PHYSICAL REVIEW A **94**, 010103(R) (2016)

Structural features of sequential weak measurements

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(Received 24 November 2015; published 7 July 2016)

We discuss the abstract structure of sequential weak measurement (WM) of general observables. In all orders, the sequential WM correlations without postselection yield the corresponding correlations of the Wigner function, offering direct quantum tomography through the moments of the canonical variables. Correlations in spin-$\frac{1}{2}$ sequential weak measurements coincide with those in strong measurements, they are constrained kinematically, and they are equivalent with single measurements. In sequential WMs with postselection, an anomaly occurs, different from the weak value anomaly of single WMs. In particular, the spread of polarization $\hat{\sigma}$ as measured in double WMs of $\hat{\sigma}$ will diverge for certain orthogonal pre- and postselected states.

DOI: 10.1103/PhysRevA.94.010103

From textbooks on quantum mechanics we learn that the ideal measurement of observable $\hat{A}$ collapses the premeasurement state $\hat{\rho}$ into an eigenstate of $\hat{A}$ hence erasing all memory of $\hat{\rho}$. If the measurement is nonideal (i.e., unsharp and imprecise), the collapse still happens, although it may keep some well-defined features of $\hat{\rho}$. On one hand, the larger the unsharpness, the more faithfully the premeasurement state will be preserved. On the other hand, the imprecision of the measurement can be compensated by measuring on a larger ensemble of identically prepared premeasurement states. The concept of weak measurement (WM) corresponds to the asymptotic limit of zero precision and infinite statistics [1] when the premeasurement state $\hat{\rho}$ would invariably survive the measurement. WM was used by Aharonov *et al.* [2] as a noninvasive quantum measurement between preselection (preparation) and postselection of the pre- and postmeasurement states, respectively. Noninvasiveness of WM is a remarkable feature both with and without postselection, and this noninvasiveness can be maintained for a succession of WMs on a single quantum system. General features of such sequential WMs form the subject of the present Rapid Communication.

WMs without postselection. We outline WM of a single observable $\hat{A}$ at the abstract level of generalized (unsharp and imprecise) measurements [3]. Consider the premeasurement state $\hat{\rho}$ and the unsharp measurement of $\hat{A}$ with precision a. Let $G_\sigma(A)$ stand for a Gaussian function of standard width a. The unnormalized postmeasurement state conditioned on the outcome A takes this form

$$\hat{\rho}_a(A) = \sqrt{G_a(A-\hat{A})}\hat{\rho}\sqrt{G_a(A-\hat{A})}, \tag{1}$$

where the outcome probability satisfies

$$p_a(A) = \mathbf{tr}\,\hat{\rho}_a(A) = \langle G_a(A-\hat{A})\rangle_{\hat{\rho}}. \tag{2}$$

If we calculate the stochastic mean $\mathbf{M}A$ of A, we get

$$\mathbf{M}A = \int p_a(A)A\,dA = \langle\hat{A}\rangle_{\hat{\rho}}. \tag{3}$$

We are interested in the WM limit of infinite imprecision $a \to \infty$, i.e., when a is so large that the difference between pre- and postmeasurement states is negligible. In practice it means $a \gg \Delta A$ where $(\Delta A)^2 = \langle\hat{A}^2\rangle_{\hat{\rho}} - (\langle\hat{A}\rangle_{\hat{\rho}})^2$. Although the relationship $\mathbf{M}A$ is independent of a, the probability distribution $p_a(A)$ diverges so that $p_\infty(A)$ does not exist. Note with Ref. [1] the WM limit of the unsharp measurement (1–2) had been used earlier for the theory of time-continuous measurement [4].

Before constructing sequential WMs, let us write the postmeasurement state (1) into the equivalent form

$$\hat{\rho}_a(A) = \exp\left(\frac{-\hat{A}_\Delta^2}{8a^2}\right)G_a(A-\hat{A}_c)\hat{\rho}, \tag{4}$$

where $\hat{A}_\Delta, \hat{A}_c$ are commuting superoperators [5] defined by $\hat{A}_\Delta\hat{O} = [\hat{A},\hat{O}]$ and $\hat{A}_c\hat{O} = \frac{1}{2}\{\hat{A},\hat{O}\}$. As an example of sequential WMs, we consider the sequence of three independent WMs of $\hat{A},\hat{B},\hat{C}$, in this order. In the WM limit, we can apply Eq. (4) without the exponential factor to construct the un-normalized postmeasurement state,

$$\hat{\rho}_a(A,B,C) = G_a(C-\hat{C}_c)G_a(B-\hat{B}_c)G_a(A-\hat{A}_c)\hat{\rho}. \tag{5}$$

