Unitary equivalence of Schrödinger and Heisenberg pictures survives in meanfield nonlinear quantum mechanics

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Abstract

The equivalence of Schrödinger, interaction, and Heisenberg pictures is often thought to break down in nonlinear quantum mechanics due to the absence of a state-independent unitary propagator. We show that for Hamiltonians depending on the state through instantaneous expectation values (meanfields), the equivalence is preserved. While the unitary intertwiner becomes initial-state dependent, it remains well-defined and ensures identical physical predictions in all three pictures.

Meanfield Couplings: Nonlinear Schrödinger Eqs.

Effective theories

Hartree-Fock, Gross-Pitevskii, Bogoliubov-de Gennes, ...

Semiclassical Gravity [Møller1962,Rosenfeld1963]

$$
\begin{aligned}
|\dot{\Psi}\rangle &= -i\hat{H}[g]|\Psi\rangle \quad \text{(nonlinear Sch. eq.)} \\
\mathrm{G}_{ab}[g] &= 8\pi G c^{-4}\langle\Psi|\hat{T}_{ab}|\Psi\rangle \quad \text{(semiclassical Einstein eq.)}
\end{aligned}
$$

Workhorse in cosmology.

Newton-Schrödinger Eq. [D1984,Penrose1996]

$$c \to \infty,\ \hat{T}_{00}/c^2 \to \hat{\mu},\ (g_{00}-1)c^2/2 \to \Phi$$

$$
\begin{aligned}
|\dot{\Psi}\rangle &= -i\left(\hat{H}_0 + \int \hat{\mu}(\mathbf{r})\Phi(\mathbf{r})d\mathbf{r}\right)|\Psi\rangle \quad \text{(nonlinear Sch. eq.)} \\
\Delta\Phi &= 4\pi G\langle\Psi|\hat{\mu}|\Psi\rangle \quad \text{(semiclassical Poisson eq.)}
\end{aligned}
$$

Workhorse for ‘quantum gravity’ in the lab.

Schrödinger vs Heisenberg picture, standard QM

Schrödinger's

$$|\dot{t}\rangle = -i\hat{H}_t|t\rangle, \quad \hat{A}_t \equiv \hat{A}_., \quad A_t = \langle t|\hat{A}_0|t\rangle$$

Heisenberg's

$$|t\rangle \equiv |0\rangle \quad \dot{\hat{A}}_t = i[\hat{H}_t, \hat{A}_t] \quad A_t = \langle 0|\hat{A}_t|0\rangle$$

Unitary map:

$$d\hat{U}_t/dt = -i[\hat{H}_t, \hat{U}_t] \quad \hat{U}_0 = \hat{1}$$

Solution satisfies $\hat{U}_t^\dagger \hat{U}_t \equiv \hat{1}$:

$$U_t = \mathcal{T}\exp\left(-i\int_0^t \hat{H}_\tau d\tau\right)$$

$$|t\rangle = \hat{U}_t|0\rangle, \qquad \hat{A}_t = \hat{U}_t^\dagger \hat{A}_0 \hat{U}_t$$

Operator algebra is preserved:

$$[\hat{A}_t, \hat{B}_t] = \hat{C}_t$$

Nonlinear Schrödinger Eq.

Hamiltonian depends on state $|t\rangle$, it is a non-linear operator. Mean-field nonlinearity: Hamiltonian depends on expectation value $O_t = \langle t|\hat{O}|t\rangle$ of an observable $\hat{O}$ (or a list of them).

$$\hat{H}_t = \hat{H}(O_t)$$

$$|\dot{t}\rangle = -i\hat{H}(O_t)|t\rangle, \qquad \hat{A}_t \equiv \hat{A}_., \qquad A_t = \langle t|\hat{A}_0|t\rangle$$

Heisenberg's

$$|t\rangle \equiv |0\rangle, \qquad \dot{\hat{A}}_t = i[\hat{H}(O_t), \hat{A}_t] \qquad A_t = \langle 0|\hat{A}_t|0\rangle$$

Unitary(?) map

$$|t\rangle = \hat{U}_t|0\rangle, \qquad \hat{A}_t = \hat{U}_t^\dagger \hat{A}_0 \hat{U}_t$$

Widespread concerns: $\hat{U}_t$ is non-unitary, not even linear! Heisenberg picture does not exist. If it does, it is not equivalent with Schrödinger's.
So what?

$\hat{U}_t$: initial-state dependent unitary

Schrödinger's

$$|\dot{t}\rangle = -i\hat{H}(O_t)|t\rangle, \quad \hat{A}_t \equiv \hat{A}_., \quad A_t = \langle t|\hat{A}_0|t\rangle.$$

Heisenberg's

$$|t\rangle \equiv |0\rangle, \quad \dot{\hat{A}}_t = i[\hat{H}(O_t), \hat{A}_t], \quad A_t = \langle 0|\hat{A}_t|0\rangle.$$

Unitary map becomes initial-state-dependent:

$$\begin{aligned} d\hat{U}_t/dt &= -i[\hat{H}(O_t), \hat{U}_t], \quad \hat{U}_0 = \hat{1}, \\ O_t &= \langle 0|\hat{U}_t^\dagger \hat{O}\hat{U}_t|0\rangle \end{aligned}$$

Solution $\hat{U}_{t;|0\rangle}$ depends on $|0\rangle$ and satisfies $\hat{U}^\dagger_{t;|0\rangle}\hat{U}_{t;|0\rangle} \equiv \hat{1}$:

$$U_{t;|0\rangle} = \mathcal{T}\exp\left(-i\int_0^t \hat{H}(O_\tau)d\tau\right).$$

$|t\rangle = \hat{U}_{t;|0\rangle}|0\rangle$ non-unitary, $\quad \hat{A}_{t;|0\rangle} = \hat{U}^\dagger_{t;|0\rangle}\hat{A}_0\hat{U}_{t;|0\rangle}$ unitary.

