GRAVITATIONAL MULTIPOLE MOMENTS

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ABSTRACT

An algorithm is described for computing the Hansen moments of an asymptotically flat, empty, stationary and axisymmetric space-time. The values of the first 12 multipole moments are given in term of the power series expansion coefficients of the Ernst potential on the symmetry axis. The moments of the first four Tomimatsu-Sato solutions are presented up to the twelfth order.

1. INTRODUCTION

Multipole moments in general relativity are expected to be useful for various purposes, and considerable effort has been made in the past few years to work out their theory. Simon and Beig [1] have shown that a given set of moments determine a solution of Einstein's equation uniquely. They provide a coordinate-free description of the space-time unlike many other methods. The multipole moments may even yield an effective means of generating solutions of the gravitational equations.

Geroch [2] has defined the multipole moment tensors in curved, static, asymptotically flat and empty space-times. Hansen [3] has generalized the notion of gravitational multipole moments to asymptotically flat stationary space-times. We can define a three-metric on the set of trajectories of the stationary Killing vector K^i by

$$h_{ij} = f g_{ij} + K_i K_j, \tag{1}$$

where g_{ij} is the metric of the spacetime, and $f = K^i K_i$.

A three dimensional manifold $\mathcal{M}$ with positive-definite metric h_{ij} is called asymptotically flat if there exists a manifold $\tilde{\mathcal{M}}$ with metric $\tilde{h}_{ij}$, such that :

(i) $\tilde{\mathcal{M}} = \mathcal{M} \cup \Lambda$, vhere Λ is a single point,

(ii) $\tilde{h}_{ik} = \Omega^2 h_{ik}$, where Ω is C^2 on $\tilde{\mathcal{M}}$,

(iii) $\Omega|_\Lambda = \Omega_{,i}|_\Lambda = 0, \qquad \tilde{D}_i \tilde{D}_k \Omega|_\Lambda = \tilde{h}_{ik}|_\Lambda .$

The following steps lead to the definition of the multipole moment tensors of a scalar potential ϕ :

(i) conformal transformation of the potential ϕ, let $\tilde{\phi} = \Omega^{-1/2}\phi$,

(ii) supposing that $\tilde{\phi}$ can be smoothly extended to the point Λ, we can define the following tensor fields on $\tilde{M}$

$$
\begin{aligned}
P^{(0)} &= \tilde{\phi} \\
P_i^{(1)} &= \tilde{D}_i P^{(0)} \\
&\vdots \\
P^{(n+1)}_{i_1 i_2 \ldots i_{n+1}} &= C\left[\tilde{D}_{i_{n+1}} P^{(n)}_{i_1 \ldots i_n} - \frac{1}{2} n(2n-1) \tilde{R}_{i_1 i_2} P^{(n-1)}_{i_3 \ldots i_{n+1}}\right],
\end{aligned} \tag{2}
$$

where C denotes the operation of taking the symmetric and trace-free part, $\tilde{D}_i$ is the derivative operator and $\tilde{R}_{ij}$ is the Ricci tensor[4] associated with the transformed metric $\tilde{h}_{ik}$,

(iii) the multipole moment tensors are the values of these tensor fields at conformal infinity,

$$M^{(n)}_{i_1 \ldots i_n} = P^{(n)}_{i_1 \ldots i_n}|_\Lambda . \tag{3}$$

If the space is flat and ϕ is the Newtonian potential, $\Delta\phi = 0$, then $M^{(n)}_{i_1 \ldots i_n}$ equals the classical multipole moment tensor.

If we calculate the moments using $\bar{\Omega} = \omega\Omega$ instead of Ω, we get [5]:

$$\bar{M}^{(n)}_{i_1 \ldots i_n} = C \sum_{k=0}^{n} \binom{n}{k} \frac{(2n-1)!!}{(2k-1)!!} (-2)^{k-n} M^{(k)}_{i_1 \ldots i_k} \tilde{D}_{i_{k+1}} \omega|_\Lambda \ldots \tilde{D}_{i_n} \omega|_\Lambda . \tag{4}$$

The change of the moments depends only on $\tilde{D}_i\omega|_\Lambda$, which is a vector of only three real components. This behaviour reflects the usual dependence of the Newtonian moments on the choice of origin. Had we left out the Ricci-tensor term from the definition of the moments, the change would have depended also on the higher derivatives of ω at the point Λ. We can make the definition of the moments unambiguous if we choose ω such that the real part of $M_i^{(1)}$ vanishes. This corresponds to the choice of a coordinate system which is at the center of mass, so the real part of the dipole moment is zero.

How to choose the potential ϕ in general relativity ? Let $f = K^i K_i$ be the norm, and $\psi_i = \epsilon_{ijkl} K^j K^{k;l}$ the curl of the stationary Killing field K^i. It follows from the vacuum Einstein equations that the curl is locally a gradient; $\psi_i = \psi_{;i}$. It is convenient to unify these potentials in the complex Ernst notation

$$\mathcal{E} = f + i\psi, \tag{5}$$

and define the complex gravitational potential

$$\xi = \frac{1-\mathcal{E}}{1+\mathcal{E}}. \tag{6}$$

We will calculate the moments of the potential $\phi = \xi$. As a result we get the mass and angular moments defined by Geroch[2] for the real and imaginary parts.

