The covariant Langevin equation

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(Dated: October 27, 2023)

The covariant form of the multivariable diffusion-drift process is described by the covariant Fokker–Planck equation using the standard toolbox of Riemann geometry. The covariant form of the equivalent Langevin stochastic equation is long sought after. We start from the simplest covariant Stratonovich stochastic differential equation depending on the local orthogonal frame (cf. vielbein). We show that this stochastic differential equation (Graham, 1977) becomes the desired covariant Langevin equation but only if we impose an additional covariant constraint: the vectors of the frame must be divergence-free.

I. FOKKER–PLANCK VS LANGEVIN EQUATION

The most common irreversible phenomena in physics are diffusive ones, modelled mathematically by the Fokker–Planck equation (FPE). If $P(x)$ is the normalized probability distribution of an abstract particle of coordinates $x = (x^1, x^2, \ldots, x^n)$ then the FPE reads [1]:

$$\frac{\partial P}{\partial t} = \tfrac{1}{2}\left(g^{ab}P\right)_{,ab} - \left(\widetilde{V}^a P\right)_{,a} \ . \tag{1}$$

The positive matrix $g^{ab}(x)$ is the matrix of diffusion and $\widetilde{V}^a(x)$ is the expectation value of the particle's velocity at position x:

$$\widetilde{V}^a(x) = \langle dx^a/dt\rangle_{P(z)=\delta(z-x)} \ . \tag{2}$$

With one eye on forthcoming considerations of covariance, we use the formalism of general relativity: summation of identical labels is understood, partial derivatives $\partial/\partial x^a$ are denoted by lower label a with the comma.

The same diffusive phenomena can be represented by stochastic processes x_t governed by the Langevin stochastic differential equation (SDE). The equivalence between the FPE and the SDE means the following relationship:

$$P_t(x) = \left\langle \delta(x - x_t)\right\rangle \ . \tag{3}$$

With n independent Wiener processes W^A, the Ito from of the Langevin SDE of x_t is the following [1]:

$$dx^a = e^a_A dW^A + \widetilde{V}^a dt \ , \tag{4}$$

where summation from 1 to n over repeated labels A is understood and the matrices $e^a_A(x)$ satisfy

$$\delta^{AB} e^a_A e^b_B = g^{ab} \ . \tag{5}$$

This condition allows for a local orthogonal gauge-freedom:

$$e^a_A \Rightarrow O^B_A e^a_B \tag{6}$$

with orthogonal matrices $O^B_A(x)$. The form of the SDE (4) is gauge-dependent but the stochastic process x_t is unique.

For completeness, we verify the relationship (3). Suppose it holds at time t, we have to show that $dP_t(x) = \langle d\delta(x - x_t)\rangle$ is satisfied if the l.h.s. is given by the FPE (1) and the r.h.s. is given by the SDE (4). Let us workout the r.h.s.:

$$\begin{aligned}
&\frac{1}{dt}\langle d\delta(x-x_t)\rangle = \\
&= \frac{1}{dt}\langle -\delta_{,a}(x-x_t)dx^a_t + \tfrac{1}{2}\delta_{,ab}(x-x_t)dx^a_t dx^b_t\rangle = \\
&= \langle -\delta_{,a}(x-x_t)V^a(x_t) + \tfrac{1}{2}\delta_{,ab}(x-x_t)g^{ab}(x_t)\rangle = \\
&= -\left(\langle\delta(x-x_t)\rangle V^a(x)\right)_{,a} + \tfrac{1}{2}\left(\langle\delta(x-x_t)\rangle g^{ab}(x)\right)_{,ab} \\
&= -\left(P(x)V^a(x)\right)_{,a} + \tfrac{1}{2}\left(P(x)g^{ab}(x)\right)_{,ab} \ .
\end{aligned} \tag{7}$$

First we calculated $d\delta(x - x_t)$ with the Ito correction, then inserted dx^a from the SDE (4). Next, we moved derivations in front of the expressions so that we could replace the argument x_t of both V^a and of g^{ab} by x, thanks to the δ-function. Finally we inserted our initial assumption that (3) holds at t. The result coincides with $dP_t(x)/dt$ calculated from the FPE (1).

