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On the earliest jump unravelling of the spatial decoherence master equation

Gábor Homa [a], Lajos Diósi [b,*]

[a] Department of Physics of Complex Systems, ELTE Eötvös Loránd University, Pázmány Péter sétány 1/A, H-1117 Budapest, Hungary
[b] Wigner Research Centre for Physics, High Energy Physics Department, H-1525 Budapest 114, P.O.B. 49, Hungary

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ABSTRACT

Solution of free particle quantum master equation with spatial decoherence can be unravelled into stochastic quantum trajectories in many ways. The first example (Diósi, 1985) proposed a piecewise deterministic jump process for the wave function. While alternative unravellings, diffusive ones in particular, proved to be tractable analytically, the jump process 1985, also called orthojump, allows for few analytic results, it needs numeric methods as well. Here we prove that, similarly to diffusive unravellings, it is localizing the quantum state.

1. Introduction

A single Schrödinger particle becomes a simple open quantum system if the particle is interacting with a thermal reservoir. Its dynamics is given by a master equation which can take the following simple form valid typically at high temperatures:

$$\frac{d\hat{\rho}}{dt}=-\frac{i}{\hbar}[\hat{H},\hat{\rho}]-\frac{D}{\hbar^2}[\hat{x},[\hat{x},\hat{\rho}]], \tag{1}$$

where $\hat{H}=(\hat{p}^2/2m)$ is the particle's Hamiltonian, $\hat{x},\hat{p}$ are its coordinate and momentum resp., and D is the diffusion constant. Joos and Zeh suggested this equation as the simplest model of spatial decoherence [1] while at the time similar single particle master equations were known from various fields, cf., e.g., [2,3]. The Wigner function of $\hat{\rho}$ satisfies

$$\frac{d\rho(x,p)}{dt}=-\frac{p}{m}\partial_x\rho(x,p)+D\partial_p^2\rho(x,p), \tag{2}$$

which coincides with the classical Fokker–Planck–Kramers equation [4] in the high-temperature (diffusion dominated, frictionless) limit. This elucidates the importance of the master equation (1) as the quantized version of momentum diffusion. Accordingly, D is the coefficient of spatial decoherence as well as of momentum diffusion: the two effects are alternative interpretations of the non-Hamiltonian mechanism in the master equation. It is well-known that the classical diffusion (2) can be equivalently described by random trajectories (x_t,p_t) in phase space. The same concept applies to the master equation (1) as well. The stochastic *quantum trajectories* are featured by state vectors Ψ_t evolving by a stochastic process such that the stochastic mean

$$\mathbf{M}\Psi_t\Psi_t^\dagger=\hat{\rho}_t \tag{3}$$

satisfies the master equation (1). Then the quantum trajectories Ψ_t are said to *unravel* the master equation.

The unravelling is never unique, one can choose diffusive unravellings, jump unravellings, or even their combinations. The earliest unravelling was the orthojump process [5]. It turned out subsequently that any master equation possesses a standard jump and a standard diffusive unravelling [6]. All possible diffusive unravellings can be parametrized uniquely [7,8], each of them corresponds to a given structure of time-continuous monitoring the system in question [8]. Similar classification is still missing for jump unravellings.

While quantum trajectories became instrumental soon for quantum optics [9–11], their invention happened earlier in studies of foundations. In the nineteen-eighties, diffusive quantum trajectories were invented by Gisin to model quantum state collapse in a discrete system [12]. One of the present authors constructed jump [5] and diffusive [13] unravellings of the master equation (1) for his gravity-related spontaneous state collapse theories [14]

* Corresponding author.
E-mail addresses: ggg.maxwell1@gmail.com (G. Homa), diosi.lajos@wigner.mta.hu (L. Diósi).

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and [15], respectively. (On three decades of various spontaneous collapse theories, all based on unravellings, see the recent review by Bassi et al. [16].)

