Towards relativistic generalization of collapse models

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Spontaneous collapse models provide a possible, testable solution to the quantum measurement problem. While experiments are providing increasingly stronger bounds on their parameters, a full-fledged relativistic extension is still missing. Previous attempts have encoutered different obstacles, such as violation of microcausality, infinite energy rate, and particle production from vacuum. Here, we propose generalization of the collapse master equation that is characterized by a local field collapse operator and a non-Markovian noise with a Lorentz invariant correlation. Our construction is able to overcome previously encountered problems and has the desirable properties in the non relativistic limit. A specific choice of the noise correlation function is also introduced and discussed.

Non-relativistic spontaneous wavefunction collapse (or simply, collapse) models provide a possible coherent solution to the well-known quantum measurement problem [1, 2]. They modify the standard, linear unitary Schrödinger equation by adding non-linear and stochastic terms, which impose the statevector localization (or collapse) into one eigenstate of a suitable collapse operator. The outcome of a subsequent measurement would provide the corresponding eigenvalue. Different realizations of the stochastic process impose different outcomes, which are distributed according to the Born rule.

While some collapse models, such as the Continuous Spontaneous Localization (CSL) [3, 4] and the Diosi-Penrose (DP) [5] models, are subject to continuous experimental and theoretical investigation [6–17], there is still an ongoing debate concerning their possible relativistic generalization [18–28]. Previous attempts have resulted in unwanted side effects [29, 30] such as violation of microcausality [31], infinite rate of energy [32], particle production from the vacuum [27], and tachyon-like dynamics [33]. In addition, a more general conceptual issue surrounds the possible compatibility between the wavefunction localization and relativity principles [1].

Here, we propose a mathematically consistent relativistic generalization of the collapse dynamics, which is characterized by a local field collapse operator and a non-Markovian stochastic noise with a Lorentz invariant correlation. We show that locality saves microcausality, while non-Markovianity and a normal ordering prescription lead to a finite rate of energy, without having particle production from the vacuum or a tachyonic behavior.

Non-linear unraveling. – We construct of our model, starting from the non-relativistic (NR), colored, non-linear collapse equation for the statevector [34, 35]. To second order in $\sqrt{\gamma}$, the equation takes a closed form in $|\psi(t)\rangle$ and reads [34]

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle=\Bigg[-\frac{i}{\hbar}\hat{H}'+\sqrt{\gamma}\sum_{i=1}^{N}\left(\hat{A}_i-\langle\hat{A}_i\rangle_t\right)\xi_i(t) \\ +\gamma\left(\hat{O}_+(t)-\langle\hat{O}_+(t)\rangle\right)\Bigg]|\psi(t)\rangle\,. \tag{1}$$

Here, $\hat{H}'=\hat{H}_0+i\hbar\gamma\hat{O}_-(t)$, with $\hat{H}_0$ being the standard quantum mechanical Hamiltonian, γ is the common coupling with the N collapse noises $\xi_i(t)$, which are real Gaussian random processes having zero mean and correlations

$$\mathbb{E}\left[\xi_i(t)\xi_j(s)\right]=D_{ij}(t,s). \tag{2}$$

Here $D_{ij}(t,s)$ is the noise correlation whose explicit form is to be determined, $\mathbb{E}$ denotes the average over different realizations of the noises $\xi_i(t)$. Further, $\hat{A}_i$ in Eq. (1) are a set of commuting self-adjoint operators that describe how the collapse occurs, and are known as collapse operators. The operators $\hat{O}_-$ and $\hat{O}_+$ are, respectively, the anti-self-adjoint and self-adjoint parts of $\hat{O}$. They read

$$\hat{O}_\pm(t)=-\sum_{i,j=1}^{N}\int_0^t\mathrm{d}s\,D_{ij}(t,s)\left[\hat{A}_i,\hat{A}_j(s-t)\right]_\pm, \tag{3}$$

where $[\,\cdot\,,\cdot\,]_-=[\,\cdot\,,\cdot\,]$ and $[\,\cdot\,,\cdot\,]_+=\{\,\cdot\,,\cdot\,\}$ denote respectively the commutator and anticommutator, while $\hat{A}_j(s-t)=\hat{U}_0^\dagger(s-t)\hat{A}_j\hat{U}_0(s-t)$ is evolved with respect to the free Hamiltonian $\hat{H}_0$, i.e. $\hat{U}_0(t)=\exp(-i\hat{H}_0t/\hbar)$. Notice that Eq. (1) is only valid up to second order in $\sqrt{\gamma}$. The prescription to obtain the equation for higher orders can be found in [34].

We aim at generalizing Eq. (1) to the relativistic regime so that the corresponding master equation is relativistically covariant. The master equation can be equivalently obtained from the linear unraveling of Eq. (1) for the

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non-normalized vector $|\phi(t)\rangle$. It reads [34]

$$\frac{\mathrm{d}}{\mathrm{d}t}|\phi(t)\rangle=-\frac{i}{\hbar}\hat{H}_0|\phi(t)\rangle+\left[\sqrt{\gamma}\sum_{i=1}^{N}\hat{A}_i\xi_i(t)\right.\\ \left.-\gamma\sum_{i,j=1}^{N}\int_0^t\mathrm{d}s\,D_{ij}(t,s)\hat{A}_i\hat{A}_j(s-t)\right]|\phi_0(t)\rangle\,, \tag{4}$$

where, to second order in $\sqrt{\gamma}$, we have $|\phi(t)\rangle=|\phi^{(0)}(t)\rangle+\sqrt{\gamma}\,|\phi^{(1)}(t)\rangle+\gamma\,|\phi^{(2)}(t)\rangle+\mathcal{O}(\gamma^{3/2})$. The standard prescription gives the corresponding master equation in the Schrödinger picture $\mathrm{d}\hat{\rho}^{(2)}(t)/\mathrm{d}t=-\frac{i}{\hbar}\left[\hat{H}_0,\hat{\rho}^{(2)}(t)\right]+\mathcal{D}[\hat{\rho}^{(0)}(t)]$, where

$$\mathcal{D}[\hat{\rho}^{(0)}(t)]=-\gamma\sum_{i,j=1}^{N}\int_0^t\mathrm{d}s\,D_{ij}(t,s)\left[\hat{A}_i,\left[\hat{A}_j(s-t),\hat{\rho}^{(0)}(t)\right]\right], \tag{5}$$

