it is argued that statistical fluctuations around the canonical ensemble can give rise to the behavior of wave-function collapse, of the kind discussed here, both energy-driven and CSL-type mass-density-driven collapse so that, with the latter, comes the Born probability interpretation of the algebra. The Hamiltonian needed for this theory to work is not provided but, as the argument progresses, its necessary features are delimited.

See also: Quantum Mechanics: Foundations.

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Quantum Mechanics: Weak Measurements

L Diósi, Research Institute for Particle and Nuclear Physics, Budapest, Hungary

Introduction

In quantum theory, the mean value of a certain observable $\hat{A}$ in a (pure) quantum state $|i\rangle$ is defined by the quadratic form:

$$\langle \hat{A} \rangle_i =: \langle i|\hat{A}|i\rangle \qquad [1]$$

Here $\hat{A}$ is Hermitian operator on the Hilbert space $\mathcal{H}$ of states. We use Dirac formalism. The above mean is interpreted statistically. No other forms had been known to possess a statistical interpretation in standard quantum theory. One can, nonetheless, try to extend the notion of mean for normalized bilinear expressions (Aharonov *et al.* 1988):

$$A_{\mathrm{w}} =: \frac{\langle f|\hat{A}|i\rangle}{\langle f|i\rangle} \qquad [2]$$

However unusual is this structure, standard quantum theory provides a plausible statistical interpretation for it, too. The two pure states $|i\rangle, |f\rangle$ play the roles of the prepared initial and the postselected final states, respectively. The statistical interpretation relies upon the concept of weak measurement. In a single weak measurement, the notorious decoherence is chosen asymptotically small. In physical terms, the coupling between the measured state and the meter is assumed asymptotically weak. The novel mean value [2] is called the (complex) weak value.

The concept of quantum weak measurement (Aharonov *et al.* 1988) provides particular

conclusions on postselected ensembles. Weak measurements have been instrumental in the interpretation of time-continuous quantum measurements on single states as well. Yet, weak measurement itself can properly be illuminated in the context of classical statistics. Classical weak measurement as well as postselection and time-continuous measurement are straightforward concepts leading to conclusions that are natural in classical statistics. In quantum context, the case is radically different and certain paradoxical conclusions follow from weak measurements. Therefore, we first introduce the classical notion of weak measurement on postselected ensembles and, alternatively, in time-continuous measurement on a single state. Certain idioms from statistical physics will be borrowed and certain not genuinely quantum notions from quantum theory will be anticipated. The quantum counterpart of weak measurement, postselection, and continuous measurement will be presented afterwards. The apparent redundancy of the parallel presentations is of reason: the reader can separate what is common in classical and quantum weak measurements from what is genuinely quantum.

Classical Weak Measurement

Given a normalized probability density $\rho(X)$ over the phase space $\{X\}$, which we call the state, the mean value of a real function $A(X)$ is defined as

$$\langle A \rangle_\rho =: \int \mathrm{d}X \, A\rho \qquad [3]$$

Let the outcome of an (unbiased) measurement of A be denoted by a. Its stochastic expectation value $E[a]$ coincides with the mean [3]:

$$E[a] = \langle A \rangle_\rho \qquad [4]$$

Performing a large number N of independent measurements of A on the elements of the ensemble of identically prepared states, the arithmetic mean $\bar{a}$ of the outcomes yields a reliable estimate of $E[a]$ and, this way, of the theoretical mean $\langle A \rangle_\rho$.

Suppose, for concreteness, the measurement outcome a is subject to a Gaussian stochastic error of standard dispersion $\sigma > 0$. The probability distribution of a and the update of the state corresponding to the Bayesian inference are described as

$$p(a) = \langle G_\sigma(a - A) \rangle_\rho \qquad [5]$$

$$\rho \to \frac{1}{p(a)} G_\sigma(a - A)\rho \qquad [6]$$

respectively. Here G_σ is the central Gaussian distribution of variance σ. Note that, as expected, eqn [5] implies eqn [4]. Nonzero σ means that the measurement is nonideal, yet the expectation value $E[a]$ remains calculable reliably if the statistics N is suitably large.