Analytic proof was found for the wave function localization in diffusive quantum trajectories [17]. The wave function is approaching a steady localized shape for long times, as we recapitulate it below. Localization in the specific jump unravellings [5] has, however, never been studied. The problem is more complicated than the diffusive case because jumps will never allow for a steady shape. An analytic proof of localization has not yet been found, we shall rely on numeric (Monte-Carlo) simulations. Jump quantum trajectories of spatial decoherence were carefully studied by Gisin and Rigo [18], and in a sequence of works by Hornberger and co-workers [19–21] for modifications of the master equation (1) which included friction. Due to friction, quantum trajectories did reach a localized steady shape, calculable analytically. The effect and proof was bound to the presence of friction. Localization in the frictionless case (1) has remained to be studied in the present work.

We are going to study localization of quantum trajectories in both position and momentum. Consider the unitary transformation of a state Ψ to its centre-of-mass frame:

$$\widetilde{\Psi}=\exp\left(i\langle\hat{x}\rangle\hat{p}-i\langle\hat{p}\rangle\hat{x}\right)\Psi, \tag{4}$$

where the centre-of-mass state satisfies $\langle\widetilde{\Psi}|\hat{x}|\widetilde{\Psi}\rangle=0$ and $\langle\widetilde{\Psi}|\hat{p}|\widetilde{\Psi}\rangle=0$ by construction. Now we can define the centre-of-mass density matrix as follows:

$$\mathbf{M}\widetilde{\Psi}_t\widetilde{\Psi}_t^\dagger=\hat{\widetilde{\rho}}_t. \tag{5}$$

This matrix is non-negative and of unit trace, like common density matrices. Its evolution, however, is non-linear, completely different from the master equation (1) of the common density matrix $\hat{\rho}_t$. Since $\hat{\widetilde{\rho}}_t$ is unravelling specific, we can use it to characterize the unravelling specific average localization of quantum trajectories Ψ_t around their individual centre-of-mass $\langle\hat{x}\rangle_t,\langle\hat{p}\rangle_t$. We can define centre-of-mass (squared) spreads by $(\Delta\widetilde{x})^2=\mathrm{Tr}(\hat{x}^2\hat{\widetilde{\rho}})$ and by $(\Delta\widetilde{p})^2=\mathrm{Tr}(\hat{p}^2\hat{\widetilde{\rho}})$.

2. Diffusive unravelling

Following [13,17], consider the stochastic Schrödinger equation [17]:

$$d\Psi=-\frac{i}{\hbar}\hat{H}\Psi dt-\frac{D}{\hbar^2}(\hat{x}-\langle\hat{x}\rangle)^2\Psi+\frac{\sqrt{2D}}{\hbar}(\hat{x}-\langle\hat{x}\rangle)\Psi dW, \tag{6}$$

where dW is the Ito-differential of the Wiener stochastic process, satisfying $\mathbf{M}dW=0$, $(dW)^2=dt$. The solutions satisfy the condition (3) of unravelling. For long time, the centre-of-mass solutions converge to the following complex Gaussian wave packet:

$$\widetilde{\Psi}_\infty(x)=\frac{1}{(2\pi\sigma_\infty^2)^{1/4}}\exp\left(-(1-i)\frac{x^2}{4\sigma_\infty^2}\right) \tag{7}$$

of squared width

$$\sigma_\infty^2=\sqrt{\frac{\hbar^3}{2Dm}}. \tag{8}$$

According to (7), the centre-of-mass density matrix (5) turns out to converge to a pure state:

$$\hat{\widetilde{\rho}}_\infty=\widetilde{\Psi}_\infty\widetilde{\Psi}_\infty^\dagger. \tag{9}$$

The coordinate and momentum spreads are given by

$$(\Delta\widetilde{x})^2=\sigma_\infty^2,\quad(\Delta\widetilde{p})^2=\frac{\hbar^2}{2\sigma_\infty^2}. \tag{10}$$

Asymptotic localization is thus the analytically calculable feature of the diffusive quantum trajectories of the simple spatial decoherence master equation (1). The centre-of-mass of $\hat{\widetilde{\rho}}_\infty$ is performing the following diffusive motion:

$$d\langle\hat{x}\rangle=\frac{1}{m}\langle\hat{p}\rangle dt+\sqrt{\frac{2\hbar}{m}}dW,\qquad d\langle\hat{p}\rangle=\sqrt{2D}dW. \tag{11}$$

Observe that the diffusion of the momentum is the classical one. On the contrary, the diffusion of the coordinate cannot happen classically, it is purely quantum.

