau} \rangle_{t-\tau}$ must be taken in the retrodicted pure state $|\Psi_S[b^*_{\text{out}}; t-\tau]\rangle$. Since pure-state retrodiction needs a minimum time delay T, the theoretical prediction, (21), of the output signal $b_{\text{out}}(t)$ can only be calculated at time $t+T$; i.e., at current time t the latest statistical retrodiction concerns $b_{\text{out}}(t-T)$.

Summary. We have applied the well-known Markovian field representation of the environmental bath at NM coupling to the embedded open system. The field in the vicinity of the system plays the role of memory responsible for the NM features; far from this vicinity it remains Markovian and subject to standard Markovian theory of monitoring. We unfolded the abstract bath into the memory and the detector part (cf. Refs. [22–24] for related approaches). A formal master equation has been derived just to confirm the Markovianity of the S + M compound. We have derived an SSE of the monitored system and pointed out its role in predicting the conditional mixed state and in retrodicting the conditional pure state. In future, Ito differential and integral calculus will have to be deployed to improve our tentative derivations.

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