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Equation of motion for estimation fidelity of monitored oscillating qubits ☆

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ABSTRACT

We study the convergence properties of state estimates of an oscillating qubit being monitored by a sequence of *discrete*, unsharp measurements. Our method derives a differential equation determining the evolution of the estimation fidelity from a single incremental step. If the oscillation frequency Ω is precisely known, the estimation fidelity converges exponentially fast to unity. For imprecise knowledge of Ω we derive the asymptotic estimation fidelity.

1. Introduction

High fidelity quantum state estimation is a key requirement in innumerable quantum control applications of quantum information processing, quantum simulation, quantum metrology, and quantum communication. Quantum state estimation [1–4] based on continuous or sequential unsharp (sometimes called weak) measurement [5–10] has opened new avenues for quantum control that obviate the need for repeated state preparation to execute tomography and allow, for example, real-time quantum closed-loop feedback. These principles have been brought to bear in different experimental platforms including microwave cavities [11] and superconducting qubits [12]. Improved sophistication in the unsharp measurement control toolbox [13] promises significant expansion beyond traditional open loop quantum control applications.

To achieve high fidelity control based on unsharp measurement the experimenter is forced to balance the benefit of allowing coherent dynamics to proceed subject to only weak perturbations, with the price of reduced information gain per measurement. As such, finding optimal estimation and control strategies are of prime importance. To make headway, detailed analytical descriptions of the measurement and estimation process are desirable.

In this paper we study the dynamics of state estimation fidelity during a sequence of discrete, unsharp measurements. Detailed analytical results are natural in the domain of *continuous* unsharp measurement, but we attempt here to place on a firmer footing the understanding of estimation dynamics during *sequential, discrete* measurements, as is natural in many experimental settings like trapped ions or microwave cavities.

We consider an estimation protocol wherein a state estimate is propagated by a Hamiltonian presumed to drive a laboratory quantum system which is also subject to sequential unsharp measurement. The state estimate is sequentially updated based on the outcome of

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each measurement on the actual quantum state with the same propagator as the system. Numerical simulations have demonstrated the convergence of the state estimate when all parameters in the Hamiltonian are precisely known [14,4] and even for the case of process tomography, treating the Hamiltonian parameters as state variables of a hybrid system [15–17].

Here we study the convergence of the state estimate for sequential unsharp measurements analytically, for the case of a two-level system undergoing Rabi oscillations. We start our study in Section 2 with a brief explanation of the state estimation protocol. In Section 3 we investigate the dynamics of the estimation fidelity, when the Hamiltonian is precisely known, or known within a specified error margin. This allows us to place bounds on the parameter space that grants high-fidelity state estimation.

2. State estimation and fidelity

A commonly used distance measure for the "closeness" of two quantum states is the estimation fidelity. It is defined as $F(\rho,\rho_e) = \mathrm{Tr}\left[\sqrt{\rho^{\frac{1}{2}}\rho_e\rho^{\frac{1}{2}}}\right]^2$, and is sometimes referred to as the *squared fidelity*. Here ρ and ρ_e are the density matrices describing the actual quantum state and the estimate of that quantum state, respectively. When both states are pure, i.e. $\rho = |\psi\rangle\langle\psi|$ and $\rho_e = |\psi^e\rangle\langle\psi^e|$, the fidelity takes the simple form $F(\psi,\psi^e) = |\langle\psi|\psi^e\rangle|^2$ [18]. The greater the fidelity, the more similar the two states are. It is 0 if, and only if, $|\psi\rangle$ and $|\psi^e\rangle$ are orthogonal pure states and it is 1 if, and only if, $|\psi\rangle = |\psi^e\rangle$.

Here we assume that $|\psi\rangle$ and $|\psi^e\rangle$ represent an *actual* and an *estimated* state, respectively. These states do not in general initially coincide, and our task is to exploit unsharp measurement with the aim of forcing $|\psi^e\rangle$ to converge onto $|\psi\rangle$ in the presence of ongoing dynamics. A single elementary step of the estimation protocol consists of a unitary time evolution of both states followed by an unsharp measurement (which is a probabilistic "filtering" operation) on the actual state. The estimated state is updated based on the result of the measurement on the actual system. Under appropriate circumstances successive repetitions of this elementary step lead to a faithful estimate of the actual state in real time.

