Earliest stochastic Schrödinger equations from foundations

Lajos Diósi

Research Institute for Particle and Nuclear Physics (transforms in 2012 into: Wigner Physics Center)

December 7, 2011
Outline

1. Early Motivations 1970-1980’s
2. 1-Shot Non-Selective Measurement, Decoherence
3. Dynamical Non-Sel. Measurement, Decoherence
4. Master Equation
5. 1-Shot Selective Measurement, Collapse
6. Dynamical Non-selective Measurement, Collapse
7. Dynamical Collapse: Diffusion or Jump
8. Dynamical Collapse: Diffusion or Jump - Proof
Interpretation of $\psi$ is statistical.
Sudden ‘one-shot’ collapse $\psi \rightarrow \psi_n$ is central.

- If collapse takes time?
- Hunt for a math model (Pearle, Gisin, Diosi)
- New physics?
1-Shot Non-Selective Measurement, Decoherence

Measurement of $\hat{A}$, pre-measurement state $\hat{\rho}$, post-measurement state, decoherence:

$$\hat{A} = \sum_n A_n \hat{P}_n; \quad \sum_n \hat{P}_n = \hat{I}, \quad \hat{P}_n \hat{P}_m = \delta_{nm} \hat{I}$$

$$\hat{\rho} \rightarrow \sum_n \hat{P}_n \hat{\rho} \hat{P}_n$$

Off-diagonal elements become zero: Decoherence.

Example: $\hat{A} = \hat{\sigma}_z = |\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow|$, $\hat{P}_\uparrow = |\uparrow\rangle \langle \uparrow|$, $\hat{P}_\downarrow = |\downarrow\rangle \langle \downarrow|$, $\hat{\rho} = \rho_{\uparrow\uparrow} |\uparrow\rangle \langle \uparrow| + \rho_{\downarrow\downarrow} |\downarrow\rangle \langle \downarrow| + \rho_{\uparrow\downarrow} |\uparrow\rangle \langle \downarrow| + \rho_{\downarrow\uparrow} |\downarrow\rangle \langle \uparrow|$

$$\rightarrow \hat{P}_\uparrow \hat{\rho} \hat{P}_\uparrow + \hat{P}_\downarrow \hat{\rho} \hat{P}_\downarrow = \rho_{\uparrow\uparrow} |\uparrow\rangle \langle \uparrow| + \rho_{\downarrow\downarrow} |\downarrow\rangle \langle \downarrow|$$

Replace 1-shot non-selective measurement (decoherence) by dynamics!
Time-continuous (dynamical) measurement of $\hat{A} = \sum_k A_k \hat{P}_k$:

$$\frac{d\hat{\rho}}{dt} = -\frac{1}{2} [\hat{A}, [\hat{A}, \hat{\rho}]]$$

Solution:

$$[\hat{A}, [\hat{A}, \hat{\rho}]] = \sum_k A_k^2 \hat{P}_k \hat{\rho} + \sum_k A_k^2 \hat{\rho} \hat{P}_k - 2 \sum_{k,l} A_k A_l \hat{P}_k \hat{\rho} \hat{P}_l$$

$$d(\hat{P}_n \hat{\rho} \hat{P}_m)/dt = -\frac{1}{2} \hat{P}_n [\hat{A}, [\hat{A}, \hat{\rho}]] \hat{P}_m = -\frac{1}{2} (A_m - A_n)^2 (\hat{P}_n \hat{\rho} \hat{P}_m)$$

Off-diagonals $\to 0$, diagonals $= \text{const}$

Example: $\hat{A} = \hat{\sigma}_z$, $d\hat{\rho}/dt = -\frac{1}{2} [\hat{\sigma}_z, [\hat{\sigma}_z, \hat{\rho}]]$

$$\hat{\rho}(t) = \rho_{\uparrow\uparrow}(0) |\uparrow\rangle \langle \uparrow| + \rho_{\downarrow\downarrow}(0) |\downarrow\rangle \langle \downarrow|$$

$$+ e^{-2t} \rho_{\uparrow\downarrow}(0) |\uparrow\rangle \langle \downarrow| + e^{-2t} \rho_{\downarrow\uparrow}(0) |\downarrow\rangle \langle \uparrow|$$

$$\to \rho_{\uparrow\uparrow}(0) |\uparrow\rangle \langle \uparrow| + \rho_{\downarrow\downarrow}(0) |\downarrow\rangle \langle \downarrow|$$
Master Equations

General non-unitary (but linear!) quantum dynamics:

\[ \frac{d\hat{\rho}}{dt} = \mathcal{L}\hat{\rho} \]

Lindblad form — necessary and sufficient for consistency:

\[ \frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \left( \hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\hat{L}^\dagger\hat{L}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}^\dagger\hat{L} \right) + \ldots \]

If \( \hat{L} = \hat{L}^\dagger = \hat{A} \):

\[ \frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] - \frac{1}{2}[\hat{A}, [\hat{A}, \hat{\rho}]] \]

Decoherence (non-selective measurement) of \( \hat{A} \) competes with \( \hat{H} \).

General case \( \hat{H} \neq 0, \hat{L} \neq \hat{L}^\dagger \): unitary, decohering, dissipative, pump mechanisms compete.
Measurement of $\hat{A} = \sum_n A_n \hat{P}_n$; $\sum_n \hat{P}_n = \hat{I}$, $\hat{P}_n \hat{P}_m = \delta_{nm} \hat{I}$

General (mixed state) and the special case (pure state), resp.