Suppose the spread of A in state ρ is finite:

$$\Delta_\rho^2 A =: \langle A^2 \rangle_\rho - \langle A \rangle_\rho^2 < \infty \qquad [7]$$

Weak measurement will be defined in the asymptotic limit (eqns [8] and [9]) where both the stochastic error of the measurement and the measurement statistics go to infinity. It is crucial that their rate is kept constant:

$$\sigma, N \to \infty \qquad [8]$$

$$\Delta^2 =: \frac{\sigma^2}{N} = \text{const.} \qquad [9]$$

Obviously for asymptotically large σ, the precision of individual measurements becomes extremely weak. This incapacity is fully compensated by the asymptotically large statistics N. In the weak measurement limit (eqns [8] and [9]), the probability distribution p_w of the arithmetic mean $\bar{a}$ of the N independent outcomes converges to a Gaussian distribution:

$$p_\mathrm{w}(\bar{a}) \to G_\Delta\left(\bar{a} - \langle A \rangle_\rho\right) \qquad [10]$$

The Gaussian is centered at the mean $\langle A \rangle_\rho$, and the variance of the Gaussian is given by the constant rate [9]. Consequently, the mean [3] is reliably calculable on a statistics N growing like $\sim \sigma^2$.

With an eye on quantum theory, we consider two situations – postselection and time-continuous measurement – of weak measurement in classical statistics.

Postselection

For the preselected state ρ, we introduce postselection via the real function $\Pi(X)$, where $0 \le \Pi \le 1$. The postselected mean value of a certain real function $A(X)$ is defined by

$$_\Pi\langle A \rangle_\rho =: \frac{\langle \Pi A \rangle_\rho}{\langle \Pi \rangle_\rho} \qquad [11]$$

where $\langle \Pi \rangle_\rho$ is the rate of postselection. Postselection means that after having obtained the outcome a regarding the measurement of A, we measure the function Π, too, in ideal measurement with random outcome π upon which we base the following random decision. With probability π, we include the current a into the statistics and we discard it

with probability $1 - \pi$. Then the coincidence of $E[a]$ and $_\Pi\langle A\rangle_\rho$, as in eqn [4], remains valid:

$$E[a] = {}_\Pi\langle A\rangle_\rho \qquad [12]$$

Therefore, a large ensemble of postselected states allows one to estimate the postselected mean $_\Pi\langle A\rangle_\rho$.

Classical postselection allows introducing the effective postselected state:

$$\rho_\Pi =: \frac{\Pi\rho}{\langle\Pi\rangle_\rho} \qquad [13]$$

Then the postselected mean [11] of A in state ρ can, by eqn [14], be expressed as the common mean of A in the effective postselected state ρ_Π:

$${}_\Pi\langle A\rangle_\rho = \langle A\rangle_{\rho_\Pi} \qquad [14]$$

As we shall see later, quantum postselection is more subtle and cannot be reduced to common statistics, that is, to that without postselection. The quantum counterpart of postselected mean does not exist unless we combine postselection and weak measurement.

Time-Continuous Measurement

For time-continuous measurement, one abandons the ensemble of identical states. One supposes that a single time-dependent state ρ_t is undergoing an infinite sequence of measurements (eqns [5] and [6]) of A employed at times $t = \delta t, t = 2\delta t, t = 3\delta t, \ldots$. The rate $\nu =: 1/\delta t$ goes to infinity together with the mean squared error σ^2. Their rate is kept constant:

$$\sigma, \nu \to \infty \qquad [15]$$

$$g^2 =: \frac{\sigma^2}{\nu} = \text{const.} \qquad [16]$$

In the weak measurement limit (eqns [15] and [16]), the infinite frequent weak measurements of A constitute the model of time-continuous measurement. Even the weak measurements will significantly influence the original state ρ_0, due to the accumulated effect of the infinitely many Bayesian updates [6]. The resulting theory of time-continuous measurement is described by coupled Gaussian processes [17] and [18] for the primitive function α_t of the time-dependent measurement outcome and, respectively, for the time-dependent Bayesian conditional state ρ_t:

$$d\alpha_t = \langle A\rangle_{\rho_t}\, dt + g\, dW_t \qquad [17]$$

$$d\rho_t = g^{-1}\left(A - \langle A\rangle_{\rho_t}\right)\rho_t\, dW_t \qquad [18]$$

Here dW_t is the Itô differential of the Wiener process.