II. COVARIANCE

Neither the FFE (1) nor the Ito–Langevin SDE (4) are covariant under general transformations of the coordinates x^a. The common reason of their non-covariance is the non-covariance of the drift vector (2). For example, if the velocity $\widetilde{V}^a$ vanishes in Euclidean coordinates it becomes non-zero in curvilinear ones.

The desired covariance is easily achieved. We borrow the toolbox of Riemann geometry well-known from general relativity [2]. Accordingly, we impose a Riemann geometry structure on the manifold of coordinates x by identifying the diffusion matrix g^{ab} with the contravariant metric tensor and we introduce the scalar probability density $\rho = P/\sqrt{g}$, of covariant normalization

$$\int \rho(x)\sqrt{g}d^n x = 1 \ . \tag{8}$$

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The covariant form of the FPE (1) follows:

$$\frac{d\rho}{dt} = \tfrac{1}{2} g^{ab}\rho_{;ab} - (V^a\rho)_{;a} \ , \tag{9}$$

where semicolons denote covariant derivatives and V^a is the covariant drift:

$$V^a = \widetilde{V}^a - \frac{1}{2\sqrt{g}}(\sqrt{g}g^{ab})_{,b} \ . \tag{10}$$

As a price of covariance, this velocity *parameter* is different from the true (2), but non-covariant, drift velocity $\widetilde{V}^a$. The covariant V^a is gauge-dependent, it coincides with the true drift $\widetilde{V}^a$ in the harmonic gauge defined just by $(x^a)_{;bc}g^{bc} = (1/\sqrt{g})(\sqrt{g}g^{ab})_{,b} = 0$ for each individual x^a.

Now we propose the covariant Langevin equation. The matrix e^a_A, introduced for the non-covariant Ito–Langevin SDE (4), is standard in Riemann geometry. It is called *frame* (or vielbein, also tetrad in the four-dimensional pseudo-Riemann space of general relativity). The condition (5) is called the frame's orthogonality condition. And now we impose our *new covariant constraint* on the frame. Namely, the covariant divergence of the frame's n orthogonal vectors should vanish each:

$$(e^a_A)_{;a} = 0 \ . \tag{11}$$

The covariant form of the non-covariant Ito–Langevin SDE (4) is, as we prove below, simple enough:

$$dx^a = e^a_A \circ dW^A + V^a dt \ , \tag{12}$$

where $\circ$ means Stratonovich differential instead of Ito's. The r.h.s. of (12) is explicit covariant. This is compatible with the covariance of the l.h.s. since the Stratonovich differentials satisfy the chain rule exactly like common differentials. In our case, if we change the coordinates for y^a then the Stratonovich differentials transform covariantly:

$$dy^a = \frac{\partial y^a}{\partial x_b} dx^b \ . \tag{13}$$

Note that the new constraints (11) restrict the full local orthogonal gauge-freedom (6), yet the choice of the frame is not unique but unique is the stochastic process x_t governed by the Stratonovich–Langevin SDE (12).

We are going to prove that the covariant Stratonovich–Langevin SDE (12) unravels the covariant FPE (9) in the sense of the relationship (3) with $P = \rho\sqrt{g}$. The proof is simple if we go back into the equivalent non-covariant framework since we already proved the relationship between the non-covariant FPE (1) and SDE (4). The missing link is the equivalence between our covariant and non-covariant SDEs. The Ito form of a Stratonovich SDE, like our (12), reads [1]:

$$\begin{aligned} dx^a &= e^a_A dW^A + \frac{1}{2}\delta^{AB}(e^a_A)_{,b} e^b_B dt + V^a dt = \qquad (14)\\ &= e^a_A dW^A + \frac{1}{2}\delta^{AB}(e^a_A)_{,b} e^b_B dt + \widetilde{V}^a - \frac{(\sqrt{g}g^{ab})_{,b}}{2\sqrt{g}} \ . \end{aligned}$$