The joint probability distribution of the three outcomes is determined by the trace of the postmeasurement state,

$$\begin{aligned} p_a(A,B,C) &= \mathbf{tr}\,\hat{\rho}_a(A,B,C) \\ &= \mathbf{tr}\,\{G_a(C-\hat{C}_c)G_a(B-\hat{B}_c)G_a(A-\hat{A}_c)\hat{\rho}\}, \end{aligned} \tag{6}$$

which, as we said already, diverges in the WM limit, and $p_\infty(A,B,C)$ does not exist. Nonetheless, the stochastic average of the product ABC is independent of a in the WM limit. Using Eq. (6), we obtain

$$\begin{aligned} \mathbf{M}ABC &= \int p_a(A,B,C)ABC\,dA\,dB\,dC \\ &= \frac{1}{8}\langle\{\hat{A},\{\hat{B},\hat{C}\}\}\rangle_{\hat{\rho}}. \end{aligned} \tag{7}$$

This important result was obtained by Bednorz and Belzig [6] assuming a quasidistribution, which this time we justify as follows.

Since the right-hand side (rhs) of the above expression is independent of a, therefore we can calculate it for $a = 0$. This means, we get the following quasidistribution from the true

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$p_a(A,B,C)$:

$$p_0(A,B,C) = \mathbf{tr}\{\delta(C-\hat{C}_c)\delta(B-\hat{B}_c)\delta(A-\hat{A}_c)\hat{\rho}\}. \tag{8}$$

This quasidistribution can have negative domains. (For the true distribution, $p_a(A,B,C) \geqslant 0$ holds in the WM limit.) The merit of this quasidistribution is that it does not contain the diverging parameter a and correctly yields the mean for the product ABC exactly like $p_a(A,B,C)$ did:

$$\begin{aligned}\mathbf{M}ABC &= \frac{1}{8}\langle\{\hat{A},\{\hat{B},\hat{C}\}\}\rangle_{\hat{\rho}}\\ &= \int p_0(A,B,C)ABC\,dA\,dB\,dC.\end{aligned} \tag{9}$$

The same is true for the means of A, B, C, AB, AC, and BC, respectively. But all other means diverge in reality, i.e., with $p_a(A,B,C)$ in the WM limit, whereas $p_0(ABC)$ suggests incorrect finite values for them.

The above results can trivially be extended for an arbitrary long sequence of WMs. Let us consider a sequence of observables $\hat{A}_1,\hat{A}_2,\ldots,\hat{A}_n$ which are weakly measured in the given order on the premeasurement state $\hat{\rho}$. If $A_1,A_2,\ldots,A_n$ denote the corresponding measurement outcomes, then

$$\mathbf{M}A_1A_2\cdots A_n = \frac{1}{2^n}\langle\{\hat{A}_1,\{\hat{A}_2,\{\cdots\{\hat{A}_{n-1},\hat{A}_n\}\cdots\}\}\}\rangle_{\hat{\rho}}. \tag{10}$$

The stochastic mean of the product of sequential WM outcomes coincides with the quantum expectation value of the *stepwise-symmetrized* (also called time-symmetric [7]) product of quantum observables. This is the central result for sequential WMs without postselection. If we regard a subset of the n outcomes and discard the rest of them, then the similar identity holds, e.g., $\mathbf{M}A_2A_7A_8 = \frac{1}{8}\langle\{\hat{A}_2,\{\hat{A}_7,\hat{A}_8\}\}\rangle_{\hat{\rho}}$. In general,

$$\mathbf{M}A_{i_1}\cdots A_{i_2}\cdots A_{i_r} = \frac{1}{2^r}\langle\{\hat{A}_{i_1},\{\hat{A}_{i_2},\ldots,\hat{A}_{i_r}\}\}\rangle_{\hat{\rho}} \tag{11}$$

holds for $(i_1,i_2,\ldots,i_r) \subseteq (1,2,\ldots,n)$, i.e., for any ordered subsets of indices from 1 to n, as it follows easily from the derivation of Eq. (10). Also we can derive all the above stochastic means from the quasidistribution,

$$\begin{aligned}&p_0(A_1,A_2,\ldots,A_n)\\ &\quad= \mathbf{tr}\,\{\delta(A_n-\hat{A}_{1,c})\delta(A_2-\hat{A}_{2,c})\cdots\delta(A_n-\hat{A}_{n,c})\hat{\rho}\}.\end{aligned} \tag{12}$$