- mixed state:
  $$\hat{\rho} \rightarrow \frac{\hat{P}_n \hat{\rho} \hat{P}_n}{p_n} \equiv \hat{\rho}_n$$
  with-prob. $p_n = \text{tr}(\hat{P}_n \hat{\rho})$

- pure state, $\hat{P}_n = |n\rangle \langle n|$
  $$|\psi\rangle \rightarrow |n\rangle \equiv |\psi_n\rangle$$
  with-prob. $p_n = |\langle n| \psi\rangle|^2$

Selective measurement is refinement of non-selective.

Mean of conditional states $= $ Non-selective post-measurement state:

$$\mathbf{M} \hat{\rho}_n = \sum_n p_n \hat{\rho}_n = \sum_n \hat{P}_n \hat{\rho} \hat{P}_n = \sum_n \hat{P}_n |\psi\rangle \langle \psi| \hat{P}_n$$

Replace 1-shot selective measurement (collapse) by dynamics!
Dynamical Non-selective Measurement, Collapse

Take pure state 1-shot measurement of $\hat{A} = \sum_n A_n |n\rangle \langle n|$ and expand it for asymptotic long times:

$$|\psi(0)\rangle \text{ evolves into } |\psi(t)\rangle \rightarrow |n\rangle$$

Construct a (stationary) stochastic process $|\psi(t)\rangle$ for $t > 0$ such that for any initial state $|\psi(0)\rangle$ the solution walks randomly into one of the orthogonal states $|n\rangle$ with probability $p_n = |\langle n|\psi(0)\rangle|^2$.

There are $\infty$ many such stochastic processes $|\psi(t)\rangle$. Luckily, for

$$\hat{\rho}(t) = M|\psi(t)\rangle \langle \psi(t)|$$

we have already constructed a possible non-selective dynamics, recall:

$$d\hat{\rho}/dt = -\frac{1}{2}[\hat{A}, [\hat{A}, \hat{\rho}]]$$

This is a major constraint for the process $|\psi(t)\rangle$. Infinite many choices still remain.
Dynamical Collapse: Diffusion or Jump

Consider the dynamical measurement of $\hat{A} = \sum_n A_n |n\rangle \langle n|$, described by dynamical decoherence (master) equation:

$$d\hat{\rho}/dt = -\frac{1}{2}[\hat{A}, [\hat{A}, \hat{\rho}]]$$

Construct stochastic process $|\psi(t)\rangle$ of dynamical collapse satisfying the master equation by $\hat{\rho}(t) = M |\psi(t)\rangle \langle \psi(t)|$.

  $$d|\psi\rangle/dt = -i\hat{H}|\psi\rangle - \frac{1}{2}(\hat{A} - \langle \hat{A} \rangle)^2 |\psi\rangle + (\hat{A} - \langle \hat{A} \rangle) |\psi\rangle w_t$$
  $$w_t : \text{standard white-noise}; \quad Mw_t = 0, \quad Mw_tw_s = \delta(t - s)$$

- Diosi’s Jump Process (1985/86):
  $$d|\psi\rangle/dt = -i\hat{H}|\psi\rangle - \frac{1}{2}(\hat{A} - \langle \hat{A} \rangle)^2 |\psi\rangle + \frac{1}{2}\langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle |\psi\rangle$$
  jumps $|\psi(t)\rangle \rightarrow \text{const.} \times (\hat{A} - \langle \hat{A} \rangle) |\psi(t)\rangle$ at rate $\langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle$
Gisin’s Diffusion Process (1984):

\[
\frac{d |\psi\rangle}{dt} = -i \hat{H} |\psi\rangle - \frac{1}{2} (\hat{A} - \langle \hat{A} \rangle)^2 |\psi\rangle + (\hat{A} - \langle \hat{A} \rangle) |\psi\rangle w_t
\]

\(w_t\) : standard white-noise; \(\mathbf{M} w_t = 0, \quad \mathbf{M} w_t w_s = \delta(t - s)\)

Diosi’s Jump Process (1985/86):

\[
\frac{d |\psi\rangle}{dt} = -i \hat{H} |\psi\rangle - \frac{1}{2} (\hat{A} - \langle \hat{A} \rangle)^2 |\psi\rangle + \frac{1}{2} \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle |\psi\rangle
\]

jumps \(|\psi(t)\rangle \rightarrow \text{const.} \times (\hat{A} - \langle \hat{A} \rangle) |\psi(t)\rangle\) at rate \(\langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle\)

If \([\hat{H}, \hat{A}] = 0\), prove:

\(\hat{\rho}(t) = \mathbf{M} |\psi(t)\rangle \langle \psi(t)\rangle\) satisfies \(d \hat{\rho}/dt = -\frac{1}{2} [\hat{A}, [\hat{A}, \hat{\rho}]]\)

\(|\psi(t)\rangle \rightarrow |n\rangle\)

\(|n\rangle\) occurs with \(p_n = |\langle n | \psi(0) \rangle|^2\)
Interpretation of $\psi$ is statistical.
Sudden ‘one-shot’ collapse $\psi \rightarrow \psi_n$ is central.

- If collapse takes time? — Why not!
- Hunt for a math model (Pearle, Gisin, Diosi) — Too many models!
- New physics?
  - No, it’s standard physics of real time-continuous measurement (monitoring).
  - Yes, it’s new!
    - to add universal non-unitary modifications to QM
    - to replace von Neumann statistical interpretation