and $\hat{\rho}^{(\alpha)}(t):=\mathbb{E}[|\phi^{(\alpha)}(t)\rangle\langle\phi^{(\alpha)}(t)|]$ with $\alpha=0,1,2$. We move to the interaction picture by using the standard relation $\dot{\hat{\rho}}_I^{(2)}(t)=\hat{U}_0(t)\left(\frac{i}{\hbar}\left[\hat{H}_0,\hat{\rho}^{(2)}(t)\right]+\dot{\hat{\rho}}^{(2)}(t)\right)\hat{U}_0^\dagger(t)$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\mathrm{I}}^{(2)}(t)=-\gamma\sum_{i,j=1}^{N}\int_0^t\mathrm{d}s\,D_{ij}(t,s)\left[\hat{A}_i(t),\left[\hat{A}_j(s),\hat{\rho}_{\mathrm{I}}^{(0)}(t)\right]\right], \tag{6}$$

whose solution reads

$$\hat{\rho}_{\mathrm{I}}^{(2)}(t)=\hat{\rho}_{\mathrm{I}}^{(0)}(t)\\ -\gamma\sum_{i,j=1}^{N}\int_0^t\mathrm{d}s\int_0^s\mathrm{d}s'\,D_{ij}(s,s')\left[\hat{A}_i(s),\left[\hat{A}_j(s'),\hat{\rho}_{\mathrm{I}}^{(0)}(t)\right]\right]. \tag{7}$$

Further, we move from the discrete label i to its continuous version $\mathbf{x}$. In doing so, we identify $\hat{A}_i(t)\leftrightarrow\hat{Q}(\mathbf{x},t)$ and $\sum_{i,j=1}^{N}D_{ij}(t,s)\;\leftrightarrow\;\iint\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_1\,\mathbb{E}[\xi(t,\mathbf{x}_2)\xi(s,\mathbf{x}_1)]=\iint\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{x}_1\,G(x_2,x_1)$, where $x_2=(t,\mathbf{x}_2)$ and $x_1=(s,\mathbf{x}_1)$. Then, the expectation value of any generic operator $\hat{O}$, to the second order in $\sqrt{\gamma}$, can be expressed as $\langle\hat{O}(z)\rangle^{(2)}=\mathrm{Tr}\left[\hat{O}_{\mathrm{I}}(z)\hat{\rho}_{\mathrm{I}}^{(2)}(z)\right]$. Its time derivative reads

$$\frac{\mathrm{d}}{\mathrm{d}z^0}\langle\hat{O}(z)\rangle^{(2)}=\frac{\mathrm{d}}{\mathrm{d}z^0}\mathrm{Tr}\left[\hat{O}_{\mathrm{I}}(z)\hat{\rho}(0)\right]-\gamma\iint_{x_2^0\le x_1^0=z^0}\mathrm{d}^4x_2\,\mathrm{d}^3\mathbf{x}_1\,G(x_2,x_1)\,\mathrm{Tr}\left[\left[\hat{Q}(x_2),\left[\hat{Q}(x_1),\hat{O}_{\mathrm{I}}(z)\right]\right]\hat{\rho}(0)\right]\\ -\gamma\fint G(x_2,x_1)\,\mathrm{Tr}\left[\left[\hat{Q}(x_2),\left[\hat{Q}(x_1),\frac{\mathrm{d}}{\mathrm{d}z^0}\hat{O}_{\mathrm{I}}(z)\right]\right]\hat{\rho}(0)\right], \tag{8}$$

where $\fint:=\iint_{x_2^0\le x_1^0\le z^0}\mathrm{d}^4x_2\,\mathrm{d}^4x_1$. Eq. (8) can also be obtained by deriving

$$\hat{O}^{(2)}(z)=\hat{O}^{(0)}(z)-\gamma\fint G(x_2,x_1)\left[\hat{Q}(x_2),\left[\hat{Q}(x_1),\hat{O}^{(0)}(z)\right]\right], \tag{9}$$

in the Heisenberg picture with respect to z^0, and taking the expectation value with respect to the initial state. Here, we have used $\hat{O}_{\mathrm{I}}(z)=\hat{O}^{(0)}(z)$. From a mathematical standpoint, the time evolved second order expression for $\hat{O}^{(2)}(z)$ in Eq. (9) can be also be obtained within the Heisenberg picture of a linear and unitary unraveling, as shown in Appendix A. In the unitary unraveling, all statistical effects of the collapse dynamics can be captured by adding a stochastic term $\hat{H}_{\mathrm{st}}(t)=\hbar\sqrt{\gamma}\int\mathrm{d}\mathbf{z}\,\hat{Q}(\mathbf{z})\xi(t,\mathbf{z})$ to the standard quantum mechanical evolution, i.e. $\hat{H}=\hat{H}_0+\hat{H}_{\mathrm{st}}(t)$.

Conceptually, however, such an unraveling is in stark contrast with the non-linear and non-unitary unraveling of collapse models. For instance, if one insists that an observed outcome in an experiment is explained objectively only by the localization of the wavefunction around a specific eigenvector, the unitary unraveling would fail to provide such an explanation. One might argue instead that even though the unitary unraveling does not lead to the collapse, it still inevitably turns a pure state into a mixed one for a macroscopic system, consistently with the Born rule. The distiction between the two unravelings is certainly important from theoretical considerations. However, since experimentally one only has access to the density matrix [36, 37], here we take a more pragmatic approach and focus only on the latter dynamics.

We emphasize again that just like their discrete counterpart, $\hat{Q}(x)$ appearing in Eq. (9) is the freely evolved operator in the Heisenberg picture and $G(x,y)$ some generic function of x and y. Next we study the requirements on $\hat{Q}(x)$ and $G(x,y)$ to have that Eq. (9) relativistically consistent.