Generalization of the relationship (9) holds

$$\begin{aligned}&\mathbf{M}A_1A_2\cdots A_n\\ &\quad= \frac{1}{2^n}\langle\{\hat{A}_1,\{\cdots\{\hat{A}_{n-1},\hat{A}_n\}\cdots\}\}\rangle_{\hat{\rho}}\\ &\quad= \int p_0(A_1,A_2,\ldots,A_n)A_1A_2\cdots A_n dA_1 dA_2\cdots dA_n.\end{aligned} \tag{13}$$

The last two WMs in a stepwise-symmetrized sequence are always interchangeable, but the rest of them are not: Order of WMs matters in general. There is, however, a remarkable class when all WMs are interchangeable. Let us discuss the example of the sequence $\hat{A}$, $\hat{B}$, and $\hat{C}$. To find a simplest sufficient condition of complete interchangeability, we require that the superoperators $\hat{A}_c$, $\hat{B}_c$, and $\hat{C}_c$ in Eq. (7) all commute. Take, e.g., the identity $[\hat{A}_c,\hat{B}_c]\hat{O} = \frac{1}{4}[[\hat{A},\hat{B}],\hat{O}]$, which says that $\hat{A}_c$ and $\hat{B}_c$ commute if $[\hat{A},\hat{B}]$ is a c number. Therefore the interchangeability of the three WMs is ensured if all three commutators $[\hat{A},\hat{B}]$, $[\hat{A},\hat{C}]$, and $[\hat{B},\hat{C}]$ are c-number. In the general case, the order of WMs within the sequence $\hat{A}_1,\hat{A}_2,\ldots,\hat{A}_n$ becomes irrelevant if

$$[\hat{A}_k,\hat{A}_l] = c\text{-number} \qquad (k,l = 1,2,\ldots,n). \tag{14}$$

This is not necessary, just a sufficient condition of complete interchangeability of the n WMs. Under this condition, the stepwise symmetrization on the rhs of Eq. (10) reduces to symmetrization,

$$\mathbf{M}A_1A_2\cdots A_n = \langle\mathcal{S}\hat{A}_1\hat{A}_2\cdots\hat{A}_n\rangle_{\hat{\rho}}, \tag{15}$$

where $\mathcal{S}$ stands for symmetrization of the operator product behind it.

Canonical observables. The conditions (14) hold typically for the linear combination of canonical variables, e.g., for the choice,

$$\hat{A}_k = u_k\hat{q} + v_k\hat{p} \qquad (k = 1,2,\ldots,n), \tag{16}$$

where $[\hat{q},\hat{p}] = i$. Then symmetrization $\mathcal{S}$ is nothing else than Weyl ordering. Since the Weyl-ordered correlation functions of canonical variables $\hat{q},\hat{p}$ or of their linear combinations, such as on the rhs of Eq. (15) coincide with the corresponding correlation functions (moments) calculated from the Wigner function $W(q,p)$ of $\hat{\rho}$, we conclude that the rhs can be rewritten in terms of Wigner function correlations,

$$\begin{aligned}\mathbf{M}A_1A_2\cdots A_n &= \int W(q,p)A_1A_2\cdots A_n dq\,dp\\ &\equiv \langle A_1A_2\cdots A_n\rangle_W.\end{aligned} \tag{17}$$

This means that, for sequential WMs of canonical observables, the generic quasidistribution (12) is redundant for $n > 2$, its role is taken over by the Wigner quasidistribution. The coincidence $p_0(q,p) = W(q,p)$ in the special case $n = 2$, $\hat{A}_1 = \hat{q}$, $\hat{A}_2 = \hat{p}$ was recognized in Ref. [6].