More concretely, an elementary step of the method can be formulated mathematically as follows. First, the actual state evolves according to the Hamiltonian dynamics of the system specified by the evolution operator $\hat{U}(\Omega)$:

$$|\psi\rangle \Rightarrow \hat{U}(\Omega)|\psi\rangle \equiv |\psi'\rangle. \tag{1}$$

If the Hamiltonian is precisely known then the estimate is propagated using the same evolution operator. However, a (classical) parameter Ω specified in the Hamiltonian, such as the Rabi frequency in the case of Rabi oscillations, may be detuned away from the actual parameter. The estimated state is thus evolved using an estimated unitary operator $U(\Omega_e)$:

$$|\psi^e\rangle \Rightarrow \hat{U}(\Omega_e)|\psi^e\rangle \equiv |\psi^{e'}\rangle. \tag{2}$$

The task of estimating Ω_e has been the subject of earlier work [17]. The actual state then undergoes the following random change (collapse) under selective measurement,

$$|\psi'\rangle \Rightarrow \frac{1}{\sqrt{p_n}}\hat{M}_n|\psi'\rangle \equiv |\psi'_n\rangle \tag{3}$$

where $\hat{M}_n$ is the Kraus operator corresponding to the nth allowed measurement outcome, and $p_n = \langle\psi|\hat{M}_n^\dagger\hat{M}_n|\psi\rangle$ is the associated probability for the outcome. The Kraus operators are constrained via $\sum_n \hat{M}_n^\dagger\hat{M}_n = \sum_n \hat{E}_n = \mathbb{I}$, where we introduced the positive effects $\hat{E}_n = \hat{M}_n^\dagger\hat{M}_n$. The estimate $|\psi^e\rangle$ is updated using the outcome of the selective measurement just done on $|\psi'\rangle$:

$$|\psi^e\rangle \Rightarrow \frac{1}{\sqrt{p_n^e}}\hat{M}_n|\psi^{e'}\rangle \equiv |\psi_n^{e'}\rangle \tag{4}$$

with $p_n^e = \langle\psi^{e'}|\hat{E}_n|\psi^{e'}\rangle$. The divisor p_n^e is merely used to re-normalize the updated $|\psi^{e'}\rangle$; the statistics of the updates are determined by the probabilities p_n to observe outcome n.

We can now define the *average* change in fidelity after a single elementary step of the estimation protocol:

$$\Delta F = \sum_n p_n|\langle\psi'_n|\psi_n^{e'}\rangle|^2 \; - \; |\langle\psi|\psi^e\rangle|^2. \tag{5}$$

Using Eq. (3) and (4) we obtain

$$\Delta F = \sum_n \frac{|\langle\psi|\hat{U}^\dagger(\delta)\hat{E}'_n|\psi^e\rangle|^2}{\langle\psi^e|\hat{E}'_n|\psi^e\rangle} \; - \; |\langle\psi|\psi^e\rangle|^2, \tag{6}$$

where $\delta = \Omega - \Omega_e$ and $\hat{E}'_n = \hat{U}^\dagger(\Omega_e)\hat{E}_n\hat{U}(\Omega_e)$ are the time-varying effects.

As is clear from Ref. [2], in the case where the Hamiltonian is precisely known ($\delta = 0$) the above update follows the spirit of the classical Bayesian update and we expect intuitively that the measured actual $|\psi\rangle$ and the updated estimate $|\psi^e\rangle$ come "closer" to each other. The method performs suitably well in various quantum estimation situations [14,4,11,19]. In what follows, we will use Eq. (6) to derive time dependence of the estimation fidelity convergence of oscillating qubits when the Hamiltonian is precisely know ($\delta = 0$) and find the asymptotic estimation fidelity when the Hamiltonian is not precisely known ($\delta \neq 0$).