Lorentz covariance.–The Lorentz covariance of Eq. (9) follows from standard quantum field theory (QFT). For the operator $\hat{O}(z)$ to evolve covariantly, it is sufficient to show that the collapse noise does not introduce any additional transformation for $\hat{O}$ in going from one reference frame to another, and that $\hat{O}$ transforms in the same manner as $\hat{O}^{(0)}$. The first term on the RHS of Eq. (9) trivially satisfies this requirement. Further, assuming that $\hat{Q}(x)$ is a Lorentz scalar, and the correlation $G(x,y)$ to be Lorentz invariant $G(x,y)=G\left((x-y)_\mu(x-y)^\mu\right)$, we see that the second term of Eq. (9), which is the contribution from the external *collapse* noise, also transforms covariantly. This holds to all orders in γ. Thus, Lorentz covariance is guaranteed as long as the collapse operator $\hat{Q}(x)$ is a Lorentz scalar and $G(x-y)$ a Lorentz invariant function.

The microcausality condition (MCC). – The MCC states that

$$\left[\hat{O}(z_2), \hat{O}(z_1)\right] = 0, \quad \text{for} \quad |z_2 - z_1| < 0, \tag{10}$$

where we are working within the $(+,-,-,-)$ convention. MCC implies that the measurement of one observable cannot influence the time evolution of any other observable outside of the lightcone corresponding to the measurement event. This request is met within standard QFT for typical interactions [38, 39]. We now study MCC for operators evolving according to a unitary unraveling of Eq. (9), where the evolution of the operators is governed by the Hamiltonian $\hat{H}(t) = \hat{H}_0 + \hat{H}_{\rm st}(t)$. Namely

$$\hat{U}_{t;t_0} = \mathcal{T}\left\{\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}t'\, \hat{H}(t')\right)\right\}. \tag{11}$$

Such a unitary operator satisfies $\hat{U}_{t_2;t_1}\hat{U}_{t_1;t_0} = \hat{U}_{t_2;t_0}$ for any Hermitian operator $\hat{H}(t)$. In the Heisenberg picture where $\hat{O}(t,\mathbf{z}) = \hat{U}^\dagger_{t;t_0}\hat{O}(t_0,\mathbf{z})\hat{U}_{t;t_0}$, the MCC condition, for $|z_2 - z_1| < 0$, requires

$$\hat{U}^\dagger_{t_1;t_0}\left[\hat{U}^\dagger_{t_2;t_1}\hat{O}(t_0,\mathbf{z}_2)\hat{U}_{t_2;t_1}, \hat{O}(t_0,\mathbf{z}_1)\right]\hat{U}_{t_1;t_0} = 0. \tag{12}$$

In general, $\hat{U}_{t_2;t_1}$ does not satisfy the time-translation property — $\hat{U}_{t_2;t_1} = \hat{U}_{\Delta t+t_0;t_0}$, where $\Delta t = t_2 - t_1$, if the Hamiltonian $\hat{H}(t)$ is time-dependent. However, since the time dependence appears in $\hat{H}_{\rm st}(t) = \hbar\sqrt{\gamma}\int \mathrm{d}\mathbf{z}\, \hat{Q}(\mathbf{z})\xi(t,\mathbf{z})$ only through the noise, we can rewrite the time integral in Eq. (11) as

$$\hat{Q}(\mathbf{z})\int_{t_1}^{t_2}\mathrm{d}t'\,\xi(t',\mathbf{z}) = \hat{Q}(\mathbf{z})\int_{t_0}^{\Delta t+t_0}\mathrm{d}s\,\tilde{\xi}(s,\mathbf{z}). \tag{13}$$

Therefore, $\hat{U}^{(\xi)}_{t_2;t_1}$ encoding the stochastic dynamics in our analysis can be written as $\hat{U}^{(\xi)}_{t_2;t_1} = \hat{U}^{(\tilde{\xi})}_{\Delta t+t_0;t_0}$. We point out that $\tilde{\xi}(t,\mathbf{z}) = \xi(t+t_1-t_0,\mathbf{z})$ has the same correlation function, due to the invariance of G under spacetime translations. Therefore, at the level of the master equation, the distinction between ξ and $\tilde{\xi}$ is unimportant and will not be retained in what follows.

In this context, since $\hat{U}_{t_2;t_1}$ is equivalent to $\hat{U}_{\Delta t+t_0;t_0}$, MCC specified in Eq. (12) is equivalent to showing that for $|z_2 - z_1| < 0$, $\left[\hat{O}(\Delta t + t_0,\mathbf{z}_2), \hat{O}(t_0,\mathbf{z}_1)\right] = 0$. Note that now only the first operator within the commutator evolves according to the collapse dynamics. To second order in $\sqrt{\gamma}$, for a given realization of the noise, the latter expression reads

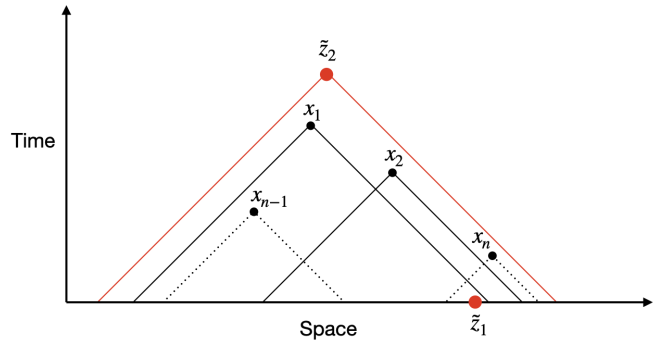

FIG. 1. Leading order causal structure which demonstrates why MCC is respected, cf. Eq. (14) and Eq. (15).

$$⨍\, \xi(x_2)\xi(x_1)\left[\left[\hat{Q}(x_2), \left[\hat{Q}(x_1), \hat{O}^{(0)}(\tilde{z}_2)\right]\right], \hat{O}^{(0)}(\tilde{z}_1)\right], \tag{14}$$