It follows from the vacuum Einstein equations, that the Ernst [6] equation holds for ξ :

$$(\xi\xi^* - 1)\Delta\xi = 2\xi^*(\nabla\xi)^2. \tag{7}$$

2. AXISTATIONARY SPACE-TIMES

The metric of a stationary axisymmetric space time can be written in the form

$$ds^2 = \frac{1}{f}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \tag{8}$$

where the functions f, γ and ω depend only on $x^1 = \rho$ and $x^2 = z$. Using (1), we get the metric on the three dimensional manifold $\mathcal{M}$,

$$h_{ij} = \begin{pmatrix} e^{2\gamma} & 0 & 0 \\ 0 & e^{2\gamma} & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}. \tag{9}$$

After the following coordinate transformation

$$\bar{\rho} = \frac{\rho}{\rho^2 + z^2} \quad , \quad \bar{z} = \frac{z}{\rho^2 + z^2} \quad , \quad \bar{\varphi} = \varphi, \tag{10}$$

the infinity is at the origin $\bar{\rho} = 0$ and $\bar{z} = 0$, and the metric is

$$h_{\bar{i}\bar{j}} = \frac{1}{\bar{r}^4}\begin{pmatrix} e^{2\gamma} & 0 & 0 \\ 0 & e^{2\gamma} & 0 \\ 0 & 0 & \bar{\rho}^2 \end{pmatrix}, \tag{11}$$

where $\bar{r}^2 = \bar{\rho}^2 + \bar{z}^2$. We drop the overbar from the coordinates and from $\bar{r}$. Let the manifold $\tilde{\mathcal{M}}$ be $\mathcal{M} \cup \Lambda$, where Λ is the origin. Let $\Omega = r^2$, then the metric on $\tilde{\mathcal{M}}$ is $\tilde{h}_{ij} = \Omega^2 h_{ij} = r^4 h_{ij}$, and the conform-transformed potential is $\tilde{\xi} = \Omega^{-1/2}\xi = \frac{1}{r}\xi$. Then from the equation (7) we get

$$(r^2\tilde{\xi}\tilde{\xi}^* - 1)\tilde{\Delta}\tilde{\xi} = 2\tilde{\xi}^*[r^2(\tilde{\nabla}\tilde{\xi})^2 + 2r\tilde{\xi}\tilde{\nabla}\tilde{\xi}\tilde{\nabla}r + \tilde{\xi}^2]. \tag{12}$$

Since the multipole tensors are invariant under a rotation about the symmetry axis $\rho = 0$, they are necessarily multiples of the symmetric trace-free outer product of the axis vector n^a with itself. Hence Hansen has defined the scalar moments by

$$P_n = \frac{1}{n!}M^{(n)}_{i_1 \dots i_n} n^{i_1} \dots n^{i_n}. \tag{13}$$

The quantities P_n uniquely determine the tensors $M^{(n)}_{i_1 \ldots i_n}$. Since on the axis $n^a = (0,1,0)$, we have

$$P_n = \frac{1}{n!} P^{(n)}_{2 \ldots 2} |_\Lambda . \tag{14}$$

It follows from equation (12) that $\tilde{\xi}$ is uniquely determined by its values on the axis. We can write $\tilde{\xi}$ there in form of power series in z,

$$\tilde{\xi}(\rho = 0) \quad = \sum_{n=0}^{\infty} m_n z^n . \tag{15}$$

It has been conjectured that $P_n = m_n$. This is true for $n = 0, 1, 2, 3$, but Hauser [7] found the conjecture to be false for $n = 4$ and 5. Hoenselaers [8] has published the values of the sixth and seventh moments.

3. GENERATING ALGORITHM

We first briefly review the results published in ref. [9]. We present an effective algorithm for generating the scalar moment P_n in terms of the coefficients m_k. For the calculation of the n^{th} moment we need $\partial_1^a \partial_2^b \tilde{\xi}|_\Lambda$, where $a + b \leq n$. So we write $\tilde{\xi}$ in the form

$$\tilde{\xi} = \sum_{i,j=0}^{\infty} a_{ij} \rho^i z^j , \tag{16}$$

where $a_{0j} = m_j$. Putting this into the equation (12) we get

$$\begin{aligned} (r+2)^2 a_{r+2,s} &= -(s+2)(s+1) a_{r,s+2} \\ &+ \sum_{\substack{k+m+p=r \\ l+n+q=s}} a_{kl} a^*_{mn} [a_{pq}(p^2 + q^2 - 4p - 5q - 2pk - 2ql - 2) \\ &+ a_{p+2,q-2}(p+2)(p+2-2k) + a_{p-2,q+2}(q+2)(q+1-2l)] \quad . \end{aligned} \tag{17}$$

Using this recursion formula we can express the constants a_{ij} by m_k.