3. Rabi oscillations

Consider a single two-level system undergoing Rabi oscillations due to the Hamiltonian

$$\hat{H}(\Omega) = \frac{\Omega}{2}\hat{\sigma}_x \tag{7}$$

where $\hbar = 1$, Ω is the Rabi frequency and $\hat{\sigma}_x$ is the Pauli matrix that generates rotations about the x-axis. The corresponding evolution operator is

$$\hat{U}(\Omega) = \exp(-i\hat{H}(\Omega)\tau). \tag{8}$$

In order to estimate the state of the system we perform symmetric unsharp measurements of the $\hat{\sigma}_z$ observable [4]. The corresponding effects are given by

$$E_0 = \tfrac{1}{2}(\mathbb{I} + \Delta p\hat{\sigma}_z) \tag{9}$$

$$E_1 = \tfrac{1}{2}(\mathbb{I} - \Delta p\hat{\sigma}_z) \tag{10}$$

where $0 \le \Delta p \le 1$ is the strength of the individual measurements. These measurements enable an unbiased estimation of the expectation value of $\hat{\sigma}_z$, i.e., $\langle\hat{\sigma}_z\rangle = (\langle E_0\rangle - \langle E_1\rangle)/\Delta p$. The time-varying effects are then

$$E_0' = \tfrac{1}{2}[\mathbb{I} + \Delta p(\cos(\Omega_e\tau)\hat{\sigma}_z + \sin(\Omega_e\tau)\hat{\sigma}_y)] \tag{11}$$

$$E_1' = \tfrac{1}{2}[\mathbb{I} - \Delta p(\cos(\Omega_e\tau)\hat{\sigma}_z + \sin(\Omega_e\tau)\hat{\sigma}_y)]. \tag{12}$$

3.1. Incremental equation

Assuming the initial states of the system and estimate are such that both remain in the yz-plane under the dynamics, we find that the change in fidelity (Eq. (6)) reduces to

$$\Delta F = \frac{\Delta p^2\left[\sin^2(\Omega_e\tau + \theta_e)\sin^2\left(\frac{\theta-\theta_e}{2}\right)\right] - (1-\Delta p^2)\left[\cos^2\left(\frac{\theta-\theta_e}{2}\right) - \cos^2\left(\frac{\theta-\theta_e}{2} + \frac{\delta\tau}{2}\right)\right]}{1 - \Delta p^2\cos^2(\Omega_e\tau + \theta_e)} \tag{13}$$

where $\theta, \theta_e \in [0, \pi]$ are the polar angles of the Bloch vectors corresponding to the states $|\psi\rangle$ and $|\psi^e\rangle$, respectively. The first term in Eq. (13) is the change in fidelity due to the measurement (which is performed after the unitary evolution), and is modulated by the measurement strength. For a projective measurement ($\Delta p^2 = 1$), we see that the change in fidelity is maximal, i.e. unity minus the initial fidelity, $\cos^2(\frac{\theta-\theta_e}{2})$. The second term is the change in fidelity due to the evolution. It does not contribute to the overall change in fidelity in two cases – when the Rabi frequency is precisely known, as well as when the measurement is projective. We notice that the right-hand side of the equation is a convex sum – the stronger the measurement, the weaker the influence of the unitary dynamics. In other words, if the level-resolution time $T_{\mathrm{lr}} = \tau/\Delta p^2$, which defines the timescale on which the state evolves due to the measurement sequence, is much briefer than the timescale $1/\delta$ of the unitary dynamics, i.e. $\tau\delta/(\Delta p^2) \ll 1$, then the measurement has a greater influence on the change in fidelity than the relative dynamics between the state and the estimate.

3.2. Differential equation

Hitherto we have restricted ourselves to a formalism appropriate for describing a sequence of discrete, unsharp measurements. Specifically, we've refrained from using stochastic Schrodinger equation or Ito calculus approaches suited to continuous measurement scenarios. To obtain analytical expressions governing the ensemble averaged behaviour of our protocol we are ultimately forced to take a time continuous limit. In this way we can derive a differential equation for the average change in fidelity after a single elementary step of the estimation protocol.

To this end we make two changes to Eq. (13). Firstly, we transform to a new coordinate system for the polar angles of the two Bloch vectors by defining, respectively, the relative half-angle, $\theta_r = (\theta - \theta_e)/2$, and the mean angle, $\bar{\theta} = (\theta + \theta_e)/2$. In addition, we can rewrite the equation in terms of the fidelity since it relates to the relative angle via $F = \cos^2\theta_r$. This yields

$$\Delta F = \frac{\Delta p^2\sin^2\left(\Omega_e\tau + \bar{\theta} - \theta_r\right)(1-F) - (1-\Delta p^2)[\pm\sqrt{F(1-F)}\sin(\delta\tau) + (F - \frac{1}{2})(1-\cos(\delta\tau))]}{1 - \Delta p^2\cos^2\left(\Omega_e\tau + \bar{\theta} - \theta_r\right)}. \tag{14}$$