Equations [17] and [18] are the special case of the Kushner–Stratonovich equations of time-continuous Bayesian inference conditioned on the continuous measurement of A yielding the time-dependent outcome value a_t. Formal time derivatives of both sides of eqn [17] yield the heuristic equation

$$a_t = \langle A\rangle_{\rho_t} + g\xi_t \qquad [19]$$

Accordingly, the current measurement outcome is always equal to the current mean plus a term proportional to standard white noise ξ_t. This plausible feature of the model survives in the quantum context as well. As for the other equation [18], it describes the gradual concentration of the distribution ρ_t in such a way that the variance $\Delta_{\rho_t} A$ tends to zero while $\langle A\rangle_{\rho_t}$ tends to a random asymptotic value. The details of the convergence depend on the character of the continuously measured function $A(X)$. Consider a stepwise $A(X)$:

$$A(X) = \sum_\lambda a^\lambda P^\lambda(X) \qquad [20]$$

The real values a^λ are step heights all differing from each other. The indicator functions P^λ take values 0 or 1 and form a complete set of pairwise disjoint functions on the phase space:

$$\sum_\lambda P^\lambda \equiv 1 \qquad [21]$$

$$P^\lambda P^\mu = \delta_{\lambda\mu} P^\lambda \qquad [22]$$

In a single ideal measurement of A, the outcome a is one of the a^λ's singled out at random. The probability distribution of the measurement outcome and the corresponding Bayesian update of the state are given by

$$p^\lambda = \langle P^\lambda\rangle_{\rho_0} \qquad [23]$$

$$\rho_0 \to \frac{1}{p^\lambda}P^\lambda\rho_0 =: \rho^\lambda \qquad [24]$$

respectively. Equations [17] and [18] of time-continuous measurement are a connatural time-continuous resolution of the "sudden" ideal measurement (eqns [23] and [24]) in a sense that they reproduce it in the limit $t \to \infty$. The states ρ^λ are trivial stationary states of the eqn [18]. It can be shown that they are indeed approached with probability p^λ for $t \to \infty$.

Quantum Weak Measurement

In quantum theory, states in a given complex Hilbert space $\mathcal{H}$ are represented by non-negative density operators $\hat\rho$, normalized by $\text{tr}\,\hat\rho = 1$. Like the

classical states ρ, the quantum state $\hat{\rho}$ is interpreted statistically, referring to an ensemble of states with the same $\hat{\rho}$. Given a Hermitian operator $\hat{A}$, called observable, its theoretical mean value in state $\hat{\rho}$ is defined by

$$\langle \hat{A} \rangle_{\hat{\rho}} = \text{tr}(\hat{A}\hat{\rho}) \qquad [25]$$

Let the outcome of an (unbiased) quantum measurement of $\hat{A}$ be denoted by a. Its stochastic expectation value $E[a]$ coincides with the mean [25]:

$$E[a] = \langle \hat{A} \rangle_{\hat{\rho}} \qquad [26]$$

Performing a large number N of independent measurements of $\hat{A}$ on the elements of the ensemble of identically prepared states, the arithmetic mean $\bar{a}$ of the outcomes yields a reliable estimate of $E[a]$ and, this way, of the theoretical mean $\langle \hat{A} \rangle_{\hat{\rho}}$. If the measurement outcome a contains a Gaussian stochastic error of standard dispersion σ, then the probability distribution of a and the update, called collapse in quantum theory, of the state are described by eqns [27] and [28], respectively. (We adopt the notational convenience of physics literature to omit the unit operator $\hat{I}$ from trivial expressions like $a\hat{I}$.)

$$p(a) = \left\langle G_\sigma(a - \hat{A}) \right\rangle_{\hat{\rho}} \qquad [27]$$

$$\hat{\rho} \rightarrow \frac{1}{p(a)} G_\sigma^{1/2}(a - \hat{A})\hat{\rho}G_\sigma^{1/2}(a - \hat{A}) \qquad [28]$$

Nonzero σ means that the measurement is nonideal, but the expectation value $E[a]$ remains calculable reliably if N is suitably large.

Weak quantum measurement, like its classical counterpart, requires finite spread of the observable $\hat{A}$ on state $\hat{\rho}$:

$$\Delta_{\hat{\rho}}^2\hat{A} =: \langle \hat{A}^2 \rangle_{\hat{\rho}} - \langle \hat{A} \rangle_{\hat{\rho}}^2 < \infty \qquad [29]$$

Weak quantum measurement, too, will be defined in the asymptotic limit [8] introduced for classical weak measurement. Single quantum measurements can no more distinguish between the eigenvalues of $\hat{A}$. Yet, the expectation value $E[a]$ of the outcome a remains calculable on a statistics N growing like $\sim\sigma^2$.