where $\tilde{z}_2 = (t_0 + \Delta t, \mathbf{z}_2)$ and $\tilde{z}_1 = (t_0, \mathbf{z}_1)$. The integral $⨍$ implies that $x_2^0 \leq x_1^0$ and $x_1^0 \leq \tilde{z}_2^0$, where $\tilde{z}_1^0$ sets the initial time. For Eq. (14) to be non-zero, its innermost commutator $[\hat{Q}(x_1), \hat{O}^{(0)}(\tilde{z}_2)]$ must also be non-zero. Since both the operators appearing in such a commutator are local QFT operators evolving with respect to the standard free Hamiltonian $\hat{H}_0$, x_1 must be inside the past lightcone of $\tilde{z}_2$. Indeed, x_1 is guaranteed not to be in the future of $\tilde{z}_2$ by the standard perturbative expansion. Similarly, for the double commutator involving $\hat{Q}(x_2)$, $\hat{Q}(x_1)$ and $\hat{O}^{(0)}(\tilde{z}_2)$ to be non-zero, x_2 must belong to the past lightcone of $\tilde{z}_2$ or x_1, which makes it necessary for x_2 to also belong to the past lightcone of $\tilde{z}_2$ as shown in Fig. 1. Finally, by following the same logic, for the outermost commutator involving $\hat{O}^{(0)}(\tilde{z}_1)$ to be non-zero, the necessary condition is that z_1 must belong to the past lightcone of $\tilde{z}_2$ or x_1 or x_2, which forces it inside the past lightcone of $\tilde{z}_2$. Thus, MCC in Eq. (10) is respected to second order due to the locality of the operator $\hat{Q}(x)$ as in standard QFT.

Following the same line of reasoning, it can further be shown that MCC is respected to all orders in $\sqrt{\gamma}$. This is because in the n-th order term

$$⨍\, \xi(x_n)...\xi(x_1)\left[\left[\hat{Q}(x_n), ..., \left[\hat{Q}(x_1), \hat{O}^{(0)}(\tilde{z}_2)\right]\right], \hat{O}^{(0)}(\tilde{z}_1)\right], \tag{15}$$

$\tilde{z}_1$ must lie inside the past lightcone of $\tilde{z}_2$ or at least one of the coordinates $x_1, x_2, ..., x_n$ of the collapse operators. Since $x_1, x_2, ..., x_n$ themselves must lie in the past lightcone of $\tilde{z}_2$, it becomes necessary for $\tilde{z}_1$ to lie inside the past lightcone of $\tilde{z}_2$ for $[\hat{O}(\tilde{z}_2), \hat{O}(\tilde{z}_1)]$ to be non-zero, and therefore z_1 to lie inside the past lightcone of z_2 for $[\hat{O}(z_2), \hat{O}(z_1)]$ to be non-zero (having assumed $t_2 \geq t_1$ without loss of generality). Therefore, the locality of $\hat{Q}$ and the standard time-ordered evolution, which imposes the past lightcone structure in Fig. 1, together ensure that MCC is respected for the relativistic stochastic dynamics as well. This proof shows that if $\hat{Q}$ is a non-local Lorentz scalar, or if the time ordering from the standard quantum dynamics is removed, then MCC might be violated.

An example of the former is the non-local collapse operator $\hat{Q} \propto \hat{\varphi}_+\hat{\varphi}_-$, where $\hat{\varphi}_+$ and $\hat{\varphi}_-$ are respectively the positive and negative frequency parts of $\hat{\varphi}$ [31]. Instead, an example of the latter is [33] where causality was found to be violated due to the removal of time-ordering, namely $⨏ \to \iint \mathrm{d}^4x\,\mathrm{d}^4y$ with $x^0, y^0 \leq z^0$, in order to remove divergences. Our analysis further shows that the noise correlations do not need to be limited to an ultra-local one, such as $G(x,y) = \delta^4(x-y)$, to preserve MCC as argued in [31]. Instead, MCC can be satisfied by taking a local collapse operator and non-divergent correlations.

In our work, motivated by the non-relativistic CSL model, we choose the collapse operator to be

$$\hat{Q}(x) = \frac{1}{2}\alpha\hat{\varphi}^2(x), \tag{16}$$

where α is a suitable free parameter. $\hat{Q}(x)$ is local, and it becomes proportional to the mass density operator $\hat{Q}_{\text{MD}}(x)$ in the NR limit.

Rate of increase of energy.–Now we show that a finite rate of energy can be achieved as long as $G(x,y)$ is well-behaved and is not divergent. Setting $\hbar = c = 1$, we compute the energy rate $\mathrm{d}E/\mathrm{d}z^0$ with Eq. (8) by taking $\hat{O}(z) = \hat{\mathcal{H}}(z)$, where

$$\hat{\mathcal{H}}(z) = \tfrac{1}{2}\hat{\pi}^2(z) + \tfrac{1}{2}\left(\nabla\hat{\varphi}(z)\right)^2 + \tfrac{1}{2}m^2\hat{\varphi}^2(z), \tag{17}$$

and then integrating over space, i.e. $\mathbf{z}$. In doing so, the first and the last terms on the RHS of Eq. (8) vanish, as the freely evolved Hamiltonian is a conserved quantity with $\mathrm{d}/\mathrm{d}z^0 \int \mathrm{d}\mathbf{z}\,\hat{\mathcal{H}}^{(0)}(z) = 0$. Therefore, we have

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}z^0} = -\gamma \iiint_{x^0\leq y^0=z^0} \mathrm{d}^4x\,\mathrm{d}^3\mathbf{y}\,\mathrm{d}^3\mathbf{z}\,G(x,y) \\ \times \mathrm{Tr}\left[\left[\hat{Q}(x),\left[\hat{Q}(y),\hat{\mathcal{H}}(z)\right]\right]\hat{\rho}(0)\right].\end{aligned} \tag{18}$$

Since $y^0 = z^0$, the inner commutator can be computed in the standard way and gives

$$\left[\hat{Q}(y),\hat{\mathcal{H}}(z)\right] = \frac{i\alpha}{2}\delta^3(\mathbf{y}-\mathbf{z})\left\{\hat{\varphi}(z),\hat{\pi}(z)\right\}. \tag{19}$$

Then, Eq. (18) becomes

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}z^0} = -\gamma\alpha^2 \iint_{x^0\leq z^0} \mathrm{d}^4x\,\mathrm{d}^3\mathbf{z}\,G(x,z) \\ \times\partial_{z^0}\left(\mathcal{N}(x,z)\mathcal{D}(x-z)\right),\end{aligned} \tag{20}$$

where

$$\begin{aligned}\mathcal{N}(x,z) &:= \frac{1}{2}\mathrm{Tr}\left[\left\{\hat{\varphi}(x),\hat{\varphi}(z)\right\}\hat{\rho}(0)\right], \\ \mathcal{D}(x-z) &:= i\left[\hat{\varphi}(x),\hat{\varphi}(z)\right].\end{aligned} \tag{21}$$