Since the $P^{(n)}_{i_1 \ldots i_n}$ tensors are symmetric, we can introduce the notation

$$P^{(n)}_{a,b} = P^{(n)}_{\underbrace{1 \ldots 1}_{a} \underbrace{2 \ldots 2}_{b} \underbrace{3 \ldots 3}_{n-a-b}} , \tag{18}$$

so we can reduce the number of the formulas at the calculation from 3^n to $\frac{1}{2}(n+1)(n+2)$. Thus the recursive definition (2) of the tensors $P^{(n)}_{i_1 \ldots i_n}$ takes the form:

$$\begin{aligned} P^{(n)}_{a,b} &= \frac{1}{n} C \Big\{ a \frac{\partial}{\partial \rho} P^{(n-1)}_{a-1,b} + b \frac{\partial}{\partial z} P^{(n-1)}_{a,b-1} - \Big[(a(a-1) + 2ab)\gamma_{,1} + 2ac\frac{1}{\rho} \Big] P^{(n-1)}_{a-1,b} \\ &- \Big[2ab + b(b-1) \Big] \gamma_{,2} P^{(n-1)}_{a,b-1} + a(a-1)\gamma_{,2} P^{(n-1)}_{a-2,b+1} + b(b-1)\gamma_{,1} P^{(n-1)}_{a+1,b-2} \\ &+ c(c-1)\rho e^{-2\gamma} P^{(n-1)}_{a+1,b} - \Big(n - \frac{3}{2}\Big) \Big[a(a-1)\tilde{R}_{11} P^{(n-2)}_{a-2,b} \\ &+ 2ab\tilde{R}_{12} P^{(n-2)}_{a-1,b-1} + b(b-1)\tilde{R}_{22} P^{(n-2)}_{a,b-2} \Big] \Big\} . \end{aligned} \tag{19}$$

where now the symbol $\mathcal{C}$ denotes the trace-free part only, the Ricci tensor is

$$\tilde{R}_{ij} = \frac{1}{D^2}(G_i G_j^* + G_i^* G_j), \tag{20}$$

where $D = r^2 \tilde{\xi}\tilde{\xi}^* - 1$, $G_1 = z\tilde{\xi}_{,1} - \rho\tilde{\xi}_{,2}$, $G_2 = \rho\tilde{\xi}_{,1} + z\tilde{\xi}_{,2} + \tilde{\xi}$, $G_3 = 0$ and

$$\gamma_{,1} = \frac{\rho}{2}(\tilde{R}_{11} - \tilde{R}_{22}) , \qquad \gamma_{,2} = \rho\tilde{R}_{12}. \tag{21}$$

For further simplification, we can define the symmetric tensor $S^{(n)}_{i_1\ldots i_n}$ such that

$$P^{(n)}_{i_1\ldots i_n} = \mathcal{C}(S^{(n)}_{i_1\ldots i_n}), \tag{22}$$

and $S^{(n)}_{i_1\ldots i_n} = 0$ if $i_k = 3$ for some $1 \leq k \leq n$. Then

$$S^{(n)}_{i_1\ldots i_n} = P^{(n)}_{i_1\ldots i_n} + \tilde{h}_{(i_1 i_2} Q^{(n-2)}_{i_3\ldots i_n)}, \tag{23}$$

where $Q^{(n-2)}_{i_3\ldots i_n}$ is a symmetric tensor. Now we can introduce the notation

$$S^{(n)}_a = S^{(n)}_{\underbrace{1\ldots1}_{a}\underbrace{2\ldots2}_{n-a}}, \tag{24}$$

so we have reduced the number of the formulas at the calculation to $n+1$. The recursion formula of the quantities $S^{(n)}_a$ takes the form

$$\begin{aligned} S^{(0)}_0 = \tilde{\xi} \quad , \quad S^{(1)}_0 &= \frac{\partial}{\partial z} S^{(0)}_0 \quad , \quad S^{(1)}_1 = \frac{\partial}{\partial \rho} S^{(0)}_0 , \\ S^{(n)}_a = \frac{1}{n}\Big\{ a\frac{\partial}{\partial\rho} S^{(n-1)}_{a-1} &+ (n-a)\frac{\partial}{\partial z} S^{(n-1)}_a + a\Big[(a+1-2n)\gamma_{,1} - \frac{a-1}{\rho}\Big] S^{(n-1)}_{a-1} \\ &+ (a-n)(a+n-1)\gamma_{,2} S^{(n-1)}_a + a(a-1)\gamma_{,2} S^{(n-1)}_{a-2} \\ &+ (n-a)(n-a-1)(\gamma_{,1} - \frac{1}{\rho}) S^{(n-1)}_{a+1} \\ &- \Big(n - \frac{3}{2}\Big)\Big[a(a-1)\tilde{R}_{11} S^{(n-2)}_{a-2} \\ &+ 2a(n-a)\tilde{R}_{12} S^{(n-2)}_{a-1} + (n-a)(n-a-1)\tilde{R}_{22} S^{(n-2)}_a \Big]\Big\}. \end{aligned} \tag{25}$$

It follows from the axisymmetry, that $S^{(n)}_a|_A \neq 0$ only if $a = 0$. For the calculation of the scalar moments, using the equation (14), we only need $P^{(n)}_{2\ldots2}|_A$. Thus we have to calculate the trace-free part of a symmetric tensor $T^{(n)}_{i_1\ldots i_n}$ for which only $T^{(n)}_{2\ldots2} \neq 0$, and we need only the 2...2 component of the result. For the scalar moments we get [9]:

$$P_n = \frac{1}{(2n-1)!!} S^{(n)}_0 . \tag{26}$$

4. THE COMPUTATION

The computation up to the m^{th} moment consists of the following steps.

1°] We express the constants a_{ij} in terms of m_i using the recursive relation (17). We need all the coefficients a_{ij} for which $i + j \leq m$.