It may be interesting to see how simple is to recover the common density matrix $\hat{\rho}_t$ in the specific case when we have $\Psi_0=\widetilde{\Psi}_\infty$ initially. Only we have to solve the stochastic equations (11) with the initial laboratory values $\langle\hat{x}\rangle_0=\langle\hat{p}\rangle_0=0$, and apply (12) to construct Ψ_t is the laboratory:

$$\Psi_t=\exp\left(-i\langle\hat{x}\rangle_t\hat{p}+i\langle\hat{p}\rangle_t\hat{x}\right)\widetilde{\Psi}_\infty. \tag{12}$$

Then we recover the common density matrix via (3).

3. Orthojump unravelling

For the sake of comparison with the diffusive unravelling, let us cast the jump unravelling of [5] into the form of a stochastic Schrödinger equation:

$$d\Psi=-\frac{i}{\hbar}\hat{H}\Psi dt-\frac{D}{\hbar^2}[(\hat{x}-\langle\hat{x}\rangle)^2-\sigma^2]\Psi dt+\left(\frac{x-\langle\hat{x}\rangle}{\sigma}-1\right)\Psi dN, \tag{13}$$

where $\sigma^2=\langle(\hat{x}-\langle\hat{x}\rangle)^2\rangle$. dN stands for the Ito-differential of a Poisson process, satisfying $\mathbf{M}dN=2D\sigma^2dt$, $(dN)^2=dN$. This equation corresponds to a piece-wise deterministic evolution of Ψ_t, interrupted by jumps at random times. In elementary terms, the mechanism is the following. Consider the deterministic non-linear Schrödinger equation

$$\frac{d\Phi}{dt}=-\frac{i}{\hbar}\hat{H}\Phi-\frac{D}{\hbar^2}[(\hat{x}-\langle\hat{x}\rangle)^2]\Phi. \tag{14}$$

[Note that this equation coincides with the deterministic part of the diffusive stochastic Schrödinger equation (6) and they share $\widetilde{\Psi}_\infty$ (7) as (normalized) steady-shape centre-of-mass solution.] Solve this non-linear Schrödinger equation for the initial condition $\Phi_0=\Psi_0$ and define the physical quantum state by $\Psi_t=\Phi_t/\|\Psi_t\|$. Note that the norm of Φ is strictly decreasing:

$$\frac{d\|\Phi\|^2}{dt}=-\frac{2D}{\hbar^2}\sigma^2. \tag{15}$$

The probability of jump-free deterministic evolution is decreasing exactly with the norm $\|\Phi\|^2$, i.e., the probability rate of jump is $(2D/\hbar^2)\sigma^2$. When a jump occurs, the smooth deterministic evolution of $\Psi/\|\Psi\|$ is interrupted by the sudden change

$$\Phi\longrightarrow(\hat{x}-\langle\hat{x}\rangle)\Phi, \tag{16}$$

rendering the new state orthogonal to what it was before the jump (cf. also [11]). After the jump, the deterministic evolution (14) restarts and continues until the next jump, etc.

4. Numeric tests of orthojumps

We have performed MC simulations of 15000 orthojump quantum trajectories. With suitable choice of physical units, we can always take trivial parameters $\hbar=m=D=1$ and that is what we did. Discretized position coordinate $x\in(-5,+5)$ into 256 bins, the time increment was 0.00075. We chose the pure state (7) deliberately as the initial pure state for (1). The centre-of-mass orthojump trajectories $\widetilde{\Psi}_t$, similarly to the diffusive ones, are going to forget the initial state after $t \gg 1$. Furthermore, the asymptotic state of the centre-of-mass diffusive trajectory (7) can be a good reference state numerically since orthojump wave packets are supposed to fluctuate (to "breath") not very far from it.

While analytic solutions for individual trajectories are not (yet) known, the analytic solution of the master equation (1) is easy [22, 23], especially for Gaussian initial states [24]. To check the robustness of our MC simulation, we shall compare the MC-simulated density matrix $\hat{\rho}_{MC}$ to the analytic solution $\hat{\rho}$ of the master equation (1).