Both in quantum theory and classical statistics, the emergence of nonideal measurements from ideal ones is guaranteed by general theorems. For completeness of this article, we prove the emergence of the nonideal quantum measurement (eqns [27] and [28]) from the standard von Neumann theory of ideal quantum measurements (von Neumann 1955). The source of the statistical error of dispersion σ is associated with the state $\hat{\rho}_M$ in the complex

Hilbert space $\mathcal{L}^2$ of a hypothetic meter. Suppose $R \in (-\infty, \infty)$ is the position of the "pointer." Let its initial state $\hat{\rho}_M$ be a pure central Gaussian state of width σ; then the density operator $\hat{\rho}_M$ in Dirac position basis takes the form

$$\hat{\rho}_M = \int dR \int dR' G_\sigma^{1/2}(R)G_\sigma^{1/2}(R')|R\rangle\langle R'| \qquad [30]$$

We are looking for a certain dynamical interaction to transmit the "value" of the observable $\hat{A}$ onto the pointer position $\hat{R}$. To model the interaction, we define the unitary transformation [31] to act on the tensor space $\mathcal{H} \otimes \mathcal{L}^2$:

$$\hat{U} = \exp(i\hat{A} \otimes \hat{K}) \qquad [31]$$

Here $\hat{K}$ is the canonical momentum operator conjugated to $\hat{R}$:

$$\exp(ia\hat{K})|R\rangle = |R + a\rangle \qquad [32]$$

The unitary operator $\hat{U}$ transforms the initial uncorrelated quantum state into the desired correlated composite state:

$$\hat{\Sigma} =: \hat{U}\hat{\rho} \otimes \hat{\rho}_M \hat{U}^\dagger \qquad [33]$$

Equations [30]–[33] yield the expression [34] for the state $\hat{\Sigma}$:

$$\hat{\Sigma} = \int dR \int dR' G_\sigma^{1/2}(R - \hat{A})\hat{\rho}G_\sigma^{1/2}$$
$$\times (R' - \hat{A}) \otimes |R\rangle\langle R'| \qquad [34]$$

Let us write the pointer's coordinate operator $\hat{R}$ into the standard form [35] in Dirac position basis:

$$\hat{R} = \int da|a\rangle\langle a| \qquad [35]$$

The notation anticipates that, when pointer $\hat{R}$ is measured ideally, the outcome a plays the role of the nonideally measured value of the observable $\hat{A}$. Indeed, let us consider the ideal von Neumann measurement of the pointer position on the correlated composite state $\hat{\Sigma}$. The probability of the outcome a and the collapse of the composite state are given by the following standard equations:

$$p(a) = \text{tr}\left[(\hat{I} \otimes |a\rangle\langle a|)\hat{\Sigma} \right] \qquad [36]$$

$$\hat{\Sigma} \rightarrow \frac{1}{p(a)}\left[(\hat{I} \otimes |a\rangle\langle a|)\hat{\Sigma}(\hat{I} \otimes |a\rangle\langle a|) \right] \qquad [37]$$

respectively. We insert eqn [34] into eqns [36] and [37]. Furthermore, we take the trace over $\mathcal{L}^2$ of both sides of eqn [37]. In such a way, as expected, eqns [36] and [37] of ideal measurement of $\hat{R}$ yield the

earlier postulated eqns [27] and [28] of nonideal measurement of $\hat{A}$.

Quantum Postselection

A quantum postselection is defined by a Hermitian operator satisfying $\hat{0} \le \hat{\Pi} \le \hat{I}$. The corresponding postselected mean value of a certain observable $\hat{A}$ is defined by

$$_{\hat{\Pi}}\langle \hat{A}\rangle_{\hat{\rho}} =: \operatorname{Re} \frac{\langle \hat{\Pi}\hat{A}\rangle_{\hat{\rho}}}{\langle \hat{\Pi}\rangle_{\hat{\rho}}} \qquad [38]$$

The denominator $\langle \hat{\Pi}\rangle_{\hat{\rho}}$ is the rate of quantum postselection. Quantum postselection means that after the measurement of $\hat{A}$, we measure the observable $\hat{\Pi}$ in ideal quantum measurement and we make a statistical decision on the basis of the outcome π. With probability π, we include the case in question into the statistics while we discard it with probability $1 - \pi$. By analogy with the classical case [12], one may ask whether the stochastic expectation value $E[a]$ of the postselected measurement outcome does coincide with

$$E[a] \stackrel{?}{=} {}_{\hat{\Pi}}\langle \hat{A}\rangle_{\hat{\rho}} \qquad [39]$$