Observe that the new drift term contains the standard partial derivatives of the frame, not the covariant ones. We are going to work it out:

$$\begin{aligned} \delta^{AB}(e^a_A)_{,b} e^b_B &= g^{ab}_{,b} - e^a_A (e^b_A)_{,b} = \\ &= g^{ab}_{,b} + e^a_A \Gamma^b_{bc} e^c_A = \\ &= g^{ab}_{,b} + g^{ac}\Gamma^b_{bc} = \\ &= \frac{1}{\sqrt{g}}(\sqrt{g}g^{ab})_{,b} \ . \end{aligned} \tag{15}$$

In the four steps we used the orthogonality (5) of the frame, the constraint (11) on its covariant divergence, then (5) again, and the identity $\Gamma^b_{ab} = (\log\sqrt{g})_{,a}$. If we insert the result in the SDE (14) we recognize the coincidence with the SDE (4). Our covariant SDE (12) yields the correct covariant drift velocity (10) of the FPE (9). It would not do so without our new constraint $(e^a_A)_{;a}$.

Construction of the frame e^a_A is simple when the Riemann space is flat. Then the coordinates x^a are functions of Euclidean coordinates y^A. Accordingly, $x^a = f^a(y)$ and the map from Euclidean to the curvilinear coordinates satisfy the tensor equation

$$\delta^{AB} f^a_{,A} f^b_{,B} = g^{ab} \ . \tag{16}$$

If we choose our frame as

$$e^a_A = f^a_{,A} \ , \tag{17}$$

it satisfies the frame's orthogonality condition (5). This frame satisfies our new covariant constraint (11) as well. The covariant divergence $(e^a_A)_{;a}$ vanishes in any curvilinear coordinates because it vanishes in the particular Euclidean coordinates where $e^a_A = \delta^a_A$.

III. DISCUSSION

All elements of our work, except our new covariant constraint (11), appeared in previous works. Riemann geometric studies of covariance of diffusive phenomena began with Graham's seminal papers [3, 4] for transport processes in non-linear and non-equilibrium thermodynamic media. Ref. [4] contains the covariant FPE (9) and the relationship (10) between the true non-covariant and covariant drifts, with the observation that their difference vanishes in harmonic coordinates. Also the covariant Stratonovich–Langevin SDE (12) is proposed but solely with the standard constraint (5), not mentioning that the constraint leaves the coefficients e^a_A ambiguous. In Ref. [5] Graham already puts the freedom (6) of the coefficients under investigation and observes that it is not gauge freedom in the Stratonovich SDE, yet without commenting on the related shortcoming of the previously proposed covariant Stratonovich SDE. Abandoning the Stratonovich form, the author constructs an Ito type covariant Langevin SDE whose covariance is based on the modification of the standard Ito differential, to contain Christoffel symbols. In Polettini's work [6], the matrix coefficient e_A^a is identified geometrically as the frame. The proposed Ito SDE, similarly to Graham's, is not of the standard covariant form since it contains Christoffel symbols. We mention an other particular from of non-standard covariance [4, 7, 8]. If a unique equilibrium state $P^{\mathrm{eq}}(x)$ exists then both the both the FPE and the SDE can be parametrized by $P^{\mathrm{eq}}(x)$ instead of the non-covariant drift $\widetilde{V}^a$.

In summary, we have shown that for a given Fokker–Planck equation there is a unique covariant Langevin stochastic process governed by the Stratonovich stochastic differential equation (12) in terms of the covariant drift vector, the covariant divergence-free orthogonal frame vectors, and covariant derivatives. Without the new divergence condition (11), the equation proposed originally by Graham in 1977, yielded frame-dependent drifts, not the unique true one.

Acknowledgments

The author thanks Laszló B. Szabados for valuable discussions. This research was funded by the Foundational Questions Institute and Fetzer Franklin Fund, a donor-advised fund of the Silicon Valley Community Foundation (Grant No's. FQXi-RFPCPW-2008, FQXi-MGA-2103), the National Research, Development and Innovation Office (Hungary) "Frontline" Research Excellence Program (Grant No. KKP133827), and the John Templeton Foundation (Grant 62099).

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