WEAK MEASUREMENTS IN CLASSICAL AND QUANTUM CONTEXTS

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Standard quantum mean value:

$$\langle \widehat{A} \rangle_i =: \langle i | \widehat{A} | i \rangle$$

interpreted statistically. No other forms had been known to possess a statistical interpretation. But the *weak value*:

$$A_{\mathsf{w}} =: \frac{\langle f | \widehat{A} | i \rangle}{\langle f | i \rangle}$$

(Aharonov, Albert and Vaidman, 1988) has plausible statistical interpretation! $|i\rangle, |f\rangle$ are prepared initial and the postselected final states. Statistical interpretation relies upon *weak measurement*s. Paradoxical application: weak measurement yields electron spin 100 instead of $\pm 1/2$. Common application: time-continuous filtering/control, classical and quantum.

State:	$\rho, \widehat{\rho}$
Measurable:	$A, \widehat{A}$
Theoretic mean:	$\langle A\rangle_\rho, \langle\widehat{A}\rangle_{\widehat{\rho}}$
Measured value:	a
Statistical error:	σ
Statistics:	N
Averaged measurement value:	$\overline{a}=(\sum_1^N a_.)/N$
Estimation of mean:	$\langle A\rangle_\rho, \langle\widehat{A}\rangle_{\widehat{\rho}} = \overline{a}$ "$\pm\frac{\sigma}{\sqrt{N}}$"

The *weak measurement limit*:

$$\sigma, N \quad \rightarrow \quad \infty \qquad\qquad \Delta^2 =: \frac{\sigma^2}{N} = \text{const}$$

In practice: σ must be greater than the whole range (max$A-$min$A$) [or (max eigenval$\widehat{A}-$min eigenval$\widehat{A}$)] while N must grow like $\sim\sigma^2$ then you achieve any fine resolution $\Delta \ll$ (max$A-$ min$A$).

Alternative to Ensemble Statistics: Single-System Temporal Statistics. Then the N weak measurements concern a single system repeatedly at frequency ν. The *weak measurement limit* for the temporal statistics:

$$\sigma, \nu \quad \rightarrow \quad \infty \qquad\qquad g^2 =: \frac{\sigma^2}{\nu} = \text{const}$$

This is time-continuous filtering/measurement (in classical theory) or time-continuous collapse/measurement (in quantum theory). It leads to *universal* state-evolution equations where $1/g$ becomes the *strength* of time-continuous filtering/measurement//collapse.

3 TIME-CONTINUOUS MEASUREMENT

A single time-dependent state ρ_t is undergoing an infinite sequence of weak measurements of A employed at times $t = \delta t, t = 2\delta t, t = 3\delta t, \dots$. The rate $\nu =: 1/\delta t$ goes to infinity together with the mean squared error σ^2. Their rate is kept constant: $g^2 =: \frac{\sigma^2}{\nu} = \text{const}$. Infinite many infitite small Baysian updates! The resulting theory: time-continuous measurement.

$$
\begin{aligned}
a_t &= \langle A \rangle_{\rho_t} + g w_t \\
\frac{\mathrm{d}\rho_t}{\mathrm{d}t} &= g^{-1} \left(A - \langle A \rangle_{\rho_t} \right) \rho_t w_t \\
< w_t w_s > &= \delta(t-s), \qquad\qquad < w_t >= 0.
\end{aligned}
$$

Special case of the Kushner-Stratonovich (1968) eq. for time-continuous Bayesian inference conditioned on the continuous measurement of A yielding the time-dependent outcome value a_t.

The first eq. is plausible: measurement outcome equals the theoretical mean plus white noise. Second eq. is state evolution: gradual shrinkage of ρ_t so that $\langle A \rangle_{\rho_t}$ tends to a random asymptotic value.

Sudden vs continuous collapse

Discrete binary distribution $\rho_t(1), \rho_t(2)$, and measurable: $A(1) = +1$, $A(2) = -1$. Alternatives: sudden collapse or continous collapse.

Sudden (Bayesian) collapse: single 'strong' (even ideal) measurement of A at, say, $t = 0$.

with prob. $\rho_0(1)$: $\quad a = +1, \quad \rho_{+0}(1) = 1,\ \rho_{+0}(2) = 0$
with prob. $\rho_0(2)$: $\quad a = -1, \quad \rho_{+0}(1) = 0,\ \rho_{+0}(2) = 1$

Continuous collapse: many-many repeated very-very weak measurements of A for $t \geq 0$.

$$
\begin{aligned}
a_t &= \langle A \rangle_{\rho_t} + g w_t \\
\frac{\mathrm{d}\rho_t}{\mathrm{d}t} &= g^{-1} \left(A - \langle A \rangle_{\rho_t}\right) \rho_t w_t
\end{aligned}
$$

Let $q = \rho(1) - \rho(2)$, then:

$$
\begin{aligned}
a_t &= q_t + g w_t \\
\frac{\mathrm{d}q_t}{\mathrm{d}t} &= g^{-1}(1 - q_t^2) w_t
\end{aligned}
$$

For $t \to \infty$: two stationary states with $q_\infty = \pm 1$ achieved with probabilities $\rho_0(1)$ and $\rho_0(2)$, respectively. Time-continuous collapse = connatural time-continuous resolution of the 'sudden' ideal measurement.