Contrary to the classical case, the quantum equation [39] does not hold. The quantum counterparts of classical equations [12]–[14] do not exist at all. Nonetheless, the quantum postselected mean $_{\hat{\Pi}}\langle \hat{A}\rangle_{\hat{\rho}}$ possesses statistical interpretation although restricted to the context of weak quantum measurements. In the weak measurement limit (eqns [8] and [9]), a postselected analog of classical equation [10] holds for the arithmetic mean $\bar{a}$ of postselected weak quantum measurements:

$$p_w(\bar{a}) \to G_\Delta\left(\bar{a} - {}_{\hat{\Pi}}\langle \hat{A}\rangle_{\hat{\rho}}\right) \qquad [40]$$

The Gaussian is centered at the postselected mean $_{\hat{\Pi}}\langle \hat{A}\rangle_{\hat{\rho}}$, and the variance of the Gaussian is given by the constant rate [9]. Consequently, the mean [38] becomes calculable on a statistics N growing like $\sim \sigma^2$.

Since the statistical interpretation of the postselected quantum mean [38] is only possible for weak measurements, therefore $_{\hat{\Pi}}\langle \hat{A}\rangle_{\hat{\rho}}$ is called the (real) weak value of $\hat{A}$. Consider the special case when both the state $\hat{\rho} = |i\rangle\langle i|$ and the postselected operator $\hat{\Pi} = |f\rangle\langle f|$ are pure states. Then the weak value $_{\hat{\Pi}}\langle \hat{A}\rangle_{\hat{\rho}}$ takes, in usual notations, a particular form [41] yielding the real part of the complex weak value A_w [1]:

$$_{f}\langle \hat{A}\rangle_{i} =: \operatorname{Re} \frac{\langle f|\hat{A}|i\rangle}{\langle f|i\rangle} \qquad [41]$$

The interpretation of postselection itself reduces to a simple procedure. One performs the von Neumann ideal measurement of the Hermitian projector $|f\rangle\langle f|$, then includes the case if the outcome is 1 and discards it if the outcome is 0. The rate of postselection is $|\langle f|i\rangle|^2$. We note that a certain statistical interpretation of $\operatorname{Im} A_w$, too, exists although it relies upon the details of the "meter."

We outline a heuristic proof of the central equation [40]. One considers the nonideal measurement (eqns [27] and [28]) of $\hat{A}$ followed by the ideal measurement of $\hat{\Pi}$. Then the joint distribution of the corresponding outcomes is given by eqn [42]. The probability distribution of the postselected outcomes a is defined by eqn [43], and takes the concrete form [44]. The constant $\mathcal{N}$ assures normalization:

$$p(\pi, a) = \operatorname{tr}\left(\delta(\pi - \hat{\Pi}) G_\sigma^{1/2}(a - \hat{A}) \hat{\rho} G_\sigma^{1/2}(a - \hat{A})\right) \quad [42]$$

$$p(a) =: \frac{1}{\mathcal{N}} \int \pi p(\pi, a)\, d\pi \qquad [43]$$

$$p(a) =: \frac{1}{\mathcal{N}} \left\langle G_\sigma^{1/2}(a - \hat{A}) \hat{\Pi} G_\sigma^{1/2}(a - \hat{A})\right\rangle_{\hat{\rho}} \qquad [44]$$

Suppose, for simplicity, that $\hat{A}$ is bounded. When $\sigma \to \infty$, eqn [44] yields the first two moments of the outcome a:

$$E[a] \to {}_{\hat{\Pi}}\langle \hat{A}\rangle_{\hat{\rho}} \qquad [45]$$

$$E[a^2] \sim \sigma^2 \qquad [46]$$

Hence, by virtue of the central limit theorem, the probability distribution [40] follows for the average $\bar{a}$ of postselected outcomes in the weak measurement limit (eqns [8] and [9]).