Suppose, for instance, we perform two WMs of $\hat{q}$ with outcomes q_1,q_2 and two WMs of $\hat{p}$ with outcomes p_1,p_2. Then independent of the orders of the four WMs, a sufficiently large statistics of outcomes allows us to determine all second-order moments of the Wigner function,

$$\begin{aligned}\langle q^2\rangle_W &= \mathbf{M}q_1q_2,\\ \langle p^2\rangle_W &= \mathbf{M}p_1p_2,\\ \langle qp\rangle_W &= \mathbf{M}q_1p_1 = \mathbf{M}q_1p_2 = \mathbf{M}q_2p_1 = \mathbf{M}q_2p_2,\end{aligned} \tag{18}$$

as well as a few higher-order ones $\langle q^2p\rangle_W$, $\langle qp^2\rangle_W$, $\langle q^2p^2\rangle_W$, and, of course, the first-order moments $\langle q\rangle_W,\langle p\rangle_W$ too.

Spin-$\frac{1}{2}$ observables. Sequential measurement of spin-$\frac{1}{2}$ observables is exceptional: Eq. (10) is valid no matter if the measurements are weak, ideal (strong), or even alternating within the sequence between the two extreme strengths. Consider the following choice of observables:

$$\hat{A}_1 = \hat{\sigma}_1, \qquad \hat{A}_2 = \hat{\sigma}_2,\ldots,\hat{A}_n = \hat{\sigma}_n, \tag{19}$$

where $\hat{\sigma}_k$ is the polarization parallel to the unit vector $\vec{e}_k$ for $k=1,2,\dots,n$. Denote the measurement outcomes by $A_1=\sigma_1$, $A_2=\sigma_2$, etc., and invoke Eq. (10) for them,

$$\mathbf{M}\sigma_1\sigma_2\cdots\sigma_n=\frac{1}{2^n}\langle\{\hat{\sigma}_1,\{\hat{\sigma}_2,\{\cdots\{\hat{\sigma}_{n-1},\hat{\sigma}_n\}\cdots\}\}\}\rangle_{\hat{\rho}}. \quad (20)$$

To confirm it for strong measurements as well, we introduce the projectors $\hat{P}_\pm=\frac{1}{2}(1\pm\hat{\sigma})$ diagonalizing the Pauli polarization matrix $\hat{\sigma}$. The standard expression for sequential strong measurements reads

$$\mathbf{M}\sigma_1\sigma_2\cdots\sigma_n=\mathbf{tr}\sum_{\sigma_n=\pm1}\sigma_n\hat{P}^{(n)}_{\sigma_n}\cdots\left[\sum_{\sigma_2=\pm1}\sigma_2\hat{P}^{(2)}_{\sigma_2}\left(\sum_{\sigma_1=\pm1}\sigma_1\hat{P}^{(1)}_{\sigma_1}\hat{\rho}\hat{P}^{(1)}_{\sigma_1}\right)\hat{P}^{(2)}_{\sigma_2}\right]\cdots\hat{P}^{(n)}_{\sigma_n}. \quad (21)$$

Observe the identity $\sum_{\sigma=\pm}\sigma\hat{P}_\sigma\hat{O}\hat{P}_\sigma=\frac{1}{2}\{\hat{\sigma},\hat{O}\}$ valid for auxiliary 2×2 matrices $\hat{O}$, and apply it n times. We obtain Eq. (20). Evaluating its rhs yields

$$\mathbf{M}\sigma_1\sigma_2\cdots\sigma_n=\begin{cases}(\vec{e}_1\vec{e}_2)(\vec{e}_3\vec{e}_4)\cdots(\vec{e}_{n-1}\vec{e}_n), & n\text{ even},\\ \langle\hat{\sigma}_1\rangle_{\hat{\rho}}(\vec{e}_2\vec{e}_3)\cdots(\vec{e}_{n-1}\vec{e}_n), & n\text{ odd}.\end{cases} \quad (22)$$

Outcome correlations of n-sequential WMs on a spin-$\frac{1}{2}$ system coincide exactly with the correlations obtained from strong measurements of the same sequence. Correlations are kinematically constrained by the chosen directions of polarization measurements. For n even, correlations are completely determined by the single mean $\langle\hat{\sigma}_1\rangle_{\hat{\rho}}$ and just independent of the premeasurement state $\hat{\rho}$ if n is even.