A single time-dependent state $\widehat{\rho}_t$ is undergoing an infinite sequence of weak measurements of $\widehat{A}$ employed at times $t = \delta t, t = 2\delta t, t = 3\delta t, \dots$. The rate $\nu =: 1/\delta t$ goes to infinity together with the mean squared error σ^2. Their rate is kept constant: $g^2 =: \frac{\sigma^2}{\nu} =$ const. Infinite many infitite small collapses! The resulting theory: time-continuous measurement (Diósi, Belavkin, 1988).

$$\begin{aligned} a_t &= \langle \widehat{A} \rangle_{\widehat{\rho}_t} + g w_t \\ \frac{\mathrm{d}\widehat{\rho}_t}{\mathrm{d}t} &= g^{-1} \left(\widehat{A} - \langle \widehat{A} \rangle_{\widehat{\rho}_t} \right) \widehat{\rho}_t w_t \\ &- \frac{1}{8} g^{-2} [\widehat{A}, [\widehat{A}, \widehat{\rho}_t]] \end{aligned}$$

This is quantum version of Kushner-Stratonovich eq. of classical time-continuous Bayesian inference. Single remarkable difference: the *decoherence term* $-[\widehat{A}, [\widehat{A}, \widehat{\rho}_t]]$. It tends to diagonalize $\widehat{\rho}_t$ in the eigenbasis of $\widehat{A}$.

Sudden vs continuous q-collapse

Qubit state $\widehat{\rho}_t$, and measurable: $\widehat{A} = \widehat{\sigma}_z$. Alternatives: sudden collapse or continous collapse.

Sudden (von Neumann-Lüders) collapse: single 'strong' (even ideal) measurement of $\widehat{A}$ at, say, $t = 0$.

with prob. $\widehat{\rho}_0(1,1)$: $\quad a = +1, \quad \widehat{\rho}_{+0}(1,1) = 1, \ \ldots$
with prob. $\widehat{\rho}_0(2,2)$: $\quad a = -1, \quad \widehat{\rho}_{+0}(2,2) = 1, \ \ldots$

Continuous collapse: many-many repeated very-very weak measurements of $\widehat{A}$ for $t \geq 0$.

$$\begin{aligned} a_t &= \langle \widehat{\sigma}_z \rangle_{\widehat{\rho}_t} + g w_t \\ \frac{d\widehat{\rho}_t}{dt} &= g^{-1} \left(\widehat{\sigma}_z - \langle \widehat{\sigma}_z \rangle_{\widehat{\rho}_t} \right) \widehat{\rho}_t w_t \\ &- \frac{1}{8} g^{-2} [\widehat{\sigma}_z, [\widehat{\sigma}_z, \widehat{\rho}_t]] \end{aligned}$$

Let $q = \mathrm{tr}(\widehat{\sigma}_z \widehat{\rho})$, then:

$$\begin{aligned} a_t &= q_t + g w_t \\ \frac{dq_t}{dt} &= g^{-1}(1 - q_t^2) w_t \end{aligned}$$

For $t \to \infty$: two stationary states with $q_\infty = \pm 1$ achieved with probabilities $\widehat{\rho}_0(1,1)$ and $\widehat{\rho}_0(2,2)$, respectively. Time-continuous collapse = connatural time-continuous resolution of the 'sudden' ideal measurement.

5 WEAK Q-MEASUREMENT, POSTSELECTION

For the preselected state ρ, we introduce postselection via the real function Π where $0 \leq \Pi \leq 1$. Postselected mean value of A is defined:

$$_{\Pi}\langle A\rangle_\rho =: \frac{\langle \Pi A\rangle_\rho}{\langle \Pi\rangle_\rho}$$

The $\langle \Pi\rangle_\rho$ is the rate of postselection. Statistical interpretation: having obtained the outcome a from measurement of A, we measure Π, too, in ideal measurement yielding random outcome π; with probability π we include the current a into the statistics and we discard it otherwise. Then, on a large postselected statistics:

$$_{\Pi}\langle A\rangle_\rho = \overline{a} \text{ ``}\pm\frac{\sigma}{\sqrt{N}}\text{''}.$$

Effective postselected state exists: $\rho_\Pi =: \frac{\Pi\rho}{\langle \Pi\rangle_\rho}$.

