# Features of Sequential Weak Measurements

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# WM vs post-selection

- In unsharp (imprecise) measurement on $\hat{\rho}$, post-measurement state preserves some well-defined features of $\hat{\rho}$.
- Imprecision $a$ of measurement can be compensated by larger ensemble statistics.
- Weak measurement (WM): asymptotic limit of zero precision $a \to \infty$ (and infinite statistics): pre-measurement state $\hat{\rho}$ invariably survives the measurement (non-invasiveness).
- WM was used by AAV as non-invasive quantum measurement between pre- and post-selected states, resp.
- *Non-invasiveness* of WM is remarkable both *with and without* post-selection, can be maintained for a succession of WMs on a single quantum system.

General features of such sequential WMs (SWMs): our topics.

# SWMs without post-selection

$\hat{A}$: measured; $A$: outcome; $\mathbf{M}$: statistical mean; $\langle\hat{A}\rangle$: q-expectation.

$\mathbf{M}A = \langle\hat{A}\rangle$ — single WM

$\mathbf{M}AB = \frac{1}{2}\langle\{\hat{A},\hat{B}\}\rangle$ — double WM: order doesn't matter

$\mathbf{M}ABC = \frac{1}{8}\big\langle\{\hat{A},\{\hat{B},\hat{C}\}\}\big\rangle$ — triple WM: $\hat{B}$, $\hat{C}$ are interchangeable

(Bednorz & Belzig 2010)

Generally:

$$\mathbf{M}A_1A_2\dots A_n = \frac{1}{2^n}\Big\langle\{\hat{A}_1,\{\hat{A}_2,\{\dots,\{\hat{A}_{n-1},\hat{A}_n\}\dots\}\}\}\Big\rangle$$

Correlation of SWM outcomes =

= Step-wise symmetrized quantum correlation of operators

Ordering in SWM matters but the last two ones are interchangeable.

Sufficient condition of full interchangeability:

$$[\hat{A}_k,\hat{A}_l] = \text{c-number} \quad (k,l = 1,2,\dots,n).$$

Then step-wise symmetrization $\Rightarrow$ symmetrization $\mathcal{S}$:

$$\mathbf{M}A_1A_2\dots A_n = \big\langle\mathcal{S}\hat{A}_1\hat{A}_2\dots\hat{A}_{n-1}\hat{A}_n\big\rangle$$

# SWM of canonical variables

$\hat{A}_k = u_k\hat{q} + v_k\hat{p} \ \ (k = 1, 2, \ldots, n)$ where $[\hat{q}, \hat{p}] = i$

Step-wise symmetrization $\Rightarrow$ symmetrization $\mathcal{S}$ = Weyl ordering!

Weyl-ordered correlation functions of $\hat{q}, \hat{p}$ =
= correlation functions (moments) of Wigner function $W(q, p)$.

$$\mathbf{M}A_1A_2\ldots A_n = \int W(q,p)A_1A_2\ldots A_n \mathrm{d}q\mathrm{d}p \equiv \langle A_1A_2\ldots A_n\rangle_W$$

(for $n = 2$: Bednorz & Belzig 2010)

Direct tomography through Wigner function moments:
Example: SWM of $\hat{q}, \hat{q}, \hat{p}, \hat{p}$ (in any order) yields

$\langle q\rangle_W = \mathbf{M}q_1 = \mathbf{M}q_2$; $\quad \langle p\rangle_W = \mathbf{M}p_1 = \mathbf{M}p_2$

$\langle q^2\rangle_W = \mathbf{M}q_1q_2$; $\quad \langle p^2\rangle_W = \mathbf{M}p_1p_2$,

$\langle qp\rangle_W = \mathbf{M}q_1p_1 = \mathbf{M}q_1p_2 = \mathbf{M}q_2p_1 = \mathbf{M}q_2p_2$

$\langle q^2p\rangle_W = \mathbf{M}q_1q_2p_1 = \mathbf{M}q_1q_2p_2$; $\quad \langle p^2q\rangle_W = \mathbf{M}p_1p_2q_1 = \mathbf{M}p_1p_2q_2$