WMs with postselection. So far, we have established the general features of outcome statistics in sequential WMs without postselection. Including postselection requires straightforward modifications. For mixed state postselection [1,8], the statistics (6) of the ABC-sequential WM modifies, such as this,

$$\begin{aligned}p_a(A,B,C|\hat{\Pi}) &= \frac{\mathbf{tr}\{\hat{\Pi}\hat{\rho}_a(A,B,C)\}}{\mathbf{tr}\{\hat{\Pi}\hat{\rho}\}}\\ &= \frac{\mathbf{tr}\{\hat{\Pi}G_a(C-\hat{C}_c)G_a(B-\hat{B}_c)G_a(A-\hat{A}_c)\hat{\rho}\}}{\mathbf{tr}\{\hat{\Pi}\hat{\rho}\}}, \end{aligned} \quad (23)$$

where $0\leqslant\hat{\Pi}\leqslant1$. Accordingly, the postselected mean (7), i.e., the mean restricted for the postselected subset $ABC|_{\text{psel}}$ of ABC, becomes

$$\mathbf{M}ABC|_{\text{psel}}=\frac{1}{8}\langle\{\hat{A},\{\hat{B},\{\hat{C},\hat{\Pi}\}\}\}\rangle_{\hat{\rho}}/\langle\hat{\Pi}\rangle_{\hat{\rho}}. \quad (24)$$

The general result must be the following:

$$\mathbf{M}A_1,A_2,\dots,A_n|_{\text{psel}}=\frac{\langle\{\hat{A}_1,\{\hat{A}_2,\dots,\{\hat{A}_n,\hat{\Pi}\}\cdots\}\}\rangle_{\hat{\rho}}}{2^n\langle\hat{\Pi}\rangle_{\hat{\rho}}}. \quad (25)$$

In the basic case, both initial and postselected states are pure states, and we are going to take this option: $\hat{\rho}=|i\rangle\langle i|$, $\hat{\Pi}=|f\rangle\langle f|$. Then, following Mitchison *et al.* [9], we introduce the sequential weak values,

$$(A_1,A_2,\dots,A_n)_w=\frac{\langle f|\hat{A}_n\hat{A}_{n-1}\cdots\hat{A}_1|i\rangle}{\langle f|i\rangle}, \quad (26)$$

and rewrite Eq. (25) in time-symmetric form [9]

$$\mathbf{M}A_1,A_2,\dots,A_n|_{\text{psel}}=\frac{1}{2^n}\sum(A_{i_1},A_{i_2},\dots,A_{i_r})_w(A_{j_1},A_{j_2},\dots,A_{j_{n-r}})^\star_w, \quad (27)$$

where the summation is understood for all partitions $(i_1,i_2,\dots,i_r)\cup(j_1,j_2\cdots j_{n-r})=(1,2,\dots,n)$ where i's and j's remain ordered. Degenerate partitions $r=0,n$ too must be counted. Certain options of reduction, shown above for sequential WMs of canonical or spin-$\frac{1}{2}$ observables, may still survive postselection, and here we are not going to discuss them. We show a particular anomaly, not present in single postselected WM but present in sequential WMs, even for the simplest ones.

Reselection. Consider the special case $|i\rangle=|f\rangle$ of postselection, and call it *reselection*. In the case of a single WM, reselection is equivalent with no postselection,

$$\mathbf{M}A=\mathbf{M}A|_{\text{rsel}}=\langle\hat{A}\rangle_{\hat{\rho}}. \quad (28)$$

Since WMs are considered noninvasive, we expect that the postmeasurement state does not differ from initial state $|i\rangle$ in the WM limit and the reselection rate tends to 1 hence the discarded outcomes would not alter the statistics. No doubt, this is the case for a single WM. As for sequential WMs, however, a glance at (27) shows that reselection does not yield equivalent results with no postselection (10). Even the simplest sequential WM will illustrate the anomaly. We consider two WMs, moreover, we consider the case when $\hat{A}_1=\hat{A}_2=\hat{A}$, i.e., we weakly measure $\hat{A}$ twice in a row, yielding outcomes A_1 and A_2, respectively. Without postselection, Eq. (10) and with reselection Eq. (27) yield, respectively,

$$\mathbf{M}A_1A_2=\langle i|\hat{A}^2|i\rangle, \quad (29)$$

$$\mathbf{M}A_1A_2|_{\text{rsel}}=\frac{1}{2}\langle i|\hat{A}^2|i\rangle+\frac{1}{2}(\langle i|\hat{A}|i\rangle)^2. \quad (30)$$