2°] We compute the polynomials $S_a^{(n)}$ from the recursive relation (25), using (20) and (21). Their respective degrees are $m - n$. We have to calculate only the $S_a^{(n)}$'s for which $a \leq m - n$.

3°] Then we can easily calculate the scalar moments using (26).

Using this algorithm even the twelfth moment could have been effortlessly computed by the computer of the Central Research Institute. I used the algebraic language REDUCE [10] for differentiation and algebraic manipulation of polynomials [11]. I have found that the computation of each subsequent moment requires about twice as much CPU time, and the length of the resulting expression is about the double of the previous. In each moment the terms tend to occur in pairs, so that it is convenient to introduce the notation

$$N_{ij} = m_i m_j - m_{i-1} m_{j+1}, \tag{27}$$

where $i \geq j + 2$. There exist algebraic identities among the quantities N_{ik} of the form

$$\begin{aligned} m_c N_{a+3,b+1} &= m_{b+1} N_{a+3,c} - m_{a+2} N_{b+2,c}, \\ m_{c+1} N_{a+3,b+1} &= m_{b+2} N_{a+3,c} - m_{a+3} N_{b+2,c}, \end{aligned} \tag{28}$$

where $a \geq b \geq c$. Using these equations we can eliminate the terms in which N_{ik} is multipled by m_j, where $j > i$ or $j < k$. I have succeded to determine some of the constant factors, and the other factors will be also simpler if at the n^{th} moment we introduce the notation

$$S_{ik} = \frac{(n-2)!\ \ (2i+1)!!\ \ (2n-2i-3)!!}{(i-2)!\ \ (n-i)!\ \ (2n-1)!!} N_{ik}. \tag{29}$$

The results of the computation of the first 12 multipole moments are :

$$P_0 = m_0$$

$$P_1 = m_1$$

$$P_2 = m_2$$

$$P_3 = m_3$$

$$P_4 = m_4 - S_{20} m_0^*$$

$$P_5 = m_5 - S_{20} m_1^{*} - S_{30} m_0^{*}$$

$$P_6 = m_6 - S_{20} m_2^{*} + \tfrac{7}{5} S_{20} m_0^{*2} m_0 - S_{30} m_1^{*} - S_{40} m_0^{*} - 2S_{31} m_0^{*}$$

$$P_7 = m_7 - S_{20} m_3^{*} + 2S_{20} m_1^{*} m_0^{*} m_0 - \tfrac{9}{5} S_{20} m_0^{*2} m_1 - S_{30} m_2^{*} + \tfrac{9}{5} S_{30} m_0^{*2} m_0 - S_{40} m_1^{*} - \tfrac{44}{25} S_{31} m_1^{*} - S_{50} m_0^{*} - \tfrac{38}{15} S_{41} m_0^{*}$$

$$P_8 = m_8 - S_{20} m_4^{*} + \tfrac{38}{21} S_{20} m_2^{*} m_0^{*} m_0 + \tfrac{11}{21} S_{20} m_1^{*2} m_0 - 2S_{20} m_1^{*} m_0^{*} m_1 - S_{20} m_0^{*3} m_0^2 - 20S_{20} m_0^{*2} m_2 - S_{30} m_3^{*} + \tfrac{18}{7} S_{30} m_1^{*} m_0^{*} m_0 + \tfrac{27}{14} S_{30} m_0^{*2} m_1 - S_{40} m_2^{*} + \tfrac{11}{5} S_{40} m_0^{*2} m_0 - \tfrac{23}{14} S_{31} m_2^{*} - S_{50} m_1^{*} - \tfrac{34}{15} S_{41} m_1^{*} - S_{60} m_0^{*} - \tfrac{11}{4} S_{51} m_0^{*} - 3S_{42} m_0^{*}$$

$$P_9 = m_9 - S_{20} m_5^{*} + \tfrac{20}{11} S_{20} m_3^{*} m_0^{*} m_0 + \tfrac{8}{11} S_{20} m_2^{*} m_1^{*} m_0 - \tfrac{106}{77} S_{20} m_2^{*} m_0^{*} m_1 - \tfrac{41}{77} S_{20} m_1^{*2} m_1 - \tfrac{21}{11} S_{20} m_1^{*} m_0^{*2} m_0^2 - \tfrac{288}{11} S_{20} m_1^{*} m_0^{*} m_2 + 2S_{20} m_0^{*3} m_1 m_0 - S_{30} m_4^{*} + \tfrac{16}{7} S_{30} m_2^{*} m_0^{*} m_0 + \tfrac{5}{7} S_{30} m_1^{*2} m_0 + \tfrac{156}{49} S_{30} m_1^{*} m_0^{*} m_1 - \tfrac{11}{7} S_{30} m_0^{*3} m_0^2 - \tfrac{1012}{49} S_{30} m_0^{*2} m_2 - S_{40} m_3^{*} + \tfrac{22}{7} S_{40} m_1^{*} m_0^{*} m_0 + \tfrac{583}{147} S_{40} m_0^{*2} m_1 - \tfrac{11}{7} S_{31} m_3^{*} + \tfrac{748}{49} S_{31} m_0^{*2} m_1 - S_{50} m_2^{*} + \tfrac{13}{5} S_{50} m_0^{*2} m_0 - \tfrac{314}{147} S_{41} m_2^{*} - S_{60} m_1^{*} - \tfrac{87}{35} S_{51} m_1^{*} - \tfrac{373}{147} S_{42} m_1^{*} - S_{70} m_0^{*} - \tfrac{20}{7} S_{61} m_0^{*} - \tfrac{148}{35} S_{52} m_0^{*}$$