Here, we have used the fact that $\partial\hat{\varphi}(z)/\partial z^0 = \hat{\pi}(z)$. The expression for rate of increase of energy simplifies further, if the initial state is such that the LHS of the first line in Eq. (21) is a function of $x - z$ (for instance when the initial state is a thermal state). Since Lorentz covariance implies $G(x,z) = G(x-z)$, the expression for energy increase would then only depend on the spatial and temporal parts $\mathbf{u} = \mathbf{x} - \mathbf{z}$ and $\tau = x^0 - z^0$ respectively, such that

$$\frac{\mathrm{d}E}{\mathrm{d}z^0} = \gamma\alpha^2 V \int \mathrm{d}^3\mathbf{u} \int_{-\infty}^{0} \mathrm{d}\tau\,G(\tau,\mathbf{u})\partial_\tau\left(\mathcal{N}(\tau,\mathbf{u})\mathcal{D}(\tau,\mathbf{u})\right), \tag{22}$$

where $V = \int \mathrm{d}\mathbf{v}$ is the full volume, with $\mathbf{v} := (\mathbf{x}+\mathbf{z})/2$.

The non-Markovian model. — To arrive at a compact expression for the rate of energy for a general non-Markovian noise, we assume that the initial state $\hat{\rho}(0)$ has a definite particle number (like the thermal state). Then, we have

$$\mathrm{Tr}\left[\hat{\rho}(0)\hat{a}_{\mathbf{q}}\hat{a}_{\mathbf{p}}\right] = \mathrm{Tr}\left[\hat{\rho}(0)\hat{a}^\dagger_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}}\right] = 0, \tag{23}$$

where $\hat{\varphi}(x) := \hat{\varphi}_+(x) + \hat{\varphi}_-(x)$, with

$$\hat{\varphi}_+(x) := \int \frac{\mathrm{d}^3\mathbf{p}}{\sqrt{2\omega_p(2\pi)^3}}\,e^{-ip.x}\hat{a}_{\mathbf{p}}, \qquad p = (\omega_p, \mathbf{p}). \tag{24}$$

Even though such an initial state is not Lorentz invariant, it serves the purpose of demonstrating that a Lorentz invariant non-Markovian noise is free of the problems of its Markovian counterpart, which is well-known to have a $\delta(\mathbf{0})$ divergence, independently of the choice of the initial state (cf. Appendix B).

The application of Eq. (23) to Eq. (21) provides the following simple expressions:

$$\begin{aligned}\mathcal{N}(x) &= \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\left(n_{\mathbf{q}} + \frac{1}{2}\right)\frac{\cos(q.x)}{\omega_q}, \\ \mathcal{D}(x) &= \int \frac{\mathrm{d}^3\mathbf{q}}{\omega_q(2\pi)^3}\sin(q.x),\end{aligned} \tag{25}$$

where $n_{\mathbf{q}}$ is the average occupancy of the mode $\mathbf{q}$ for the initial state, such that $\int \mathrm{d}^3\mathbf{q}\,n_{\mathbf{q}} \to \frac{(2\pi)^{3/2}}{L^3}\sum n_q = (2\pi)^{3/2}N/V$, N being the total number of particles. The factor of $1/2$ in the first expression captures the standard QFT vacuum divergence. This factor leads to a divergent particle production rate from vacuum, which was already found in Ref. [33]. There, the removal of time-ordering was proposed to obtain a finite expression. However, as argued before, this might violate causality, which is also reflected in the tachyonic behavior reported in Ref. [33]. Therefore, here, we propose to use a normal-ordering prescription, where all the observables of interest are normal-ordered at all times, i.e. $\hat{O}(z) \to :\hat{O}(z):$ and thus the $1/2$ term is dropped. Note that this prescription leaves the MCC analysis unchanged, as shown in Appendix C.

By combining Eqs. (25) and (22), and computing explicitly the z^0 derivative, we can derive the following normal-ordered (NO) expression

$$\left.\frac{\mathrm{d}E}{\mathrm{d}z^0}\right|_{\mathrm{NO}} = \frac{\gamma\alpha^2 V}{(2\pi)^6}\int \mathrm{d}^3\mathbf{x}\int_{-\infty}^{0}\mathrm{d}\tau\, G(\tau,\mathbf{x})\iint \mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}\, n_{\mathbf{q}} \times\left[\frac{\cos(p.x)\cos(q.x)}{\omega_q}-\frac{\sin(p.x)\sin(q.x)}{\omega_p}\right]. \tag{26}$$

Since this integrand remains unchanged under reflection $x\to -x$, Eq. (26) can be written more compactly as

$$\left.\frac{\mathrm{d}E}{\mathrm{d}z^0}\right|_{\mathrm{NO}} = \frac{\gamma\alpha^2 V}{2(2\pi)^4}\iint \mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}\, n_{\mathbf{q}} \times \Re\left[\frac{\mathcal{G}(p+q)+\mathcal{G}(p-q)}{2\omega_q}+\frac{\mathcal{G}(p+q)-\mathcal{G}(p-q)}{2\omega_p}\right], \tag{27}$$

where $\mathcal{G}(p)$ is the four-dimensional Fourier transform of the correlation $G(x)$. Notably, the multiplicative $\delta(\mathbf{0})$ divergence present the white noise expression does not appear in the non white case. Further, as long as one is only concerned with obtaining a finite rate of energy, any choice of $\mathcal{G}(q)$ for which Eq. (27) is finite becomes a viable relativistic non-Markovian collapse correlation within the normal-ordering prescription.

