Comment on 'Underlining some limitations of the statistical formalism in quantum mechanics' by Fratini and Hayrapetyan

András Bodor[1]
Department of Physics of Complex Systems, Eötvös University
H-1117 Budapest, Pázmány Péter sétány 1/A, Hungary

Lajos Diósi[2*]
Wigner Research Center for Physics
H-1525 Budapest 114, POB 49, Hungary
(Dated: February 14, 2012)

We point out that Fratini and Hayrapetyan ignored the randomness of mixing, a basic request to prepare a statistical ensemble rather than an ensemble in general. Their analysis is irrelevant for standard statistical ensembles, their conclusions about the limitations of standard theory become unjustified.

Fratini and Hayrapetyan (FH) claimed recently that "ensembles of states which are represented by the same density operator in quantum mechanics can behave differently in experiments" [1]. To prove this, they choose two totally unpolarized ensembles $\mathcal{A}$ and $\mathcal{B}$ of spin-1/2 particles. $\mathcal{A}$ is made of an equal number $N/2$ of particles having full polarization $\hbar/2$ and $-\hbar/2$, respectively, along the x axis. $\mathcal{B}$ is prepared the same way just using particles fully polarized along the z axis. FH claim correctly that $\mathcal{A}$ and $\mathcal{B}$ behave differently in experiments. But their conclusion that "the statistical description of an ensemble of states, as given by its density matrix, ..., should be considered as incomplete" is, to our opinion, incorrect.

FH are apparently unaware of their ensembles are different from what standard statistics as well as standard quantum mechanics understand as *statistical ensembles*. FH prepare the ensemble $\mathcal{A}$ of an *equal number* $N/2$ of $\pm\hbar/2$ polarizations which is obviously no legitimate construction for a *statistical ensemble*. Statistical ensembles must consist of *independent* states, this requires *random* mixing [4]. Statistical formalism will then attribute a *unique* statistical ensemble to the unpolarized state, different from both ensembles $\mathcal{A}$ and $\mathcal{B}$ of FH.

Totally unpolarized statistical ensembles are extensively used in quantum information theory. If the statistical (i.e.: random) mixture of $|S_z, \pm 1\rangle$ were distinguishable from the statistical mixture of $|S_x, \pm 1\rangle$ then standard information and communication protocols [2, 3] would not work at all — they are just based on the fact that these two statistical ensembles are identical.

The mistake of FH has nothing to do with quantum mechanics rather it is a classical statistical misconception. If we mix $N/2$ black balls and $N/2$ white balls in an urn then the emerging ensemble is not a statistical ensemble. To prepare the statistical ensemble, we should have mixed N black or white balls *chosen at random* each. The resulting statistical ensemble will contain black and white balls according to the balanced binomial distribution:

$$p(N_{black}) = \binom{N}{N_{black}} 2^{-N}.$$

This statistical ensemble is what standard statistics assigns to the uniform distribution of black and white balls. The urn containing an equal number of them is not a statistical ensemble in the above sense, it does not represent the uniform probability distribution. To be sure, the distribution of black balls is peaked at $N/2$:

$$p(N_{black}) = \delta_{N_{black},N/2}$$

and this is sharply different from the previous correct one. Therefore, in the spirit of [1], we could have proposed similar (mistaken) limitations of classical statistical formalism: Since two ensembles with the same one-ball probability distribution behave differently in experiments, the ensembles can not be fully characterized by their one-ball distributions. We would obviously be wrong: one of the ensembles in question is not a legitimate statistical ensemble, its experimental difference from the correct statistical ensemble is neither a surprise nor a point to claim limitations of standard statistics.

Quite similarly, neither $\mathcal{A}$ nor $\mathcal{B}$ are correct statistical ensembles, their experimental difference from each other (and from the correct statistical ensemble!) is not a surprise, the ensembles $\mathcal{A}, \mathcal{B}$ don't provide any limitation or

*Electronic address: diosi.lajos@wigner.mta.hu; URL: www.rmki.kfki.hu/~diosi

alteration to the standard statistical formalism nor to the role of the density operator. Had FH used the correct mixing to construct the two ensembles they would have left with no measurable difference between them which fact is fundamental in the quantum theory and is particularly well understood in quantum informatics.