$$P_{10} = m_{10} - S_{20} m_6^{*} + \tfrac{274}{143} S_{20} m_4^{*} m_0^{*} m_0 + \tfrac{262}{429} S_{20} m_3^{*} m_1^{*} m_0 - \tfrac{146}{143} S_{20} m_3^{*} m_0^{*} m_1 + \tfrac{71}{429} S_{20} m_2^{*2} m_0 - \tfrac{106}{143} S_{20} m_2^{*} m_1^{*} m_1 - \tfrac{727}{429} S_{20} m_2^{*} m_0^{*2} m_0^2 - \tfrac{4930}{231} S_{20} m_2^{*} m_0^{*} m_2 - \tfrac{37}{39} S_{20} m_1^{*2} m_0^{*} m_0^2 - \tfrac{1685}{231} S_{20} m_1^{*2} m_2 + \tfrac{42}{13} S_{20} m_1^{*} m_0^{*2} m_1 m_0 + \tfrac{7}{13} S_{20} m_0^{*4} m_0^3 + \tfrac{252}{13} S_{20} m_0^{*3} m_2 m_0 - \tfrac{125}{13} S_{20} m_0^{*3} m_1^2 - S_{30} m_5^{*} + \tfrac{74}{33} S_{30} m_3^{*} m_0^{*} m_0 + \tfrac{34}{33} S_{30} m_2^{*} m_1^{*} m_0 + \tfrac{101}{33} S_{30} m_2^{*} m_0^{*} m_1 + \tfrac{127}{132} S_{30} m_1^{*2} m_1 - 3S_{30} m_1^{*} m_0^{*2} m_0^2 - \tfrac{109}{4} S_{30} m_1^{*} m_0^{*} m_2 - \tfrac{5}{4} S_{30} m_0^{*3} m_1 m_0 - \tfrac{1317}{56} S_{30} m_0^{*2} m_3 - S_{40} m_4^{*} + \tfrac{58}{21} S_{40} m_2^{*} m_0^{*} m_0 + \tfrac{19}{21} S_{40} m_1^{*2} m_0 + \tfrac{55}{9} S_{40} m_1^{*} m_0^{*} m_1 - \tfrac{143}{63} S_{40} m_0^{*3} m_0^2 - \tfrac{682}{49} S_{40} m_0^{*2} m_2 - \tfrac{67}{44} S_{31} m_4^{*} + \tfrac{589}{28} S_{31} m_1^{*} m_0^{*} m_1 - \tfrac{33}{7} S_{31} m_0^{*2} m_2 - S_{50} m_3^{*} + \tfrac{26}{7} S_{50} m_1^{*} m_0^{*} m_0 + \tfrac{39}{7} S_{50} m_0^{*2} m_1 - \tfrac{37}{18} S_{41} m_3^{*} + \tfrac{8833}{588} S_{41} m_0^{*2} m_1 - S_{60} m_2^{*} + 3S_{60} m_0^{*2} m_0 - \tfrac{33}{14} S_{51} m_2^{*} - \tfrac{4063}{1764} S_{42} m_2^{*} - S_{70} m_1^{*} - \tfrac{13}{5} S_{61} m_1^{*} - \tfrac{713}{196} S_{52} m_1^{*} - S_{80} m_0^{*} - \tfrac{35}{12} S_{71} m_0^{*} - \tfrac{339}{70} S_{62} m_0^{*} - \tfrac{1553}{392} S_{53} m_0^{*}$$

$$P_{11} = m_{11} - S_{20} m_7^{*} + \tfrac{134}{65} S_{20} m_5^{*} m_0^{*} m_0 + \tfrac{38}{65} S_{20} m_4^{*} m_1^{*} m_0 - \tfrac{10}{13} S_{20} m_4^{*} m_0^{*} m_1 + \tfrac{2}{13} S_{20} m_3^{*} m_2^{*} m_0 - \tfrac{22}{39} S_{20} m_3^{*} m_1^{*} m_1 - \tfrac{23}{13} S_{20} m_3^{*} m_0^{*2} m_0^2 - \tfrac{1256}{65} S_{20} m_3^{*} m_0^{*} m_2 - \tfrac{11}{39} S_{20} m_2^{*2} m_1 - \tfrac{86}{65} S_{20} m_2^{*} m_1^{*} m_0^{*} m_0^2 - \tfrac{52}{5} S_{20} m_2^{*} m_1^{*} m_2 + \tfrac{446}{195} S_{20} m_2^{*} m_0^{*2} m_1 m_0 - \tfrac{9}{65} S_{20} m_1^{*3} m_0^2 + \tfrac{62}{39} S_{20} m_1^{*2} m_0^{*} m_1 m_0 + \tfrac{84}{65} S_{20} m_1^{*} m_0^{*3} m_0^3 + \tfrac{2296}{65} S_{20} m_1^{*} m_0^{*2} m_2 m_0 - \tfrac{1071}{65} S_{20} m_1^{*} m_0^{*2} m_1^2 - \tfrac{7}{5} S_{20} m_0^{*4} m_1 m_0^2 + \tfrac{308}{5} S_{20} m_0^{*3} m_2 m_1 - S_{30} m_6^{*} + \tfrac{30}{13} S_{30} m_4^{*} m_0^{*} m_0 + \tfrac{34}{39} S_{30} m_3^{*} m_1^{*} m_0 + \tfrac{370}{117} S_{30} m_3^{*} m_0^{*} m_1 + \tfrac{11}{39} S_{30} m_2^{*2} m_0 + \tfrac{170}{117} S_{30} m_2^{*} m_1^{*} m_1 - \tfrac{101}{39} S_{30} m_2^{*} m_0^{*2} m_0^2 - \tfrac{6038}{273} S_{30} m_2^{*} m_0^{*} m_2 - \tfrac{61}{39} S_{30} m_1^{*2} m_0^{*} m_0^2 - \tfrac{2138}{273} S_{30} m_1^{*2} m_2 - \tfrac{40}{13} S_{30} m_1^{*} m_0^{*2} m_1 m_0 -$$