The analytic solution in coordinate representation reads:

$$\begin{aligned}\langle x|\,\hat{\rho}_t\,|y\rangle = \frac{1}{\sqrt{2\pi}\,\Sigma(t)}\exp\Bigg\{&-\frac{1}{8\Sigma^2(t)}(x+y)^2\\ &-\frac{1+2\sqrt{2}t+2t^2+\frac{2\sqrt{2}}{3}t^3+\frac{1}{3}t^4}{8\Sigma^2(t)}(x-y)^2\\ &-i\frac{1+\sqrt{2}t+t^2}{4\Sigma^2(t)}(x^2-y^2)\Bigg\}\end{aligned} \tag{17}$$

where the squared spatial spread is

$$\Sigma^2(t) = \frac{1}{\sqrt{2}} + t + \frac{t^2}{\sqrt{2}} + \frac{t^3}{3}. \tag{18}$$

With the same initial pure state, we MC-generated 3×5000 quantum trajectories $\{\Psi_t^{(n)}; n = 1, 2, \ldots, 15000\}$ and determined $\hat{\rho}_{MC,t}$ numerically:

$$\langle x|\,\hat{\rho}_{MC,t}\,|y\rangle = \frac{1}{N}\sum_n^N \Psi_t^{(n)}(x)\Psi_t^{(n)*}(y), \tag{19}$$

where N stands for the number of trajectories (not to be confused with the Poisson process N in Sec. 3): Displaying its normalised distance

$$\frac{\left(\mathrm{Tr}(\hat{\rho}-\hat{\rho}_{MC})^2\right)^{1/2}}{(\mathrm{Tr}\hat{\rho}^2)^{1/2}} \tag{20}$$

from the analytic $\hat{\rho}_t$ (17) in the range $t \in (0, 5)$. Data taken on three independent statistics of 5000 trajectories verify that statistical errors stay about the order of 0.001. This suggests a decent stability and precision of simulation on the unified statistics of 15000 trajectories (Fig. 1). For qualitative comparison, Fig. 2 shows the MC-simulated Wigner function and the exact one at $t = 5$. These checks confirm that 15000 trajectories will suffice to test the basic feature of interest: localization by orthojump unravelling.

We determined the centre-of-mass density matrix

$$\langle x|\,\hat{\widetilde{\rho}}_{CM,t}\,|y\rangle = \frac{1}{N}\sum_n^N \widetilde{\Psi}_t^{(n)}(x)\widetilde{\Psi}_t^{(n)*}(y) \tag{21}$$

on three increasing statistics. Our main results are shown in Fig. 3, where the time-evolution of spatial and momentum spreads $\Delta\widetilde{x}, \Delta\widetilde{p}$ are displayed for $t \in (0, 5)$. Initial values are known analytically: $\Delta\widetilde{x}_0 = \Delta\widetilde{p}_0 = 1/2^{1/4} \approx 0.84$. For times longer than the characteristic time scale 1 (when $\hbar = m = D = 1$) of the master equation (1), localization takes place asymptotically, both in coordinate and momentum. Both $\Delta\widetilde{x}$ and $\Delta\widetilde{p}$ converge to constants, their conservative estimates are

$$\Delta\widetilde{x}_\infty = 1.62 \pm 0.01, \quad \Delta\widetilde{p}_\infty = 1.63 \pm 0.01. \tag{22}$$

The errors ± 0.01 mark a loose upper-bound on fluctuations of the flat parts of the simulated curves in Fig. 3.