To characterize the similarity and difference of the two ensembles $\mathcal{A}$ and $\mathcal{B}$, let us note, that the one-particle density matrix of both ensembles is identical to $\varrho_1 = \frac{1}{2}I$, i.e., 1/2 times the unit matrix: $\varrho_1^{\mathcal{A}} = \varrho_1^{\mathcal{B}} = \frac{1}{2}I$. It represents the completely unpolarized state, indeed, as it should. The two-particle density matrix of the correct statistical ensemble should be [5]

$$\varrho_2 = \frac{1}{4} I \otimes I.$$

The two-particle density matrices of the ensembles of FH differ from each other:

$$\varrho_2^{\mathcal{A}} = \frac{\frac{1}{2}N(\frac{1}{2}N-1)}{N(N-1)} \left(\left|S_x^{++}\right\rangle \left\langle S_x^{++}\right| + \left|S_x^{--}\right\rangle \left\langle S_x^{--}\right| \right) + \frac{N^2/4}{N(N-1)} \left(\left|S_x^{+-}\right\rangle \left\langle S_x^{+-}\right| + \left|S_x^{-+}\right\rangle \left\langle S_x^{-+}\right| \right)$$

$$\begin{aligned}
\varrho_2^{\mathcal{B}} =& \frac{\frac{1}{2}N(\frac{1}{2}N-1)}{N(N-1)} \left(\left|S_z^{++}\right\rangle \left\langle S_z^{++}\right| + \left|S_z^{--}\right\rangle \left\langle S_z^{--}\right| \right) + \frac{N^2/4}{N(N-1)} \left(\left|S_z^{+-}\right\rangle \left\langle S_z^{+-}\right| + \left|S_z^{-+}\right\rangle \left\langle S_z^{-+}\right| \right) \\
=& \frac{\frac{1}{2}N(\frac{1}{2}N-1)+N^2/4}{2N(N-1)} \left(\left|S_x^{++}\right\rangle \left\langle S_x^{++}\right| + \left|S_x^{+-}\right\rangle \left\langle S_x^{+-}\right| + \left|S_x^{-+}\right\rangle \left\langle S_x^{-+}\right| + \left|S_x^{--}\right\rangle \left\langle S_x^{--}\right| \right) + \\
& \frac{\frac{1}{2}N(\frac{1}{2}N-1)-N^2/4}{2N(N-1)} \left(\left|S_x^{++}\right\rangle \left\langle S_x^{--}\right| + \left|S_x^{--}\right\rangle \left\langle S_x^{++}\right| + \left|S_x^{+-}\right\rangle \left\langle S_x^{-+}\right| + \left|S_x^{-+}\right\rangle \left\langle S_x^{+-}\right| \right)
\end{aligned}$$

where $|S_x^{++}\rangle = |S_x, +1\rangle \otimes |S_x, +1\rangle$, $|S_x^{+-}\rangle = |S_x, +1\rangle \otimes |S_x, -1\rangle$, e.t.c., stand for the corresponding two-particle states. The coefficients $\frac{\frac{1}{2}N(\frac{1}{2}N-1)}{N(N-1)}$ and $\frac{N^2/4}{N(N-1)}$ are 1/2 times the relative frequency of particle pairs with respectively parallel and anti-parallel polarization. In the second equation we used the identites like, e.g.:

$$\left|S_z^{++}\right\rangle \left\langle S_z^{++}\right| = \frac{1}{2} \left(\left|S_x^{++}\right\rangle \left\langle S_x^{++}\right| + \left|S_x^{--}\right\rangle \left\langle S_x^{--}\right| \right).$$

These density matrices also differ from the form $\varrho_2 = \frac{1}{4} I \otimes I$ of the correct statistical ensemble. While the one-particle density matrices in $\mathcal{A}$,$\mathcal{B}$ and in the correct statistical ensemble are identical, $\varrho_1^{\mathcal{A}} = \varrho_1^{\mathcal{B}} = \varrho_1$, this is not true for the two- and multi-particle density matrices. The variances, considered by FH, are intrinsically multi-particle quantities, so the N-particle density matrix must be used in their computation. Therefore the outcomes for $\mathcal{A}$, for $\mathcal{B}$ and for the standard statistical ensemble will differ from each other exactly as standard quantum mechanics predicts.

The authors thank Tamás Geszti for his valuable remarks. This work was supported by the Hungarian OTKA Grant No. 75129.

[1] Fratini F and Hayrapetyan A G 2011 *Phys. Scr.* **84** 035008
[2] Nielsen M A Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge University Press)
[3] Diósi L 2011 *Short Course on Quantum Information Theory* (Berlin: Springer)
[4] See paragraphs 2.2,4.2 in [3] and Exercise 2.2 in particular.
[5] If $\rho = \rho_1$ is the (one-particle) density matrix, the multi-particle density matrices of the corresponding statistical ensemble are $\rho_2 = \rho \otimes \rho$, $\rho_3 = \rho \otimes \rho \otimes \rho$, e.t.c..