$$\frac{2824}{91} S_{30} m_1^* m_0^* m_3 + S_{30} m_0^{*4} m_0^3 + 24 S_{30} m_0^{*3} m_2 m_0 - \frac{37}{3} S_{30} m_0^{*3} m_1^2 - S_{40} m_5^* +$$
$$\frac{8}{3} S_{40} m_3^* m_0^* m_0 + \frac{4}{3} S_{40} m_2^* m_1^* m_0 + \frac{50}{9} S_{40} m_2^* m_0^* m_1 + \frac{17}{9} S_{40} m_1^{*2} m_1 - \frac{13}{3} S_{40} m_1^* m_0^{*2} m_0^2 -$$
$$\frac{3433}{189} S_{40} m_1^* m_0^* m_2 - \frac{104}{27} S_{40} m_0^{*3} m_1 m_0 - \frac{5993}{189} S_{40} m_0^{*2} m_3 - \frac{58}{39} S_{31} m_5^* + \frac{14530}{819} S_{31} m_2^* m_0^* m_1 +$$
$$\frac{5021}{819} S_{31} m_1^{*2} m_1 - \frac{536}{91} S_{31} m_1^* m_0^* m_2 - 77 S_{31} m_0^{*2} m_3 - S_{50} m_4^* + \frac{68}{21} S_{50} m_2^* m_0^* m_0 +$$
$$\frac{23}{21} S_{50} m_1^{*2} m_0 + \frac{533}{63} S_{50} m_1^* m_0^* m_1 - \frac{65}{21} S_{50} m_0^{*3} m_0^2 - \frac{507}{49} S_{50} m_0^{*2} m_2 - 2 S_{41} m_4^* +$$
$$\frac{7993}{378} S_{41} m_1^* m_0^* m_1 + \frac{4927}{252} S_{41} m_0^{*2} m_2 - S_{60} m_3^* + \frac{30}{7} S_{60} m_1^* m_0^* m_0 + 7 S_{60} m_0^{*2} m_1 - \frac{41}{18} S_{51} m_3^* +$$
$$\frac{27989}{1764} S_{51} m_0^{*2} m_1 - \frac{233}{108} S_{42} m_3^* - S_{70} m_2^* + \frac{17}{5} S_{70} m_0^{*2} m_0 - \frac{52}{21} S_{61} m_2^* - \frac{841}{252} S_{52} m_2^* - S_{80} m_1^* -$$
$$\frac{8}{3} S_{71} m_1^* - \frac{59}{14} S_{62} m_1^* - \frac{119}{36} S_{53} m_1^* - S_{90} m_0^* - \frac{62}{21} S_{81} m_0^* - \frac{467}{90} S_{72} m_0^* - \frac{125}{21} S_{63} m_0^*$$

