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NORMAL ORDERING THE SQUEEZE OPERATOR BY GENERALIZED WICK THEOREM

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Abstract

The recently proposed generalized Wick theorem offers a simple way to normal ordering the squeeze operator, shown by the present notes dedicated to the memory of J. Janszky.

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In 1988, I translated a popular article *Squeezed Light* [1] for the Hungarian version of Scientific American. How shall we say *squeezed* in our language from now on? At the time, Janszky was already Hungary's top quantum optics theorist. He excelled in squeezed light research as well [2, 3], so I did not want to decide without asking him. He voted for the word *préselt* [prayshalt], and it worked [4] and became standard in our talking and writing about squeezed light. To his memory, let me dedicate an easy route to normal ordering the squeeze operator, based on my recent extension of the Wick theorem for general orderings of q and p.

In quantum (as well as in classical) optics, a single electromagnetic mode is equivalent to a harmonic oscillator whose dynamics is invariant for rotations of the phase plane spanned by coordinate $\hat{q}$ and momentum $\hat{p}$. The usual representation is the complex one in terms of $\hat{c} = (\hat{q} + i\hat{p})/\sqrt{2}$ and its conjugate $\hat{c}^\dagger$, hence rotations simply become phase shifts of $\hat{c}$ and $\hat{c}^\dagger$. Squeezing $\hat{q}$ into $\mathrm{e}^{-r}\hat{q}$ corresponds to the unitary operator

$$\hat{S} = \exp\left(\frac{r}{2}(\hat{c}^2 - \hat{c}^{\dagger 2})\right). \tag{1}$$

To confirm this, we can start from the elementary notion of squeezing: we have to rescale $\hat{q}$ by a factor $1/\mu$ and $\hat{p}$ by μ. Rewrite $\hat{S}$ in terms of $\hat{q}, \hat{p}$,

$$\hat{S} = \exp\left(\frac{i}{2} r\{\hat{q}, \hat{p}\}\right) \tag{2}$$

yielding

$$\hat{S}^\dagger \hat{q} \hat{S} = \mathrm{e}^{-r}\hat{q}, \qquad \hat{S}^\dagger \hat{p} \hat{S} = \mathrm{e}^{-r}\hat{p}, \tag{3}$$

which is the desired squeezing with $\mu = \mathrm{e}^r$.

If we are interested in the coordinate representation of $\hat{S}$, we have a definitive expression [5], as well as a complementary one in the momentum basis

$$\hat{S} = \frac{1}{\sqrt{\mu}} \int |q/\mu\rangle\langle q| dq = \sqrt{\mu} \int |\mu p\rangle\langle p|\, dp. \tag{4}$$

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To see their equivalence with the standard forms (1), (2), we take the derivative of each side. Both expressions of $\hat{S}$ satisfy the same simple differential equation

$$\frac{d\hat{S}}{d\mu} = \left(-\frac{1}{2\mu} + \frac{i}{\mu}\hat{p}\hat{q}\right)\hat{S} = \frac{i}{2\mu}\{\hat{q},\hat{p}\}\hat{S}. \tag{5}$$

With the initial condition $\hat{S} = \hat{1}$ at $\mu = 1$, we obtain the unique solution coinciding with (1) and (2).

For quantum optics, the normal $\mathcal{N}$ ordered form of $\hat{S}$ is of interest. It was derived, e.g., in [5–8]. We shall nicely reduce the budget of calculations if we rely on the generalized Wick theorem (GWT) [9] connecting two different orderings $\mathcal{O}, \mathcal{O}'$ of the exponential characteristic function $\exp(\hat{X})$,

$$\mathcal{O}'\mathrm{e}^{\hat{X}} = \mathrm{e}^{C}\mathcal{O}\mathrm{e}^{\hat{X}}, \tag{6}$$

where $\hat{X}$ is any linear combination of $\hat{q}$, $\hat{p}$ (or $\hat{c}$, $\hat{c}^\dagger$). The pre-factor is a c-number because the exponent is the c-number,

$$C = \frac{1}{2}(\mathcal{O}' - \mathcal{O})\hat{X}^2. \tag{7}$$

We call it the *general contraction* between $\mathcal{O}'$ and $\mathcal{O}$.