$\langle q^2p^2\rangle_W = \mathbf{M}q_1q_2p_1p_2$

# SWM of spin-$\frac{1}{2}$ observable

SQM of $\hat{A}_1 = \hat{\sigma}_1,\ \hat{A}_2 = \hat{\sigma}_2,\ \ldots, \hat{A}_n = \hat{\sigma}_n;\quad (\hat{\sigma}_k = \hat{\vec{\sigma}}\vec{e}_k,\ |\vec{e}_k| = 1)$
Outcomes $A_1 = \sigma_1,\ A_2 = \sigma_2,\ \ldots, A_n = \sigma_n$
Surprize:

$$\mathbf{M}\sigma_1\sigma_2\ldots\sigma_n = \frac{1}{2^n}\left\langle\{\hat{\sigma}_1,\{\hat{\sigma}_2,\{\ldots,\{\hat{\sigma}_{n-1},\hat{\sigma}_n\}\ldots\}\}\}\right\rangle \qquad (*)$$

is valid no matter the measurements are weak or strong (ideal).

R.h.s. for SSM (with $\hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\sigma})$:

$$\mathbf{tr}\sum_{\sigma_n=\pm1}\sigma_n\hat{P}^{(n)}_{\sigma_n}\ldots\left(\sum_{\sigma_2=\pm1}\sigma_2\hat{P}^{(2)}_{\sigma_2}\left(\sum_{\sigma_1=\pm1}\sigma_1\hat{P}^{(1)}_{\sigma_1}\hat{\rho}\hat{P}^{(1)}_{\sigma_1}\right)\hat{P}^{(2)}_{\sigma_2}\right)\ldots\hat{P}^{(n)}_{\sigma_n}$$

Key identity $\sum_{\sigma=\pm}\sigma\hat{P}_\sigma\hat{O}\hat{P}_\sigma = \frac{1}{2}\{\hat{\sigma},\hat{O}\}$, using it $n$-times yields $(*)$!

Evaluating r.h.s. yields

$$\mathbf{M}\sigma_1\sigma_2\ldots\sigma_n = \begin{cases} (\vec{e}_1\vec{e}_2)(\vec{e}_3\vec{e}_4)\ldots(\vec{e}_{n-1}\vec{e}_n) & n \text{ even} \\ \langle\hat{\sigma}_1\rangle(\vec{e}_2\vec{e}_3)\ldots(\vec{e}_{n-1}\vec{e}_n) & n \text{ odd} \end{cases}$$

Correlations are kinematically constrained:

- $n$ even — correlations are independent of $\hat{\rho}$
- $n$ odd — correlations depend on $\hat{\rho}$ but via $\langle\hat{\sigma}_1\rangle$

# Testing SWM in Time-Continuous Measurement

- TCM is standard theory.
- TCMs are standard in lab.
- TCMs have WM regime!

TCM of Heisenberg $\hat{A}_t$ in state $\hat{\rho}$, outcomes (signal) $A_t$:

$$A_t = \langle \hat{A}_t \rangle + \sqrt{\alpha} w_t;$$

$\alpha$ : precision/unsharpness of TCM
$w_t$ : standard white-noise

TCM is invasive on the long run but it remains non-invasive as long as

$$\int_0^t \langle (\Delta \hat{A}_s)^2 \rangle ds \ll \alpha.$$

That's where SQM applies to signal's auto-correlation:

$$\mathbf{M} A_{t1} A_{t2} = \frac{1}{2} \langle \{ \hat{A}_{t1}, \hat{A}_{t2} \} \rangle$$

$$\mathbf{M} A_{t1} A_{t2} A_{t3} = \frac{1}{2} \langle \{ \hat{A}_{t1}, \{ \hat{A}_{t2}, \hat{A}_{t3} \} \} \rangle \quad \text{etc.}$$

Recall r.h.s.'s must be Wigner function moments if $\hat{A}$ is harmonic, kinematically constrained if $\hat{A}$ is spin-$\frac{1}{2}$.