Operator algebra is preserved:

$$[\hat{A}_t, \hat{B}_t] = \hat{C}_t.$$

Spin-$\frac{1}{2}$ in 'magnetic' mean-field

Nonlinear spin Hamiltonian [Gisin1990]: $\hat{H}(\sigma_t^x) = \frac{1}{2}\omega\sigma_t^x\,\hat{\sigma}^z$,
z-precession frequency $\propto \sigma_t^x = \langle t|\hat{\sigma}^x|t\rangle$ of the Bloch vector.

$$|\dot{t}\rangle = -\tfrac{i}{2}\omega\,\sigma_t^x\,\hat{\sigma}^z\,|t\rangle.$$

Solution for $\hat{\boldsymbol{\sigma}}_t = \langle t|\hat{\boldsymbol{\sigma}}|t\rangle$:

$$\sigma_t^x = \rho\cos\phi_t, \qquad \sigma_t^y = \rho\sin\phi_t, \qquad \sigma_t^z = \sigma_0^z$$

$$\phi_t = \phi_0 + \arcsin\tanh(\rho\omega t),$$

Unitary propagator is $|0\rangle$-dependent z-rotation:

$$\hat{U}_{t;|0\rangle} = \hat{U}_{t;\sigma^x,\sigma^y,\sigma^z} = \exp\left[-\frac{i}{2}\Big(\phi_t - \phi_0\Big)\hat{\sigma}^z\right].$$

Heisenberg observables:

$$\begin{aligned}
\hat{\sigma}_t^x &= \cos\left(\phi_t - \phi_0\right)\hat{\sigma}^x - \sin\left(\phi_t - \phi_0\right)\hat{\sigma}^y \\
\hat{\sigma}_t^y &= \cos\left(\phi_t - \phi_0\right)\hat{\sigma}^y + \sin\left(\phi_t - \phi_0\right)\hat{\sigma}^x \\
\hat{\sigma}_t^z &= \hat{\sigma}^z
\end{aligned}$$

Mean-field directional force

Free particle of momentum $\hat{p}$ and position $\hat{q}$, directional force $\propto$ mean position $q_t = \langle t|\hat{q}|t\rangle$.

$$\hat{H}(q_t) = \tfrac{1}{2}\hat{p}^2 + \omega^2 q_t \hat{q}.$$

(Classical limit: central harmonic oscillator of frequency ω.)
Heisenberg operators:

$$\hat{p}_t = \hat{p} + (p_t - p_0), \qquad \hat{q}_t = \hat{q} + t\hat{p} + (q_t - q_0 - p_0 t).$$

Unitary propagator is (free) plus $|0\rangle$-dependent shift & boost:

$$\hat{U}_{t;|0\rangle} = \hat{U}_{t;q_0,p_0} = \exp\Big[-i(p_t - p_0)\hat{q}\Big].\exp\Big[i(q_t - q_s 0 - p_0 t)\hat{p}\Big]\exp[-it\hat{p}^2/2]$$

depends on $|0\rangle$ only through the initial expectation values

$$q_0 = \langle 0|\hat{q}|0\rangle, \quad p_0 = \langle 0|\hat{p}|0\rangle.$$

Meanfield coupling in local field theory

$$\hat{H}_t^{nonlin} = \lambda \int O_t(\mathbf{x})\hat{\phi}(\mathbf{x})d\mathbf{x}, \quad O_t(\mathbf{x}) = \langle t|\hat{O}(\mathbf{x})|t\rangle$$

Map $|t\rangle = \hat{U}_t|0\rangle$ is nonlinear, state-dependent, not unitary in the usual sense, does not act as algebra automorphism.
Ennoying implications: breakdown of microcausality $[\hat{A}_t(\mathbf{x}), \hat{B}_t(\mathbf{x}')] = 0$ for $\mathbf{x} \neq \mathbf{x}'$, breakdown of covariance, and instantaneous entanglement generation between separate locations.

[D.H.Hsu, PLB872.140053(2026)]

I felt compelled to object, since Semiclassical Gravity (mean-field theory) can retain microcausality, covariance, and should not induce entanglement. I showed that $\hat{U}_t$ depends only on $|0\rangle$ (in addition to t), hence $\hat{U}_{t;|0\rangle}$ does act as algebra authomorphism, ensures microcausality, and covariance.

[L.D, preprint arXiv:2602.06845]

Summary

We have shown that the standard equivalence between Schrödinger and Heisenberg pictures survives in nonlinear quantum mechanics, provided the nonlinearity enters through expectation values. The key is to recognize that while a universal, state-independent unitary propagator $\hat{U}_t$ no longer exists, a trajectory-wise unitary $\hat{U}_{t;|0\rangle}$ can always be constructed for any given initial state $|0\rangle$. The evolution

$$|t\rangle = \hat{U}_{t;|0\rangle}|0\rangle$$

of the Schrödinger state is non-unitary but the evolution

$$\hat{A}_t = \hat{U}^{\dagger}_{t;|0\rangle}\hat{A}_0\hat{U}_{t;|0\rangle}$$

of the Heisenberg operators is unitary, although it depends on the constant Heisenberg state $|0\rangle$.