To prepare the application of this theorem, consider the (first) expression of $\hat{S}$ in (4). For convenience, introduce $\varkappa = 1 - 1/\mu$ and insert the identity

$$|q/\mu\rangle = \mathrm{e}^{i\varkappa q\hat{p}}|q\rangle \tag{8}$$

into (4). Observe that it leads to the PQ-ordered form

$$\hat{S} = \frac{1}{\sqrt{\mu}}\mathcal{O}_{PQ}\mathrm{e}^{i(1-1/\mu)\hat{p}\hat{q}}, \tag{9}$$

where $\mathcal{O}_{\mathrm{PQ}}$ puts the $\hat{p}$'s to the left of the $\hat{q}$'s. To apply GWT (6), (7), we first unravel the bilinear form $\hat{p}\hat{q}$ in the exponent. Let us insert the identity

$$\mathrm{e}^{i\varkappa\hat{p}\hat{q}} = \int \mathrm{e}^{iz\hat{p}\pm z^\star\hat{q}}\exp\left(-\frac{|z|^2}{|\varkappa|}\right)\frac{d^2z}{\pi|\varkappa|} \tag{10}$$

into (9), yielding

$$\hat{S} = \int \mathcal{O}_{PQ}\mathrm{e}^{iz\hat{p}\pm z^\star\hat{q}}\exp\left(-\frac{|z|^2}{|\varkappa|}\right)\frac{d^2z}{\pi\sqrt{\mu}|\varkappa|}, \tag{11}$$

where the sign $\pm$ is the sign of $\varkappa$.

Now we can apply our GWT (6), (7)

$$\mathcal{O}_{PQ}\mathrm{e}^{iz\hat{p}\pm z^\star\hat{q}} = \mathrm{e}^{C}\mathcal{N}\mathrm{e}^{iz\hat{p}\pm z^\star\hat{q}}, \tag{12}$$

$$C = \frac{1}{2}(\mathcal{O}_{PQ} - \mathcal{N})(iz\hat{p} \pm z^\star\hat{q})^2. \tag{13}$$

Using the relations

$$\mathcal{N}\hat{q}^2 \equiv \mathcal{N}\frac{(\hat{c}+\hat{c}^\dagger)^2}{2} = \hat{q}^2 - \frac{1}{2}, \qquad \mathcal{N}\hat{p}^2 \equiv \mathcal{N}\frac{(\hat{c}-\hat{c}^\dagger)^2}{2i} = \hat{p}^2 - \frac{1}{2}, \qquad \mathcal{N}\hat{q}\hat{p} = \frac{1}{2}\{\hat{q},\hat{p}\}, \tag{14}$$

together with $\mathcal{O}_{\rm PQ}\hat{q}\hat{p} = \mathcal{O}_{\rm PQ}\hat{p}\hat{q} = \hat{p}\hat{q}$, we obtain the c-number contraction

$$C = \frac{1}{4}(z^{\star 2} - z^2) \pm \frac{1}{2}|z|^2. \tag{15}$$

Now (11) takes the form

$$\hat{S} = \mathcal{N}\int \exp\left(iz\hat{p} \pm z^\star \hat{q} + \frac{z^{\star 2} - z^2}{4} - \left|\frac{1}{\varkappa} - \frac{1}{2}\right| |z|^2\right) \frac{d^2 z}{\pi\sqrt{\mu}|\varkappa|}. \tag{16}$$

Thanks to GWT, we only needed elementary calculational steps so far, and the evaluation of the Gaussian integral yields the desired normal ordered squeeze operator,

$$\hat{S} = \sqrt{\frac{2\mu}{1+\mu^2}}\,\mathcal{N}\exp\left\{\frac{i(\mu^2-1)\hat{p}\hat{q} - \frac{1}{2}(\mu-1)^2(\hat{p}^2+\hat{q}^2)}{1+\mu^2}\right\}, \tag{17}$$

where we expressed $\varkappa$ as $1 - 1/\mu$.