$$P_{12} = m_{12} - S_{20} m_8^* + \frac{38}{17} S_{20} m_6^* m_0^* m_0 + \frac{134}{221} S_{20} m_5^* m_1^* m_0 - \frac{626}{1105} S_{20} m_5^* m_0^* m_1 +$$
$$\frac{178}{2431} S_{20} m_4^* m_2^* m_0 - \frac{482}{1105} S_{20} m_4^* m_1^* m_1 - \frac{285}{143} S_{20} m_4^* m_0^{*2} m_0^2 - \frac{225084}{12155} S_{20} m_4^* m_0^* m_2 -$$
$$\frac{79}{2431} S_{20} m_3^{*2} m_0 - \frac{106}{221} S_{20} m_3^* m_2^* m_1 - \frac{2826}{2431} S_{20} m_3^* m_1^* m_0^* m_0^2 - \frac{9404}{1105} S_{20} m_3^* m_1^* m_2 +$$
$$\frac{4374}{2431} S_{20} m_3^* m_0^{*2} m_1 m_0 - \frac{717}{2431} S_{20} m_2^{*2} m_0^* m_0^2 - \frac{38762}{12155} S_{20} m_2^{*2} m_2 - \frac{621}{2431} S_{20} m_2^* m_1^{*2} m_0^2 +$$
$$\frac{26124}{12155} S_{20} m_2^* m_1^* m_0^* m_1 m_0 + \frac{2796}{2431} S_{20} m_2^* m_0^{*3} m_0^3 + \frac{2421943}{85085} S_{20} m_2^* m_0^{*2} m_2 m_0 -$$
$$\frac{160349}{12155} S_{20} m_2^* m_0^{*2} m_1^2 + \frac{2826}{12155} S_{20} m_1^{*3} m_1 m_0 + \frac{2286}{2431} S_{20} m_1^{*2} m_0^{*2} m_0^3 +$$
$$\frac{1584983}{85085} S_{20} m_1^{*2} m_0^* m_2 m_0 - \frac{96061}{12155} S_{20} m_1^{*2} m_0^* m_1^2 - \frac{252}{85} S_{20} m_1^* m_0^{*3} m_1 m_0^2 +$$
$$\frac{588}{5} S_{20} m_1^* m_0^{*2} m_2 m_1 - \frac{21}{85} S_{20} m_0^{*5} m_0^4 - \frac{1176}{85} S_{20} m_0^{*4} m_2 m_0^2 + \frac{504}{85} S_{20} m_0^{*4} m_1^2 m_0 -$$
$$\frac{672}{17} S_{20} m_0^{*3} m_2^2 - S_{30} m_7^* + \frac{158}{65} S_{30} m_5^* m_0^* m_0 + \frac{586}{715} S_{30} m_4^* m_1^* m_0 + \frac{483}{143} S_{30} m_4^* m_0^* m_1 +$$
$$\frac{50}{143} S_{30} m_3^* m_2^* m_0 + \frac{417}{325} S_{30} m_3^* m_1^* m_1 - \frac{1857}{715} S_{30} m_3^* m_0^{*2} m_0^2 - \frac{70877}{3575} S_{30} m_3^* m_0^* m_2 +$$
$$\frac{2811}{7150} S_{30} m_2^{*2} m_1 - \frac{1602}{715} S_{30} m_2^* m_1^* m_0^* m_0^2 - \frac{40597}{3575} S_{30} m_2^* m_1^* m_2 - \frac{22281}{7150} S_{30} m_2^* m_0^{*2} m_1 m_0 -$$
$$\frac{178687}{7150} S_{30} m_2^* m_0^* m_3 - \frac{171}{715} S_{30} m_1^{*3} m_0^2 - \frac{13161}{7150} S_{30} m_1^{*2} m_0^* m_1 m_0 - \frac{453113}{50050} S_{30} m_1^{*2} m_3 +$$
$$\frac{12}{5} S_{30} m_1^* m_0^{*3} m_0^3 + \frac{1098}{25} S_{30} m_1^* m_0^{*2} m_2 m_0 - \frac{576}{25} S_{30} m_1^* m_0^{*2} m_1^2 + \frac{33}{50} S_{30} m_0^{*4} m_1 m_0^2 +$$
$$\frac{749}{25} S_{30} m_0^{*3} m_3 m_0 + \frac{403}{5} S_{30} m_0^{*3} m_2 m_1 - S_{40} m_6^* + \frac{386}{143} S_{40} m_4^* m_0^* m_0 + \frac{162}{143} S_{40} m_3^* m_1^* m_0 +$$
$$\frac{11722}{2145} S_{40} m_3^* m_0^* m_1 + \frac{57}{143} S_{40} m_2^{*2} m_0 + \frac{2062}{715} S_{40} m_2^* m_1^* m_1 -$$
$$\frac{525}{143} S_{40} m_2^* m_0^{*2} m_0^2 - \frac{1962203}{135135} S_{40} m_2^* m_0^* m_2 - \frac{333}{143} S_{40} m_1^{*2} m_0^* m_0^2 - \frac{712249}{135135} S_{40} m_1^{*2} m_2 -$$
$$\frac{124}{15} S_{40} m_1^* m_0^{*2} m_1 m_0 - \frac{39944}{945} S_{40} m_1^* m_0^* m_3 + \frac{5}{3} S_{40} m_0^{*4} m_0^3 + \frac{2552}{135} S_{40} m_0^{*3} m_2 m_0 -$$
$$\frac{478}{27} S_{40} m_0^{*3} m_1^2 - \frac{3956}{135} S_{40} m_0^{*2} m_4 - \frac{73}{50} S_{31} m_6^* + \frac{11784}{715} S_{31} m_3^* m_0^* m_1 + \frac{6432}{715} S_{31} m_2^* m_1^* m_1 -$$
$$\frac{2823}{650} S_{31} m_2^* m_0^* m_2 - \frac{8079}{4550} S_{31} m_1^{*2} m_2 - \frac{2586}{25} S_{31} m_1^* m_0^* m_3 - \frac{177}{5} S_{31} m_0^{*3} m_1^2 - S_{50} m_5^* +$$
$$\frac{34}{11} S_{50} m_3^* m_0^* m_0 + \frac{18}{11} S_{50} m_2^* m_1^* m_0 + \frac{831}{110} S_{50} m_2^* m_0^* m_1 + \frac{587}{220} S_{50} m_1^{*2} m_1 -$$
$$\frac{65}{11} S_{50} m_1^* m_0^{*2} m_0^2 - \frac{45656}{3465} S_{50} m_1^* m_0^* m_2 - \frac{871}{132} S_{50} m_0^{*3} m_1 m_0 - \frac{1079}{45} S_{50} m_0^{*2} m_3 - \frac{382}{195} S_{41} m_5^* +$$
$$\frac{220741}{12285} S_{41} m_2^* m_0^* m_1 + \frac{78461}{12285} S_{41} m_1^{*2} m_1 + \frac{26252}{945} S_{41} m_1^* m_0^* m_2 - \frac{3364}{45} S_{41} m_0^{*2} m_3 - S_{60} m_4^* +$$
$$\frac{26}{7} S_{60} m_2^* m_0^* m_0 + \frac{9}{7} S_{60} m_1^{*2} m_0 + \frac{74}{7} S_{60} m_1^* m_0^* m_1 - \frac{85}{21} S_{60} m_0^{*3} m_0^2 - \frac{506}{63} S_{60} m_0^{*2} m_2 -$$
$$\frac{489}{220} S_{51} m_4^* + \frac{156559}{6930} S_{51} m_1^* m_0^* m_1 + \frac{21619}{990} S_{51} m_0^{*2} m_2 - \frac{401}{195} S_{42} m_4^* + \frac{548}{9} S_{42} m_0^{*2} m_2 -$$
$$S_{70} m_3^* + \frac{34}{7} S_{70} m_1^* m_0^* m_0 + \frac{1751}{210} S_{70} m_0^{*2} m_1 - \frac{12}{5} S_{61} m_3^* + \frac{5392}{315} S_{61} m_0^{*2} m_1 - \frac{1039}{330} S_{52} m_3^* -$$
$$S_{80} m_2^* + \frac{19}{5} S_{80} m_0^{*2} m_0 - \frac{107}{42} S_{71} m_2^* - \frac{409}{105} S_{62} m_2^* - \frac{5867}{1980} S_{53} m_2^* - S_{90} m_1^* - \frac{474}{175} S_{81} m_1^* -$$
$$\frac{478}{105} S_{72} m_1^* - \frac{1588}{315} S_{63} m_1^* - S_{100} m_0^* - \frac{119}{40} S_{91} m_0^* - \frac{27}{5} S_{82} m_0^* - \frac{1487}{210} S_{73} m_0^* - \frac{439}{90} S_{64} m_0^*$$