Note that the two possible signs in front of the square root come about when relating a cosine and sine product to the fidelity since $\cos\theta_r\sin\theta_r = \sqrt{F(1-F)}$ or $\cos\theta_r\sin\theta_r = -\sqrt{F(1-F)}$ depending on the particular value of θ_r. In the regime where the sequential measurement strength is weak compared to the unitary dynamics, a few approximations are appropriate. Firstly, since $\Delta p \ll 1$, a Taylor expansion of ΔF can be truncated, omitting terms of order $\mathcal{O}(\Delta p^4)$. Secondly, assuming $\delta\tau \ll 1$, we can approximate the sine and cosine of the angle $\delta\tau$ by $\delta\tau$ itself and 1, respectively. Lastly, we note that the fidelity $F = \cos^2\theta_r$ and thus, for small Δp, θ_r will change little over the course of a single Rabi oscillation, while the mean angle $\bar{\theta}$ varies approximately over the full 2π range. Therefore, we can average over all possible mean angles using $\overline{\sin^2\bar{\theta}} = \overline{\cos^2\bar{\theta}} = 1/2$ and $\overline{\sin\bar{\theta}} = \overline{\cos\bar{\theta}} = 0$, where the overline indicates this average over one oscillation period. In order to obtain a differential equation we divide by the change in time after a single elementary step, τ, and take the limit that τ tends to zero. Simultaneously we require that Δp tends to zero such that $\lim_{\substack{\Delta p\to 0\\ \tau\to 0}}\Delta p^2/\tau = \gamma$, the so-called continuum limit of the sequence of unsharp measurements [20]. We thus arrive at:

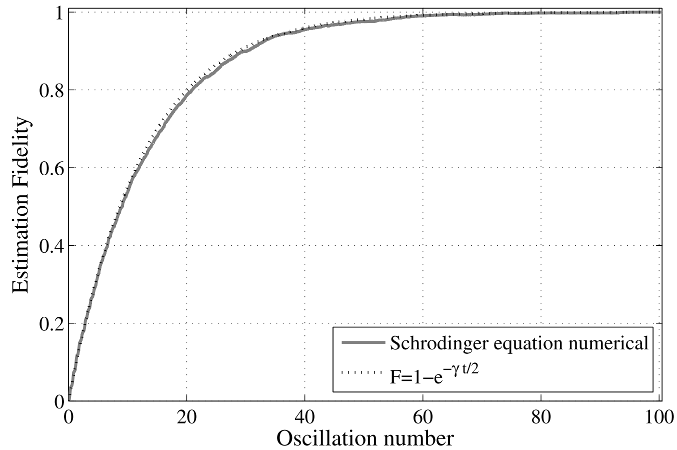

Fig. 1. Convergence of the estimation fidelity for known dynamics. The solid grey line shows the averaged fidelity over 1000 numerical simulations, while the dotted line represents the theoretical prediction Eq. (16). We used $\Delta p = 0.04$ and $\tau = \pi/50$ so that $\gamma = 0.0255$, and averaged over 1000 runs.