Quantum postselection is subtle! The quantum counterpart of postselected mean, i.e.:

$$_{\widehat{\Pi}}\langle \hat{A}\rangle_{\hat{\rho}} =: \mathrm{Re}\frac{\langle \widehat{\Pi}\hat{A}\rangle_{\hat{\rho}}}{\langle \widehat{\Pi}\rangle_{\hat{\rho}}}$$

has no statistical interpretation unless the measurement of $\hat{A}$ is *weak measurement*. Then it goes like the classical one:

$$_{\widehat{\Pi}}\langle \hat{A}\rangle_{\hat{\rho}} = \overline{a} \text{ ``}\pm\frac{\sigma}{\sqrt{N}}\text{''}$$

The quantum weak value anomaly

Special case: both the state $\widehat{\rho} = |i\rangle\langle i|$ and the post-selected operator $\widehat{\Pi} = |f\rangle\langle f|$ are pure states. Then ${}_{\widehat{\Pi}}\langle\widehat{A}\rangle_{\widehat{\rho}}$ reduces to:

$$ {}_f\langle\widehat{A}\rangle_i =: \mathrm{Re}\frac{\langle f|\widehat{A}|i\rangle}{\langle f|i\rangle} $$

The rate of postselection is $|\langle f|i\rangle|^2$. Choose:

$$ \begin{aligned} |i\rangle &= \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\phi/2} \\ e^{-i\phi/2} \end{bmatrix} \\ |f\rangle &= \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\phi/2} \\ e^{i\phi/2} \end{bmatrix} \end{aligned} $$

Postselection rate: $\cos^2\phi$. Let us weakly measure:

$$ \widehat{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} $$

Its *weak value*:

$$ {}_f\langle\widehat{A}\rangle_i = \frac{1}{\cos\phi} \tag{1} $$

lies *outside* the range of the eigenvalues of $\widehat{A}$. The anomaly can be arbitrary large if the rate $\cos^2\phi$ of postselection decreases.

Striking consequences follow from this anomaly if we turn to the statistical interpretation. For concreteness, suppose $\phi = 2\pi/3$ so that ${}_f\langle\widehat{A}\rangle_i = 2$. On average, seventy-five percents of the statistics N will be lost in postselection. The arithmetic mean $\overline{a}$

of the postselected outcomes of independent weak measurements converges stochastically to the weak value upto the fluctuation Δ:

$$\overline{a} = 2 \text{ ``}\pm\Delta\text{''}$$

Choose $\sigma = 10$ which is already well beyond the scale of the eigenvalues ± 1 of the observable $\widehat{A}$. Then:

$$\Delta^2 = \sigma^2/N(post) = 400/N$$

Accordingly, if $N = 3600$ independent quantum measurements of precision $\sigma = 10$ are performed regarding the observable $\widehat{A}$ then the arithmetic mean $\overline{a}$ of the ~ 900 postselected outcomes a will be 2 ± 0.33. This exceeds significantly the largest eigenvalue of the measured observable $\widehat{A}$. Quantum postselection appears to bias the otherwise unbiassed non-ideal weak measurements.

SUMMARY AND RELATED CONTEXTS

I discussed two particular applications of weak measurement: in postselection and in time-continuous measurement, There are further real variants of the weak measurement limit. Like the usual thermodynamic limit in standard statistical physics. Then weak measurements concern a certain additive microscopic observable (e.g.: the spin) of each constituent and the weak value represents the corresponding additive macroscopic parameter (e.g.: the magnetization) in the infinite volume limit. This example indicates that weak values have natural interpretation despite the apparent artificial conditions of their definition. It is important that the weak value, with or without postselection, plays the physical role similar to that of the common mean $\langle\widehat{A}\rangle_{\widehat{\rho}}$. If, between their pre- and postselection, the states $\widehat{\rho}$ become weakly coupled with the state of another quantum system via the observable $\widehat{A}$ their average influence will be as if $\widehat{A}$ took the weak value ${}_{\widehat{\Pi}}\langle\widehat{A}\rangle_{\widehat{\rho}}$. Weak measurements also open a specific loophole to circumvent quantum limitations related to the irreversible disturbances that quantum measurements cause to the measured state. Non-commuting observables become simultaneously measurable in the weak limit: simultaneous weak values of non-commuting observables will exist.

Literally, weak measurement had been coined in 1988 for quantum measurements with (pre- and) postselection, and became the tool of a certain time-symmetric statistical interpretation of quantum states. Foundational applications target the paradoxical problem of pre- and retrodiction in quantum theory. In a broad sense, however, the very principle of weak measurement encapsulates the trade between asymptotically weak precision and asymptotically large statistics. Its relevance in different fields has not yet been fully explored. Growing number of foundational, theoretical, and experimental applications are being considered in the literature – predominantly in the context of quantum physics.