Reselection decreases $\mathbf{M}A_1A_2$ by half of the squared quantum uncertainty $(\Delta A)^2$ in state $|i\rangle$,

$$\mathbf{M}A_1A_2-\mathbf{M}A_1A_2|_{\text{rsel}}=\frac{1}{2}(\Delta A)^2. \quad (31)$$

This is an unexpected anomaly. The reason must lie in the contribution of outcomes *discarded* by reselection, i.e., $\mathbf{M}A_1A_2|_{\text{disc}}\times$ (discard rate) $\to\frac{1}{2}(\Delta A)^2$ must be satisfied.

As an example, consider a spin-$\frac{1}{2}$ system in the upward polarized initial state $|i\rangle=|\uparrow\rangle$. Let us begin with a single WM of $\hat{\sigma}\equiv\hat{\sigma}_x$ with outcome σ_1. The contribution of the discarded outcomes reads

$$\mathbf{M}\sigma_1|_{\text{disc}}=\frac{\langle\downarrow|\left[\exp\left(-\frac{1}{8}\hat{\sigma}^2_\Delta/a^2\right)\hat{\sigma}_c|\uparrow\rangle\langle\uparrow|\right]|\downarrow\rangle}{\langle\downarrow|\left[\exp\left(-\frac{1}{8}\hat{\sigma}^2_\Delta/a^2\right)|\uparrow\rangle\langle\uparrow|\right]|\downarrow\rangle}, \quad (32)$$

where we use the exact expression of the post-WM state with the exponential factor as in Eq. (4) otherwise we get 0 for the rate of discarded events. This rate is just the denominator in the above fraction, yielding $\sim\frac{1}{4}a^{-2}$ asymptotically. This rate goes to zero in the WM limit, but $\mathbf{M}\sigma_1|_{\rm disc}$ vanishes anyway since the numerator is zero identically. Now, let us weakly measure $\hat\sigma\equiv\hat\sigma_x$ twice in a sequence, yielding outcomes σ_1,σ_2. Since the quantum spread is $\Delta\sigma_x=1$ in state $|\uparrow\rangle$, we have to prove that in reselection the contribution of the discarded events satisfies $\mathbf{M}\sigma_1\sigma_2|_{\rm disc}\times$ (discard rate) $\to\frac{1}{2}$. Its analytic form can be written as

$$\mathbf{M}\sigma_1\sigma_2|_{\rm disc}=\frac{\langle\downarrow|\big[\exp\big(-\frac{1}{4}\hat\sigma_\Delta^2/a^2\big)\hat\sigma_c^2|\uparrow\rangle\langle\uparrow|\big]|\downarrow\rangle}{\langle\downarrow|\big[\exp\big(-\frac{1}{4}\hat\sigma_\Delta^2/a^2\big)|\uparrow\rangle\langle\uparrow|\big]|\downarrow\rangle}. \qquad (33)$$

The denominator yields rate $\sim\frac{1}{2}a^{-2}$ of discards, and it is vanishing in the WM limit. The exponential factor in the numerator can be neglected in the WM limit, and we get the following result:

$$\begin{aligned}\mathbf{M}\sigma_1\sigma_2|_{\rm disc}&=2a^2\langle\downarrow|(\hat\sigma_c^2|\uparrow\rangle\langle\uparrow|)|\downarrow\rangle\\&=2a^2\frac{1}{4}\langle\downarrow|\{\hat\sigma,\{\hat\sigma,|\uparrow\rangle\langle\uparrow|\}\}|\downarrow\rangle\\&=a^2.\end{aligned} \qquad (34)$$

As we see, the correlation of two subsequent $\hat\sigma_x$ polarization WMs diverges on the discarded events in reselection. This is in itself a different and stronger anomaly than the paradigmatic large but finite mean values obtained in single WMs with postselection [2]. What we wished to confirm here is that the divergent mean a^2 compensates the vanishing rate $\frac{1}{2}a^{-2}$ to yield the finite contribution $\frac{1}{2}$ of the discarded outcomes in reselection.