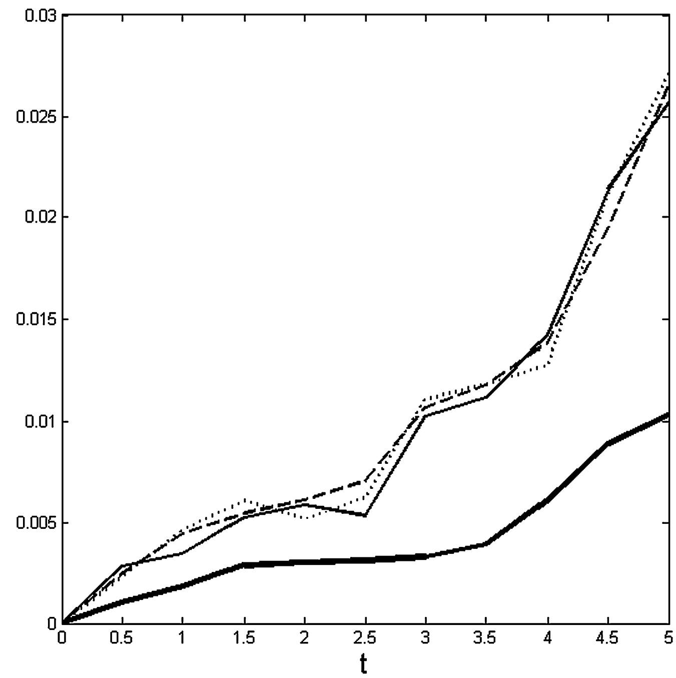

Fig. 1. Normalized distance $\sqrt{\mathrm{Tr}(\hat{\rho}_{MC}-\hat{\rho})^2}/\sqrt{Tr(\hat{\rho}^2)}$ between MC-simulated density matrix $\hat{\rho}_{MC}$ and the exact $\hat{\rho}$ in the time interval $t \in (0, 5)$, taken on three-times 5000 trajectories (solid, dot, dash, resp.), and on the overall 15000 trajectories (lower solid).

This is the first numeric evidence, in lack of analytic ones, for localization of orthojump trajectories in frictionless spatial decoherence.

5. Summary

We have studied the localization of wave function in orthojump unravelling of the simplest and paradigmatic spatial decoherence master equation of a free particle. Localization in diffusive unravellings became proven analytically long ago. This time we were able to prove and calculate localization of the orthojump unravelling — using MC simulations. We used 15000 MC-simulated quantum trajectories to confirm localization both in coordinate ($\Delta\widetilde{x}$) and momentum ($\Delta\widetilde{p}$), which we demonstrated on the centre-of-mass density matrix $\hat{\widetilde{\rho}}$. The obtained numeric values (22) are about twice as large as those (10) in diffusive unravelling. Such slightly looser localization may be explained heuristically. The asymptotic centre-of-mass density matrix $\hat{\widetilde{\rho}}_\infty$ contains randomness because the centre-of-mass wave function $\widetilde{\Psi}_t$ never ceases to undergo jumps, i.e., it is "breathing" at random times, whereas in diffusive unravelling Ψ_t acquires a constant shape for large t hence $\hat{\widetilde{\rho}}_\infty$ does not contain randomness, diffusive features contribute to the centre-of-mass motion (11) only.

Our work was restricted for the demonstration of stability and localization of the orthojump trajectories for the frictionless decoherent dynamics of a Schrödinger particle. Further studies should explore more details of orthojump trajectories' rich structure. Numeric methods seem instrumental. However, similar to the diffusive case (6), a possible power of the Ito formalism (13) remains to be explored for analytic calculations.

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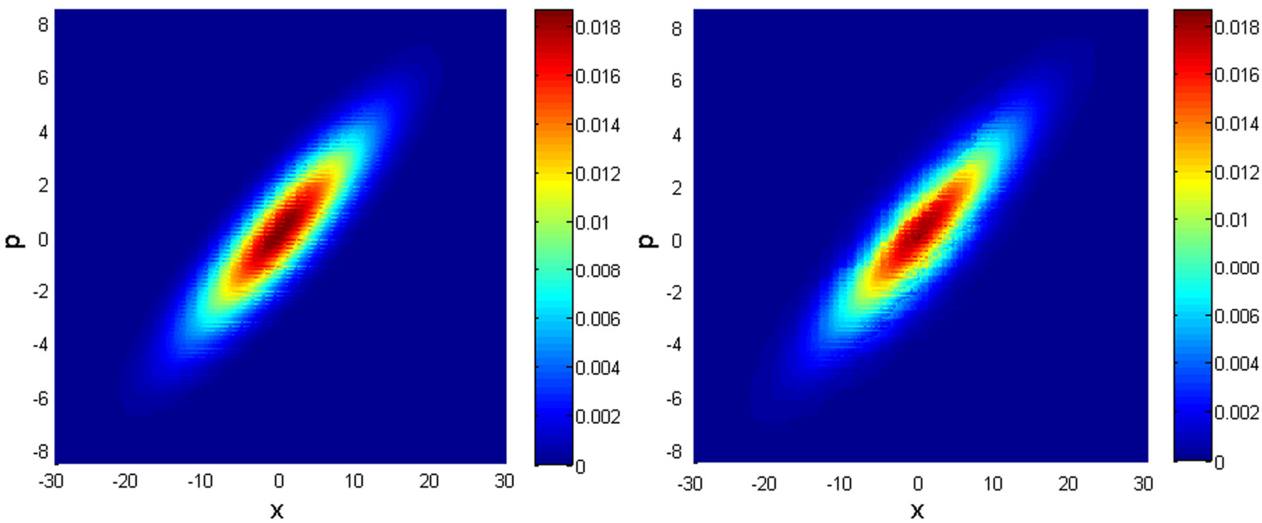