Non-relativistic and relativistic limits. – We now discuss the non-relativistic limit to motivate specific choices for $\mathcal{G}(q^2)$. Such a limit can be understood as the one for which $n_{\mathbf{q}}=0$, for $\mathbf{q}^2 \gtrsim m^2$. Namely, only particles whose kinetic energy is well below the rest mass energy are retained. This also implies $\omega_q \approx m$. Notably, the $\mathbf{p}$ integral in Eq. (27) still runs over all the $\mathbb{R}^3$ values. We divide the $\mathbf{p}$ integral into the NR ($|\mathbf{p}|^2 \ll m^2$ and $\omega_p\approx m$) and the relativistic ($|\mathbf{p}|^2 \gg m^2$ and $\omega_p \approx |\mathbf{p}|$) regimes. Since the noise correlation is Lorentz invariant, with the following structure $\mathcal{G}(p\pm q) = \mathcal{G}\left[(\omega_p\pm\omega_q)^2-(\mathbf{p}\pm\mathbf{q})^2\right]$, in the NR regime we have $\mathcal{G}_{\mathrm{NR}}(p+q)\approx\mathcal{G}(4m^2)$ and $\mathcal{G}_{\mathrm{NR}}(p-q)/\omega_q - \mathcal{G}_{\mathrm{NR}}(p-q)/\omega_p \approx 0$. On the other hand, when $\mathbf{p}$ is in the relativistic regime, we have $\mathcal{G}_{\mathrm{rel}}(p\pm q)\approx \mathcal{G}(\pm 2|\mathbf{p}|m)$. Since $\mathcal{G}(q)$ is an even function — which follows from the requirement that $G(x)$ must be real — $\mathcal{G}_{\mathrm{rel}}(p+q)-\mathcal{G}_{\mathrm{rel}}(p-q)\approx 0$, and the rate of increase of energy becomes

$$\left.\frac{\mathrm{d}E}{\mathrm{d}z^0}\right|_{\mathrm{NO}} \approx \frac{\gamma\alpha^2 V}{2(2\pi)^4}\int_0^m \mathrm{d}\mathbf{q}\, n_{\mathbf{q}} \times\left[\int_0^m \mathrm{d}\mathbf{p}\,\frac{\mathcal{G}(4m^2)}{m}+\int_m^\infty \mathrm{d}\mathbf{p}\,\frac{\mathcal{G}(2|\mathbf{p}|m)}{m}\right], \tag{28}$$

where the first integral gives $\mathcal{G}(4m^2)/m\times 4/3\pi m^3$. Thus, in the NR limit, we find that the energy rate is proportional to the total number of particles N, as the overall factor of V in Eq. (28) cancels out due to the relation $\int \mathrm{d}\mathbf{q}\, n_{\mathbf{q}} = (2\pi)^{3/2}N/V$. Such a feature is shared with the NR collapse models that can be found in literature [13]. Further, the choice of $\mathcal{G}(q)$ is now only constrained by the convergence of the second integral in Eq. (28), which can be easily achieved by a suitable choice of $\mathcal{G}(q)$. Nevertheless, for the relativistic white noise $G(x)=\delta^4(x)$, $\mathcal{G}(q)$ is a constant and we get that Eq. (28) diverges as $\delta(\mathbf{0})$, as expected and also shown in Appendix B.

In the fully relativistic limit, for which the corresponding distribution $n(\mathbf{q})$ is non-zero only when $|\mathbf{q}|\gtrsim m$, we can find a finite expression for the energy rate. By following a similar reasoning as before, Eq. (27) can be approximated to

$$\left.\frac{\mathrm{d}E}{\mathrm{d}z^0}\right|_{\mathrm{NO}} \approx \frac{\gamma\alpha^2 V}{2(2\pi)^4}\int_m^\infty \mathrm{d}\mathbf{q}\,\frac{n_{\mathbf{q}}}{|\mathbf{q}|} \times\left[\int_0^m \mathrm{d}\mathbf{p}\,\mathcal{G}(2m|\mathbf{q}|)+\int_m^\infty \mathrm{d}\mathbf{p}\,\mathcal{G}\left(4|\mathbf{p}||\mathbf{q}|\sin^2\theta_{pq}\right)\right], \tag{29}$$

where θ_{pq} is the angle between the four-dimensional vectors p and q, and we can substitute the first integral with $\mathcal{G}(2m|\mathbf{q}|)\times 4/3\pi m^3$. The convergence of Eq. (29) puts rather mild constraints on $\mathcal{G}(q)$ (and thus on $G(x)$).

Outlook.–The convergence of the energy rate in Eq. (27) can be achieved, for instance, with the following choice

$$\mathcal{G}(q) = \exp\left(-q^4/\beta^4\right), \tag{30}$$

where β is another free parameter of the model. The proposed exponential form is motivated from its resemblance to the Gaussian correlation of the CSL model. The quartic dependence q^4 is considered in place of a quadratic dependence since q^2 becomes negative for the spacelike regions and thus $\exp(-q^2)$ diverges. Conversely, $\mathcal{G}(q)$ in Eq. (30) is finite in any point in the Fourier space. Further, $\mathcal{G}(q)\to 0$ fast enough as $q^2\to\pm\infty$ thereby guaranteeing the convergence of Eq. (27). Moreover, in $(1+1)$ dimension, one can also obtain an analytic expression for the corresponding $G(x)$, which is given in terms of the Meijer G-function

$$G(x) = \frac{\beta^2}{2}G^{2,0}_{0,3}\left(\frac{1}{256}\beta^4(x^2-t^2)^2\,\middle|\, 0,0,\frac{1}{2}\right). \tag{31}$$

Such an expression features a peak at a characteristic length scale $x=t$ (with the speed of light $c=1$) and goes to zero for large spatial separations. While an analytic expression for $G(x)$ in $(1+3)$ dimensions is hard to obtain, it is reasonable to assume that $G(x)$ would still have the desired properties as its $(1+1)$ counterpart.

Along with the choice of the collapse operator, the choice of $G(x)$, such as that in Eq. (31), completely fixes the relativistic stochastic dynamics up to the free parameters α, β and γ. These parameters, like in the CSL model, will be subject to experimental investigation. We emphasize again that this $\mathcal{G}$ is just one of the many possible choices that can be made. An exploration of other possible noise correlations is outside the scope of the present work but nevertheless is an interesting subject for future research.

Discussion.–We have shown that a consistent relativistic stochastic dynamics can be constructed, which leads to a finite rate of increase of energy and also respects the microcausality condition. For a collapse operator that is a quadratic local field operator, a collapse noise which has a non-Markovian Lorentz invariant correlation, and with an additional normal ordering prescription, such a dynamics is free of the problems found in previous works and has a desirable CSL like behavior in the non-relativistic limit.