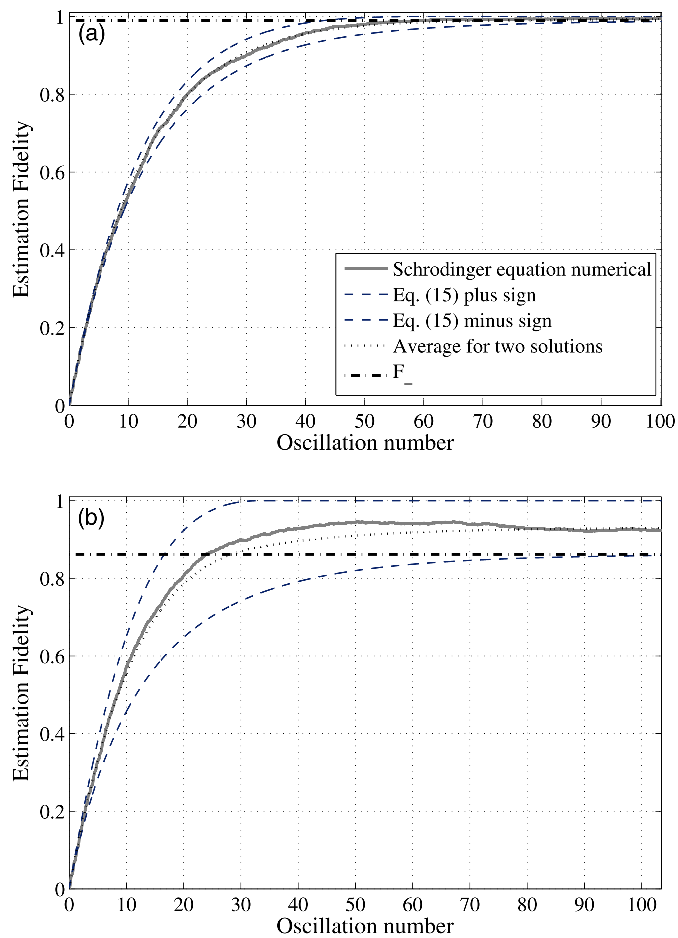

Fig. 2. Convergence of the estimation fidelity for finite δ. The solid grey line shows the numerical solution to the Schödinger equation, the top and bottom dashed lines the numerical solutions to Eq. (15) for the plus and minus signs respectively, while the dotted line is the average of the latter two curves. The horizontal dot-dashed line is the asymptotic limit F_-. In (a) $\delta = \gamma/20$ and in (b) $\delta = \gamma/5$. In both figures $\Delta p = 0.04$ and $\tau = \pi/50$ so that $\gamma = 0.0255$.

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\gamma}{2}(1-F) \pm \delta\sqrt{F(1-F)}. \tag{15}$$

The parameter γ characterizes the strength of the measurement sequence [20] and is related to the level resolution time $T_{\mathrm{lr}} = 1/\gamma$. Implicit in this derivation is the assumption $\delta\tau \ll 1$. If the Rabi frequency is known precisely (i.e. $\delta = 0$), this equation has the simple solution

$$F = 1 - \exp\left(-\tfrac{\gamma}{2}t\right). \tag{16}$$

The average estimation fidelity of a sequential unsharp measurement thus converges exponentially fast to unity. This result was already conjectured from numerical simulations in [4]. Fig. 1 compares the theoretical prediction (16) to the averaged estimation fidelity obtained from 1000 numerical simulations of a qubit undergoing Rabi oscillations governed by Hamiltonian (7), while being subjected to unsharp measurements at periodicity τ. We used $\Delta p = 0.04$ and $\tau = \pi/50$, yielding $\gamma = 0.0255$ in units of the Rabi frequency Ω.

If the estimate of the Rabi frequency is different from the actual value, i.e. $\delta \neq 0$, the fidelity of estimation will erode. Solving Eq. (15) at steady state, $\frac{dF}{dt} = 0$, we find that the asymptotic average fidelity is

$$F_+ = 1 \quad \text{or} \quad F_- = \frac{\gamma^2}{\gamma^2 + 4\delta^2}, \tag{17}$$

where the subscripts indicate the asymptotic solutions for the corresponding signs of the two possibilities in Eq. (15). The ensemble averaged behaviour is expected to be simply the average of the two solutions to Eq. (15) tending asymptotically to $\bar{F} = (F_+ + F_-)/2$. This expectation is clearly borne out as illustrated in Fig. 2 where we use the same parameters as in Fig. 1. In Fig. 2(a) $\delta = \gamma/20$, while in (b) $\delta = \gamma/5$. The solid grey line again is the numerical solution to the Schrödinger equation in the presence of unsharp measurement. The dashed curves show the two possible solutions of differential equation (15) corresponding to the two allowed signs. The dotted curve, which closely follows the numerical solution, is the average of these two solutions. The horizontal dot-dashed line indicates the asymptotic limit F_-. The slight overshoot in the numerical simulation compared to the analytical result for larger δ around the time $\sim 1/\gamma$, is persistent in our simulations and not accounted for in this model. However, very faithful correspondence between our analytical results and the numerical simulation is observed in the asymptotic regime.

4. Conclusion

The techniques employed in this manuscript establish a systematic approach toward analytical description of an oscillating qubit undergoing sequential, discrete unsharp measurement and can in principle be extended to other systems of interest. Employing the same approach is expected to yield detailed understanding of process and parameter estimation dynamics. Our results clearly delineate parameter regimes in which sequential unsharp measurement can be usefully employed as a state estimation tool for oscillating qubits.

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