Quantum Weak-Value Anomaly

Unlike in classical postselection, effective postselected quantum states cannot be introduced. We can ask whether eqn [47] defines a correct postselected quantum state:

$$\hat{\rho}_{\hat{\Pi}}^? =: \operatorname{Herm} \frac{\hat{\Pi}\hat{\rho}}{\langle \hat{\Pi}\rangle_{\hat{\rho}}} \qquad [47]$$

This pseudo-state satisfies the quantum counterpart of the classical equation [14]:

$$_{\hat{\Pi}}\langle \hat{A}\rangle_{\hat{\rho}} = \operatorname{tr}\left(\hat{A}\hat{\rho}_{\hat{\Pi}}^?\right) \qquad [48]$$

In general, however, the operator $\hat{\rho}_{\hat{\Pi}}^?$ is not a density operator since it may be indefinite. Therefore, eqn [47] does not define a quantum state. Equation [48] does not guarantee that the quantum weak value

$_{\hat{\Pi}}\langle\hat{A}\rangle_{\hat{\rho}}$ lies within the range of the eigenvalues of the observable $\hat{A}$.

Let us see a simple example for such anomalous weak values in the two-dimensional Hilbert space. Consider the pure initial state given by eqn [49] and the postselected pure state by eqn [50], where $\phi \in [0, \pi]$ is a certain angular parameter.

$$|i\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\phi/2} \\ e^{-i\phi/2} \end{bmatrix} \qquad [49]$$

$$|f\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\phi/2} \\ e^{i\phi/2} \end{bmatrix} \qquad [50]$$

The probability of successful postselection is $\cos^2\phi$. If $\phi \neq \pi/2$, then the postselected pseudo-state follows from eqn [47]:

$$\hat{\rho}_{\hat{\Pi}}^{?} = \frac{1}{2} \begin{bmatrix} 1 & \cos^{-1}\phi \\ \cos^{-1}\phi & 1 \end{bmatrix} \qquad [51]$$

This matrix is indefinite unless $\phi = 0$, its two eigenvalues are $1 \pm \cos^{-1}\phi$. The smaller the postselection rate $\cos^2\phi$, the larger is the violation of the positivity of the pseudo-density operator. Let the weakly measured observable take the form

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad [52]$$

Its eigenvalues are ± 1. We express its weak value from eqns [41], [49], and [50] or, equivalently, from eqns [48] and [51]:

$$_{f}\langle\hat{A}\rangle_i = \frac{1}{\cos\phi} \qquad [53]$$

This weak value of $\hat{A}$ lies outside the range of the eigenvalues of $\hat{A}$. The anomaly can be arbitrarily large if the rate $\cos^2\phi$ of postselection decreases.

Striking consequences follow from this anomaly if we turn to the statistical interpretation. For concreteness, suppose $\phi = 2\pi/3$ so that $_{f}\langle\hat{A}\rangle_i = 2$. On average, 75% of the statistics N will be lost in postselection. We learnt from eqn [40] that the arithmetic mean $\bar{a}$ of the postselected outcomes of independent weak measurements converges stochastically to the weak value upto the Gaussian fluctuation Δ, as expressed symbolically by

$$\bar{a} = 2 \pm \Delta \qquad [54]$$

Let us approximate the asymptotically large error σ of our weak measurements by $\sigma = 10$ which is already well beyond the scale of the eigenvalues ± 1 of the observable $\hat{A}$. The Gaussian error Δ derives from eqn [9] after replacing N by the size of the postselected statistics which is approximately $N/4$:

$$\Delta^2 = 400/N \qquad [55]$$

Accordingly, if $N = 3600$ independent quantum measurements of precision $\sigma = 10$ are performed regarding the observable $\hat{A}$, then the arithmetic mean $\bar{a}$ of the ~ 900 postselected outcomes a will be 2 ± 0.33. This exceeds significantly the largest eigenvalue of the measured observable $\hat{A}$. Quantum postselection appears to bias the otherwise unbiased nonideal weak measurements.

Quantum Time-Continuous Measurement

The mathematical construction of time-continuous quantum measurement is similar to the classical one. We consider the weak measurement limit (eqns [15] and [16]) of an infinite sequence of nonideal quantum measurements of the observable $\hat{A}$ at $t = \delta t, 2\delta t, \ldots$, on the time-dependent state $\hat{\rho}_t$. The resulting theory of time-continuous quantum measurement is incorporated in the coupled stochastic equations [56] and [57] for the primitive function α_t of the time-dependent outcome and the conditional time-dependent state $\hat{\rho}_t$, respectively (Diósi 1988):

$$d\alpha_t = \langle\hat{A}\rangle_{\hat{\rho}_t} dt + g dW_t \qquad [56]$$

$$d\hat{\rho}_t = -\frac{1}{8} g^{-2} [\hat{A}, [\hat{A}, \hat{\rho}_t]] dt \\ + g^{-1} \mathrm{Herm}\left(\hat{A} - \langle\hat{A}\rangle_{\hat{\rho}_t}\right) \hat{\rho}_t dW_t \qquad [57]$$