However, even after such technical difficulties have been overcome, the question of whether such a dynamics can be consistently interpreted as a relativistic collapse model is still open. It has been shown in Sec. 14.2 of [1] that even though a consistent relativistic stochastic dynamics might be constructed at the level of the density matrix (obtained after averaging over different realizations of the collapse noise), for a single specific realization of the collapse noise, the quantum expectation value can in general be different when computed along different hypersurfaces by observers moving relative to each other. This poses a conceptual problem if one wishes to assign an objective meaning to the statevector at all times for each realization of the stochastic noise. One might not find such a difficulty surprising, given that related conceptual issues concerning relativistic quantum monitoring have also been pointed in [40–42].

Nevertheless, we emphasize that the dynamics presented here can still be viewed as a consistent relativistic stochastic dynamics corresponding to the linear and unitary unraveling of the wavefunction (cf. Sec. 14.2 of [1]), which predicts the suppression of macroscopic superpositions due to the coupling of matter with an underlying stochastic field (within the standard decoherence formalism). However, to have a convincing resolution of the measurement problem being consistent with relativity, it remains to be seen if such conceptual issues can be also overcome, or if they would turn out to be fundamental and unavoidable.

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Appendix A: The unitary unraveling

We consider the quantum dynamics governed by the Hamiltonian $\hat{H}$,

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm st}(t), \tag{A1}$$

where $\hat{H}_0$ is the standard Hamiltonian of the real free Klein-Gordon scalar field, while $\hat{H}_{\rm st}(t) = \hbar\sqrt{\gamma}\int \mathrm{d}\mathbf{z}\,\hat{Q}(\mathbf{z})\xi(t,\mathbf{z})$ is the contribution from the collapse noise within the unitary unraveling. The effect of the latter can be studied perturbatively, as in standard quantum field theory (QFT). For that, as it is well-known, the full unitary operator $\hat{U}_{t;t_0}$ can be factorized as

$$\hat{U}_{t;t_0} = \hat{U}_0(t,t_0)\times\hat{U}_{\rm st}(t,t_0), \qquad \hat{U}_0(t,t_0) = \exp\left[-\frac{i}{\hbar}\hat{H}_0(t-t_0)\right], \qquad \hat{U}_{\rm st}(t,t_0) = \mathcal{T}\left\{\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\,\hat{H}^{\rm I}_{\rm st}(t')\right]\right\}, \tag{A2}$$

where $\hat{H}^{\mathrm{I}}_{\mathrm{st}}(t)=\hat{U}_0^\dagger(t,t_0)\hat{H}_{\mathrm{st}}(t)\hat{U}_0(t,t_0)$. In the Heisenberg picture, given Eq. (A2), a generic operator $\hat{O}(z)$ is given by

$$\hat{O}(\mathbf{z},t)=\left[\hat{1}+\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\hat{H}^{\mathrm{I}}_{st}(t')-\frac{1}{\hbar^2}\int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t'}\mathrm{d}t''\hat{H}^{\mathrm{I}}_{\mathrm{st}}(t'')\hat{H}^{\mathrm{I}}_{\mathrm{st}}(t')+\text{higher order}\right]\times\hat{O}_0(\mathbf{z},t)\times$$
$$\times\left[\hat{1}-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\hat{H}^{\mathrm{I}}_{\mathrm{st}}(t')-\frac{1}{\hbar^2}\int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t'}\mathrm{d}t''\hat{H}^{\mathrm{I}}_{st}(t')\hat{H}^{\mathrm{I}}_{\mathrm{st}}(t'')+\text{higher order}\right]. \tag{A3}$$

The time evolution of $\hat{O}$ can then be written in terms of the freely evolved (0^{th} order) term $\hat{O}^{(0)}(t)=\hat{O}_0(t)=\hat{U}_0^\dagger(t,t_0)\hat{O}\hat{U}_0(t,t_0)$ as

$$\hat{O}(\mathbf{z},t)=\hat{O}^{(0)}(\mathbf{z},t)+\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\left[\hat{H}^{\mathrm{I}}_{\mathrm{st}}(t'),\hat{O}^{(0)}(\mathbf{z},t)\right]-\frac{1}{\hbar^2}\int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t'}\mathrm{d}t''\left[\hat{H}^{\mathrm{I}}_{\mathrm{st}}(t''),\left[\hat{H}^{\mathrm{I}}_{\mathrm{st}}(t'),\hat{O}^{(0)}(\mathbf{z},t)\right]\right]+\text{higher order}\,. \tag{A4}$$

Substituting the expression for $\hat{H}_{\mathrm{st}}$ and averaging over the stochastic realizations of the noise $\xi(x)$, we get

$$\begin{aligned}\mathbb{E}[\hat{O}^{(2)}(z)]&=\hat{O}^{(0)}(z)-\gamma\iint_{x^0\le y^0\le z^0}\mathrm{d}^4x\,\mathrm{d}^4y\,\mathbb{E}[\xi(x)\xi(y)]\,\left[\hat{Q}(x),\left[\hat{Q}(y),\hat{O}^{(0)}(z)\right]\right]\\&=\hat{O}^{(0)}(z)-\gamma\iint_{x^0\le y^0\le z^0}\mathrm{d}^4x\,\mathrm{d}^4y\,G(x-y)\left[\hat{Q}(x),\left[\hat{Q}(y),\hat{O}^{(0)}(z)\right]\right],\end{aligned} \tag{A5}$$

whose derivative with respect to z^0 gives Eq. (8) of the main text.

Appendix B: The white noise

The request of having a Lorentz invariant noise correlation together with the request of white noise, i.e. $\mathbb{E}[\xi(x^0,\mathbf{x})\xi(z^0,\mathbf{z})]=\delta(\tau)f(\mathbf{u})$, $\tau=x^0-z^0$, $\mathbf{u}=\mathbf{x}-\mathbf{z}$, fixes the noise correlation to $G(\tau,\mathbf{u})=\delta(\tau)\delta(\mathbf{u})$. For $G(x)=\delta^4(x)$, the rate of increase in energy given by Eq. (22) becomes

$$\frac{\mathrm{d}E}{\mathrm{d}z^0}=\frac{\gamma\alpha^2V}{2}\left(\partial_\tau\mathcal{N}(\tau,0)\mathcal{D}(0,0)+\mathcal{N}(0,0)\partial_\tau\mathcal{D}(\tau,0)\right)\big|_{\tau=0}. \tag{B1}$$