# SWM with post-selection

Outcome correlations:

$$\mathbf{M}A_1, A_2, \ldots, A_n|_{psel} = \frac{\langle\{\hat{A}_1, \{\hat{A}_2, \ldots, \{\hat{A}_n, \hat{\Pi}\}\ldots\}\}\rangle}{2^n\langle\hat{\Pi}\rangle}.$$

Generic post-selection (D. 2006, Silva & al. 2014): $0 \leq \hat{\Pi} \leq 1$.
For pure state pre/post-selection $\hat{\rho} = |i\rangle\langle i|, \hat{\Pi} = |f\rangle\langle f|$, introduce sequential weak values:

$$(A_1, A_2, \ldots, A_n)_w = \frac{\langle f|\hat{A}_n\hat{A}_{n-1}\ldots\hat{A}_1|i\rangle}{\langle f|i\rangle}$$

$$\mathbf{M}A_1, A_2, \ldots, A_n|_{psel} = \frac{1}{2^n}\sum(A_{i_1}, A_{i_2}, \ldots, A_{i_r})_w(A_{j_1}, A_{j_2}, \ldots, A_{j_{n-r}})^\star_w$$

$\Sigma$ for all partitions $(i_1, i_2, \ldots, i_r) \cup (j_1, j_2, \ldots, j_{n-r}) = (1, 2, \ldots, n)$ where $i$'s and $j$'s remain ordered. Degenerate partitions $r = 0, n$, too, must be counted. (Mitchison, Jozsa, Popescu 2007)

- $n = 1$ reduces to AAV 1988.
- $n = 2$ contains a new paradox.

# Re-selection paradox

Special post-selection: $|i\rangle = |f\rangle$, call it *re-selection*.
For single WM, re-selection is equivalent with no-post-selection:

$$\mathbf{M}A = \mathbf{M}A|_{rsel} = \langle\hat{A}\rangle$$

WMs are non-invasive, we expect re-selection and *no-post-selection* are equivalent. But they aren't, already for $n=2$ and $\hat{A}_1=\hat{A}_2=\hat{A}$:

$$\begin{aligned}\mathbf{M}A_1A_2 &= \langle i|\hat{A}^2|i\rangle,\\ \mathbf{M}A_1A_2|_{rsel} &= \frac{1}{2}\langle i|\hat{A}^2|i\rangle + \frac{1}{2}(\langle i|\hat{A}|i\rangle)^2\end{aligned}$$

Re-selection decreases $\mathbf{M}A_1A_2$ by half of $(\Delta A)^2$ in state $|i\rangle$:

$$\mathbf{M}A_1A_2 - \mathbf{M}A_1A_2|_{rsel} = \frac{1}{2}(\Delta A)^2. \tag{1}$$

Unexpected anomaly! Reason is *finite* contribution of outcomes *discarded* by re-selection.

# Example: spin-$\frac{1}{2}$

$R_{disc}$ — rate of discards; $a$ — precision/unsharpness of measurements

$$\mathbf{M}\ldots|_{rsel} = \mathbf{M}\cdots - \lim_{a\to\infty}\left(R_{disc}\mathbf{M}\ldots|_{disc}\right)$$

In WM limit $a\to\infty$ of re-selection: $R_{disc}\to 0$.

Single WM of $\hat\sigma\equiv\hat\sigma_x$, outcome $\sigma_1$ with re-selection $|i\rangle=|f\rangle=|\uparrow\rangle$:

- $R_{disc}\sim(1/4a^2)\to 0$.
- $\mathbf{M}\sigma_1|_{disc}=0$ hence $R_{disc}\mathbf{M}\sigma_1|_{disc}=0$ anyway.

SWM of $\hat\sigma_1=\hat\sigma_2\equiv\hat\sigma_x$, outcomes $\sigma_1,\sigma_2$ with re-selection $|i\rangle=|f\rangle=|\uparrow\rangle$:

- $R_{disc}\sim(1/2a^2)\to 0$.
- $\mathbf{M}\sigma_1\sigma_2|_{disc}=a^2$ hence $R_{disc}\mathbf{M}\sigma_1\sigma_2|_{disc}\to 1/2$, QED.

Correlation of double $\hat\sigma_x$ WM in state $|\uparrow\rangle$ diverges on the discarded events in re-selection. Explains why re-selection differs from no-post-selection. Novel SWM anomalies add to AAV88.

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