Summary and discussion. Superoperator formalism has helped us to determine the correlation functions of sequential WMs in terms of the quantum expectation values of the stepwise symmetric product of the corresponding observables. The condition of interchangeability of WMs within the sequence has been found. Canonical variables are interchangeable, and without postselection their WM correlation functions coincide with the corresponding correlation functions of the Wigner function. It follows from our result how all nth-order correlation functions (moments) of the Wigner function can, in principle, be determined directly on the outcome statistics of the sequence of n WMs. This makes sequential WMs a tool of direct quantum state tomography (limited normally by the highest available order n in a given experiment). Sequential WMs may demonstrate quantum paradoxes since the negativity of the Wigner function leads to nonclassical statistics of sequential WMs, such as in Ref. [6], see also Ref. [10]. Earlier suggestions associated outcomes of single postselected WMs with Bohmian velocities [11]. As for the outcomes of sequential WMs, our result suggests Wigner phase space coordinates as the natural interpretation. (This interpretation proves to be universal if the sequential WM of spin-$\frac{1}{2}$ observables is related to the Wigner function in the Grassmann variables introduced in Ref. [12], an issue we leave open here.) Spin-$\frac{1}{2}$ observables behave very differently. Two polarization WMs yield no new information at all compared to single measurements since the correlation is determined by the angle between the two polarizers and independent of the quantum state, just like for two strong (ideal) polarization measurements. This is more than resemblance. We found that a sequence of n weak or, alternatively, strong spin-$\frac{1}{2}$ measurements yield identical n-order correlation functions, respectively.

Finally, we studied the marginal case $|f\rangle=|i\rangle$ of postselection which we called re-selection and found that in sequential WMs it is not equivalent with lack of postselection. This means that in sequential WMs with reselection the discarded statistics matters however close we are to ideal WMs. This unexpected effect roots in a novel weak value anomaly this time referring to the anomalous (divergent) value of the weakly measured (i.e., in sequential WM) autocorrelation on the statistics discarded by reselection. This phenomenon is a robust feature of sequential WMs, and it is not tractable in terms of standard weak values. As an example, we have shown that the correlation of outcomes in double WMs of $\hat\sigma_x$ in the preselected state $|i\rangle=|\uparrow\rangle$ and postselected on $|f\rangle=|\downarrow\rangle$ will diverge whereas any correlation larger than $\|\sigma_x\|^2=1$ is counterintuitive.

Acknowledgments. This work was supported by the Hungarian Scientific Research Fund under Grant No. 103917 and by the EU COST Action No. MP1209.

[1] L. Diósi, in *Encyclopedia of Mathematical Physics*, edited by J.-P. Françoise, G. L. Naber, and S. T. Tsou (Elsevier, Oxford, 2006), Vol. 4, p. 276.
[2] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
[3] K. Kraus, *States, Effects and Operations* (Springer, Berlin, 1983); M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
[4] A. Barchielli, L. Lanz, and G. M. Prosperi, Nuovo Cimento B **72**, 79 (1982); C. M. Caves and G. J. Milburn, Phys. Rev. A **36**, 5543 (1987); L. Diósi, Phys. Lett. A **129**, 419 (1988).
[5] K. Chou, Z. Su, B. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985); L. Diósi, Phys. Rev. A **42**, 5086 (1990); Quant. Semiclass. Opt. **8**, 309 (1996).
[6] A. Bednorz and W. Belzig, Phys. Rev. Lett. **105**, 106803 (2010).
[7] B. Berg, L. I. Plimak, A. Polkovnikov, M. K. Olsen, M. Fleischhauer, and W. P. Schleich, Phys. Rev. A **80**, 033624 (2009).
[8] R. Silva, Y. Guryanova, N. Brunner, N. Linden, A. J. Short, and S. Popescu, Phys. Rev. A **89**, 012121 (2014).
[9] G. Mitchison, R. Jozsa, and S. Popescu, Phys. Rev. A **76**, 062105 (2007).
[10] A. Bednorz and W. Belzig, Phys. Rev. A **83**, 052113 (2011).
[11] H. M. Wiseman, New J. Phys. **9**, 165 (2007); S. Kocsis, B. Braverman, S. Ravets, M. J. Stevens, R. P. Mirin, L. K. Shalm, and A. M. Steinberg, Science **332**, 1170 (2011).
[12] K. E. Cahill and R. J. Glauber, Phys. Rev. A **59**, 1538 (1999).