It is worthwhile to notice that, apart from the powerful GWT [9], which is a consequence of the Baker–Campbell–Haussdorff theorem [10–12] so common in quantum optics, our method is akin to Fan's "integration within ordered products" [5–8]. Also the technical formula

$$\int \exp(\zeta|z|^2 + \xi z + \eta z^\star + f z^2 + g z^{\star 2})\frac{d^2 z}{\pi} = \frac{1}{\sqrt{\zeta^2 - 4fg}}\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg} \tag{18}$$

to evaluate our (16) can be borrowed from [8].

The natural representation of normal ordered expressions is in terms of $\hat{c}$ and $\hat{c}^\dagger$, of course, rather than in $\hat{q}$ and $\hat{p}$. Let us rewrite (17) accordingly; we arrive at

$$\hat{S} = \sqrt{\frac{2\mu}{1+\mu^2}}\mathcal{N}\exp\left\{\frac{\frac{1}{2}(\mu^2-1)(\hat{c}^2 - \hat{c}^{\dagger 2}) + (\mu-1)^2\hat{c}^\dagger\hat{c}}{\mu^2+1}\right\}, \tag{19}$$

which, using the standard squeezing parameter r, takes this form

$$\begin{aligned} \hat{S} &= \frac{1}{\sqrt{\coth r}}\mathcal{N}\exp\left\{\frac{1}{2}\tan r(\hat{c}^2 - \hat{c}^{\dagger 2}) + \left(\frac{1}{\coth r} - 1\right)\hat{c}^\dagger\hat{c}\right\} \\ &= \frac{1}{\sqrt{\coth r}}\exp\left(-\frac{1}{2}\tan r\hat{c}^{\dagger 2}\right)\left(\frac{1}{\coth r}\right)^{\hat{c}^\dagger\hat{c}}\exp\left(\frac{1}{2}\tan r\hat{c}^2\right) \end{aligned} \tag{20}$$

coinciding with (8.13) of [7] — as it should.

This short derivation, an expansion of the example already outlined in [9], gives us the opportunity to visualize how GWT reduces the calculational budget in general. We start from the PQ-ordered l.h.s. of (12) (with notation $a = z^\star, b = iz$) and normal order it without using the GWT,

$$\begin{aligned} \mathcal{O}_{\rm PQ}{\rm e}^{a\hat{q}+b\hat{p}} &= {\rm e}^{(b/i\sqrt{2})(\hat{c}-\hat{c}^\dagger)}{\rm e}^{(a/\sqrt{2})(\hat{c}+\hat{c}^\dagger)} \\ &= {\rm e}^{(a^2+b^2)/4}{\rm e}^{-(b/i\sqrt{2})\hat{c}^\dagger}{\rm e}^{(b/i\sqrt{2})\hat{c}}{\rm e}^{a\sqrt{2})\hat{c}^\dagger}{\rm e}^{(a\sqrt{2})\hat{c}} \\ &= {\rm e}^{(a^2+b^2)/4-(i/2)ab}{\rm e}^{-(b/i\sqrt{2})\hat{c}^\dagger}{\rm e}^{a\sqrt{2})\hat{c}^\dagger}{\rm e}^{(b/i\sqrt{2})\hat{c}}{\rm e}^{(a\sqrt{2})\hat{c}} \\ &= {\rm e}^{(a^2+b^2)/4-(i/2)ab}\mathcal{N}{\rm e}^{a\hat{q}+b\hat{p}}. \end{aligned} \tag{21}$$

Here the Baker–Campbell–Haussdorff theorem applies three times step by step. This can be spared by applying the GWT in a single step, as we did before.

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