From the standard equal-time commutation relations of $\hat{\varphi}$, which enter the definitions of $\mathcal{N}$ and $\mathcal{D}$, we get $\mathcal{D}(0,0)=0$ and $\partial_\tau\,\mathcal{D}(\tau,0)|_{\tau=0}=-\partial_{z^0}\,\mathcal{D}(x,z)|_{x=z}=\delta(\mathbf{0})$. One can also obtain this result directly from the expression of $\mathcal{D}(x)$ in Eq. (25). This implies that the increase in energy diverges as

$$\frac{\mathrm{d}E}{\mathrm{d}z^0}=\delta(\mathbf{0})\frac{\gamma\alpha^2V\,\mathrm{Tr}\left[\hat{\varphi}^2(0)\hat{\rho}(0)\right]}{2}. \tag{B2}$$

Here we have used the relation $\mathcal{N}(0,0)=\mathrm{Tr}\left[\hat{\varphi}^2(0)\hat{\rho}(0)\right]$. Note that, in Eq. (B2), in addition to the standard QFT divergence (of the type one might encounter in computing $\langle\mathrm{in}|\,\hat{\varphi}^2\,|\mathrm{in}\rangle$), there is a multiplicative diverging factor $\delta(\mathbf{0})$. Even if the standard QFT divergence is removed by standard procedures, for example by imposing a normal ordering on the energy rate ($:\mathrm{d}/\mathrm{d}z^0\int\mathrm{d}\mathbf{z}\,\hat{\mathcal{H}}^{(0)}(z)=0:$), Eq. (B2) still remains divergent due to the multiplicative factor $\delta(\mathbf{0})$. This difficulty with relativistic Markovian models, which has already been pointed out in previous works [29–32], serves as a motivation to work with a more generic, non-Markovian noise function.

Appendix C: MCC with normal ordering

In the main text MCC was shown to be satisfied to all orders in ξ, by showing that

$$\fint\xi(x_n)...\xi(x_1)\left[\left[\hat{Q}(x_n),...,\left[\hat{Q}(x_1),\hat{O}^{(0)}(\tilde{z}_2)\right]\right],\hat{O}^{(0)}(\tilde{z}_1)\right]=0, \tag{C1}$$

whenever $|\tilde{z}_2-\tilde{z}_1|<0$, and hence $|z_2-z_1|<0$. Here we will show that this condition remains unchanged if the time-evolved operator $\hat{O}(z)$ is normal ordered. That is

$$\fint\xi(x_n)...\xi(x_1)\left[:\left[\hat{Q}(x_n),...,\left[\hat{Q}(x_1),\hat{O}^{(0)}(\tilde{z}_2)\right]\right]:,\hat{O}^{(0)}(\tilde{z}_1)\right]=0,\quad\text{for}\quad|\tilde{z}_2-\tilde{z}_1|<0. \tag{C2}$$

This is straightforward to see for operators $\hat{O}^{(0)}(\tilde{z}_2)$ that are linear or at most quadratic in creation and annihilation operators, such as $\hat{O}^{(0)} \propto \hat{\varphi}$ or $\hat{O}^{(0)} \propto \hat{\varphi}^2$, or the free Hamiltonian $\hat{H}_0$. This is because the collapse operator $\hat{Q}(x) = \frac{1}{2}\alpha\hat{\varphi}^2(x)$ is quadratic, and therefore the commutator $\left[\hat{Q}(x_1), \hat{O}^{(0)}(\tilde{z}_2)\right]$ would be of the same order in creation and annihilation operators as $\hat{O}^{(0)}$. Thus, the additional normal ordering can only add a constant, leaving the outermost commutator with $\hat{O}^{(0)}(\tilde{z}_1)$ in Eq. (C2) unchanged, when $\hat{O}^{(0)}$ is at most quadratic in creation and annihilation operators.

For higher order operators $\hat{O}^{(0)}$, such as $\hat{O}^{(0)}(x) \propto \hat{\varphi}^4(x)$, the value of the commutator would in general be different with normal ordering when $|\tilde{z}_2 - \tilde{z}_1| \geq 0$. However, it can be seen with the help of Wick's theorem that the commutator would still be zero when $|z_2 - z_1| < 0$. To show this, we first point out that for $\left[\hat{Q}(x_n), ..., \left[\hat{Q}(x_1), \hat{O}^{(0)}(\tilde{z}_2)\right]\right]$ to be non-zero, x_n, x_{n-1},..., x_1 must all be in the past lightcone of $\tilde{z}_2$, as argued in the main text. This does not change with the normal ordering that we impose. We now look at one of the many terms, such as

$$\oint \xi(x_n)...\xi(x_1)\left[: \hat{Q}(x_n)...\hat{Q}(x_1)\hat{O}^{(0)}(\tilde{z}_2) :, \hat{O}^{(0)}(\tilde{z}_1)\right], \tag{C3}$$

that contributes to the commutator in Eq. (C2). The product $: \hat{Q}(x_n)...\hat{Q}(x_1)\hat{O}^{(0)}(\tilde{z}_2) :$ can be written using Wick's theorem as [43]

$$: \hat{Q}(x_n)...\hat{Q}(x_1)\hat{O}^{(0)}(\tilde{z}_2) := \mathcal{T}\{\hat{Q}(x_n)...\hat{Q}(x_1)\hat{O}^{(0)}(\tilde{z}_2)\} - \text{all posible contractions.} \tag{C4}$$

The time ordering would simply change the ordering of the operators as $\mathcal{T}\{\hat{Q}(x_n)...\hat{O}^{(0)}(\tilde{z}_2)\} \to \hat{O}^{(0)}(\tilde{z}_2)...\hat{Q}(x_n)$. It does not change the statement that for Eq. (C3) to be non-zero, it is necessary for $\tilde{z}_1$ to be inside the lightcone of at least one of the spacetime points $x_1, x_2, ..., x_n, \tilde{z}_2$. Again, since x_n, x_{n-1},..., x_1 all lie inside the past lightcone of $\tilde{z}_2$, it implies that $\tilde{z}_1$ must lie inside the past lightcone of $\tilde{z}_2$, for Eq. (C3) to be non-zero. The same argument holds for the terms involving the contractions in Eq. (C4). Thus, Eq. (C2), and hence MCC, is satisfied even in the presence of the additional normal ordering that we impose to make the rate of increase of energy non-divergent.