Equation [56] and its classical counterpart [17] are perfectly similar. There is a remarkable difference between eqn [57] and its classical counterpart [18]. In the latter, the stochastic average of the state is constant: $E[d\rho_t] = 0$, expressing the fact that classical measurements do not alter the original ensemble if we "ignore" the outcomes of the measurements. On the contrary, quantum measurements introduce irreversible changes to the original ensemble, a phenomenon called decoherence in the physics literature. Equation [57] implies the closed linear first-order differential equation [58] for the stochastic average of the quantum state $\hat{\rho}_t$ under time-continuous measurement of the observable $\hat{A}$:

$$\frac{dE[\hat{\rho}_t]}{dt} = -\frac{1}{8} g^{-2} [\hat{A}, [\hat{A}, E[\hat{\rho}_t]]] \qquad [58]$$

This is the basic irreversible equation to model the gradual loss of quantum coherence (decoherence) under time-continuous measurement. In fact, the very equation models decoherence under the influence of a large class of interactions, for example, with thermal reservoirs or complex environments. In

two-dimensional Hilbert space, for instance, we can consider the initial pure state $\langle i| =: [\cos\phi, \sin\phi]$ and the time-continuous measurement of the diagonal observable [59] on it. The solution of eqn [58] is given by eqn [60]:

$$\hat{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad [59]$$

$$E[\hat{\rho}_t] = \begin{bmatrix} \cos^2\phi & e^{-t/4g^2}\cos\phi\sin\phi \\ e^{-t/4g^2}\cos\phi\sin\phi & \sin^2\phi \end{bmatrix} \qquad [60]$$

The off-diagonal elements of this density matrix go to zero, that is, the coherent superposition represented by the initial pure state becomes an incoherent mixture represented by the diagonal density matrix $\hat{\rho}_\infty$.

Apart from the phenomenon of decoherence, the stochastic equations show remarkable similarity with the classical equations of time-continuous measurement. The heuristic form of eqn [56] is eqn [61] of invariable interpretation with respect to the classical equation [19]:

$$a_t = \langle\hat{A}\rangle_{\hat{\rho}_t} + g\xi_t \qquad [61]$$

Equation [57] describes what is called the time-continuous collapse of the quantum state under time-continuous quantum measurement of $\hat{A}$. For concreteness, we assume discrete spectrum for $\hat{A}$ and consider the spectral expansion

$$\hat{A} = \sum_\lambda a^\lambda \hat{P}^\lambda \qquad [62]$$

The real values a^λ are nondegenerate eigenvalues. The Hermitian projectors $\hat{P}^\lambda$ form a complete orthogonal set:

$$\sum_\lambda \hat{P}^\lambda \equiv \hat{I} \qquad [63]$$

$$\hat{P}^\lambda \hat{P}^\mu = \delta_{\lambda\mu}\hat{P}^\lambda \qquad [64]$$

In a single ideal measurement of $\hat{A}$, the outcome a is one of the a^λ's singled out at random. The probability distribution of the measurement outcome and the corresponding collapse of the state are given by

$$p^\lambda = \langle\hat{P}^\lambda\rangle_{\hat{\rho}_0} \qquad [65]$$

$$\hat{\rho}_0 \rightarrow \frac{1}{p^\lambda}\hat{P}^\lambda\hat{\rho}_0\hat{P}^\lambda =: \hat{\rho}^\lambda \qquad [66]$$

respectively. Equations [56] and [57] of continuous measurements are an obvious time-continuous

resolution of the "sudden" ideal quantum measurement (eqns [65] and [66]) in a sense that they reproduce it in the limit $t \rightarrow \infty$. The states $\hat{\rho}^\lambda$ are stationary states of eqn [57]. It can be shown that they are indeed approached with probability p^λ for $t \rightarrow \infty$ (Gisin 1984).