Fig. 2. Wigner function solving the master equation (1) in units $\hbar = m = D = 1$ at $t = 5$ with initial state (7): analytic solution (left), MC solution on 15000 trajectories (right). (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

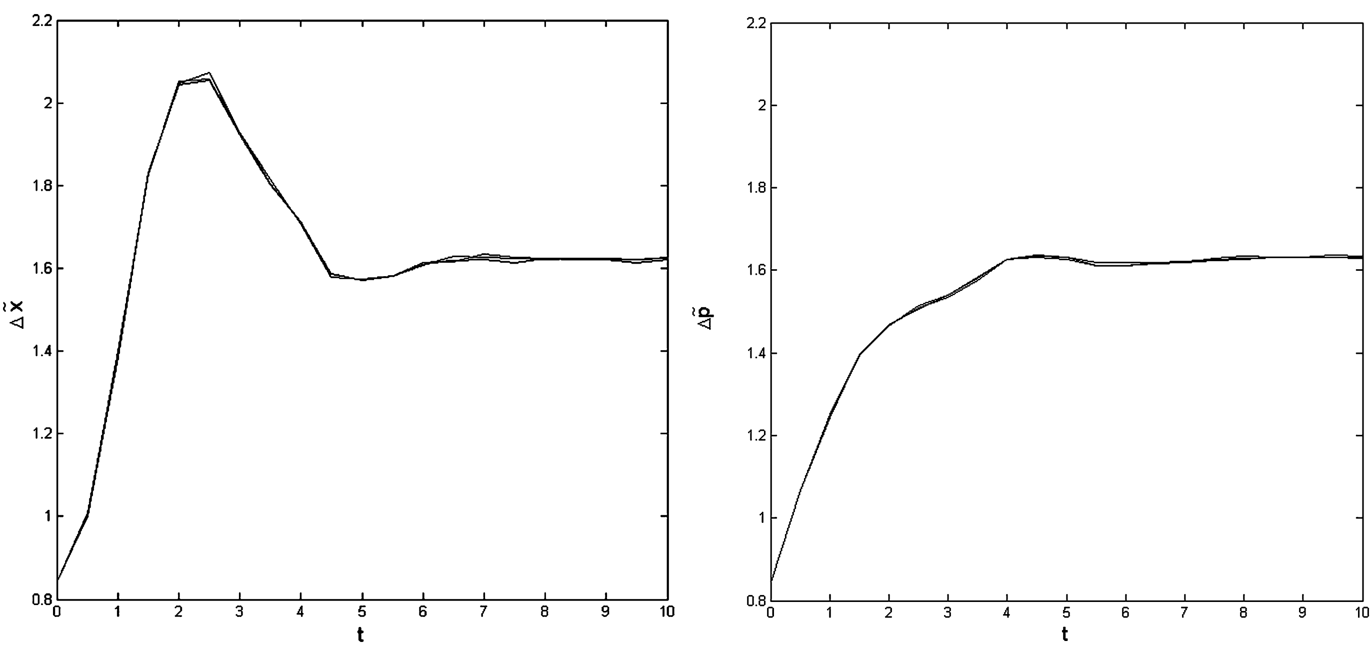

Fig. 3. Centre-of-mass spreads $\Delta\tilde{x}$ (a) and $\Delta\tilde{p}$ (b) in MC-simulated density matrix $\hat{\tilde{\rho}}_{MC}$ in time interval $(0, 5)$, taken on 5000, 10000, and 15000 trajectories. Values are overlapping within 0.01.

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