Related Contexts

In addition to the two particular examples as in postselection and in time-continuous measurement, respectively, presented above, the weak measurement limit itself has further variants. A most natural example is the usual thermodynamic limit in standard statistical physics. Then weak measurements concern a certain additive microscopic observable (e.g., the spin) of each constituent and the weak value represents the corresponding additive macroscopic parameter (e.g., the magnetization) in the infinite volume limit. This example indicates that weak values have natural interpretation despite the apparent artificial conditions of their definition. It is important that the weak value, with or without postselection, plays the physical role similar to that of the common mean $\langle\hat{A}\rangle_{\hat{\rho}}$. If, between their pre- and postselection, the states $\hat{\rho}$ become weakly coupled with the state of another quantum system via the observable $\hat{A}$, their average influence will be as if $\hat{A}$ took the weak value $_{\hat{\Pi}}\langle\hat{A}\rangle_{\hat{\rho}}$. Weak measurements also open a specific loophole to circumvent quantum limitations related to the irreversible disturbances that quantum measurements cause to the measured state. Noncommuting observables become simultaneously measurable in the weak limit: simultaneous weak values of noncommuting observables will exist.

Literally, weak measurement had been coined in 1988 for quantum measurements with (pre- and) postselection, and became the tool of a certain time-symmetric statistical interpretation of quantum states. Foundational applications target the paradoxical problem of pre- and retrodiction in quantum theory. In a broad sense, however, the very principle of weak measurement encapsulates the trade between asymptotically weak precision and asymptotically large statistics. Its relevance in different fields has not yet been fully explored and a growing number of foundational, theoretical, and experimental applications are being considered in the literature – predominantly in the context of quantum physics. Since specialized monographs or textbooks on quantum weak measurement are not yet available, the reader is mostly referred to research articles, like the recent one by Aharonov and Botero (2005), covering many topics of postselected quantum weak values.

Nomenclature

a	measurement outcome
$\bar{a}$	arithmetic mean of measurement outcomes
$\hat{A}$	Hermitian operator, quantum observable
$A(X)$	real phase-space function
$E[\ldots]$	stochastic expectation value
$\langle f|\hat{A}|i\rangle$	matrix element
$\langle f|i\rangle$	inner product
$\mathcal{H}$	Hilbert space
$\mathcal{L}^2$	space of Lebesgue square-integrable complex functions
p	probability distribution
tr	trace
$\hat{U}$	unitary operator
W_t	Wiener process
ξ_t	white noise process
$_\Pi\langle\ldots\rangle_\rho$	postselected mean value
$\hat{\rho}$	density operator
$\rho(X)$	phase-space distribution
$\otimes$	direct product
$\dagger$	operator adjoint
$|\ldots\rangle$	state vector
$\langle\ldots|$	adjoint state vector
$\langle\ldots\rangle_\rho$	mean value

Further Reading

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Quantum *n*-Body Problem

R G Littlejohn, University of California at Berkeley, Berkeley, CA, USA

Introduction

This article concerns the nonrelativistic quantum mechanics of isolated systems of n particles interacting by means of a scalar potential, what we shall call the "quantum n-body problem." Such systems are described by the kinetic-plus-potential Hamiltonian,

$$H = T + V = \sum_{\alpha=1}^{n} \frac{|\boldsymbol{P}_\alpha|^2}{2m_\alpha} + V(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_n) \qquad [1]$$

where $\boldsymbol{R}_\alpha, \boldsymbol{P}_\alpha, \alpha = 1, \ldots, n$ are the positions and momenta of the n particles in three-dimensional space, m_α are the masses, and V is the potential energy. This Hamiltonian also occurs in the "classical n-body problem," in which V is usually assumed to consist of the sum of the pairwise gravitational interactions of the particles. In this article, we shall only assume that V (hence H) is invariant under translations, proper rotations, parity, and permutations of identical particles. The Hamiltonian H is also invariant under time reversal. This Hamiltonian describes the dynamics of isolated atoms, molecules, and nuclei, with varying degrees of approximation, including the case of molecules in the Born–Oppenheimer approximation, in which V is the Born–Oppenheimer potential. We shall ignore the spin of the particles, and treat the wave function Ψ as a scalar. We assume that Ψ is an eigenfunction of H, $H\Psi = E\Psi$. In practice, the value of n typically ranges from 2 to several hundred. Often the cases $n = 3$ and $n = 4$ are of special interest. In this article, we shall assume that $n \geq 3$, since $n = 2$ is the trivial case of central-force motion. The quantum n-body problem is not to be confused with the "quantum