For the computation up to the 12^{th} moment, I needed 4 megabyte memory, and one hour CPU time.

Using theese fomulas for the multipole moments one can search for those solutions of the Einstein's equations for which only the first few moments are not zero. This is not true for the Kerr solution, since then $N_{ik} = 0$, and $P_n = m_n = M(ia)^n$, where M is the mass parameter, and a is the angular momentum parameter.

Can we get simpler expressions for the moments if we use another conform factor $\tilde{\Omega} = \omega\Omega$ instead of Ω? To answer this question we do not have to carry out the whole calculation again. Putting the form (14) of the scalar moments into the (4) transformation formula of the moment tensors, we get

$$\tilde{P}_n = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} P_k, \tag{30}$$

where $\alpha = -\frac{1}{2}\tilde{D}_2\omega|_\Lambda$. The change of the moments depends only on the real number α, so there is not much way to make the results simpler. I have not found a simpler form.

5. THE MOMENTS OF THE TOMIMATSU-SATO SOLUTIONS

If we know the value of the complex potential ξ on the symmetry axis for a given space-time, we can calculate the coefficients m_i in the power series expansion of $\tilde{\xi}$ there, and using the formulas for the scalar moments, or using the general algorithm, we can calculate the momets P_n. The value of ξ on the axis for the δ^{th} Tomimatsu-Sato solution is [12]:

$$\frac{1}{\xi} = p\frac{(x+1)^\delta + (x-1)^\delta}{(x+1)^\delta - (x-1)^\delta} - iq, \tag{31}$$

where x and y are prolate spheroidal coordinates, on the axis $y = 1$, and p and q are parameters, $p^2 + q^2 = 1$. The cylindrical coordinate $z = \kappa xy$, where $\kappa = \frac{Mp}{\delta}$. Since on the axis $y = 1$ and $z = 1/\bar{z}$,

$$x = \frac{\delta}{Mp\bar{z}}. \tag{32}$$

The power series expansion of a quotient of polynomials can be easily calculated using a recursion formula. If

$$\sum_{k=0}^{\infty} c_k \bar{z}^k = \frac{\sum_{i=0}^{\infty} a_i \bar{z}^i}{\sum_{j=0}^{\infty} b_j \bar{z}^j}, \tag{33}$$

then

$$c_k = \frac{a_k}{b_0} - \frac{1}{b_0}\sum_{i=0}^{k-1} c_i b_{k-i}. \tag{34}$$

It is interesting that for the second Tomimatsu-Sato solution

$$N_{jk} = -\frac{1}{4^{k+1}} p^{2k+2} M^{2k+3} m_{i-k-2}. \tag{35}$$

The moments of the first four Tomimatsu-Sato solutions up to the 12^{th} order are:

$\delta = 1$:

Kerr solution, $P_n = M(ia)^n$ for all n,

$\delta = 2$:

$P_0 = M,$

$P_1 = iM^2 q,$

$P_2 = M^3(\frac{3}{4}p^2 - 1),$

$P_3 = iM^4 q(\frac{1}{2}p^2 - 1),$

$P_4 = M^5(\frac{5}{16}p^4 - \frac{17}{14}p^2 + 1),$

$P_5 = iM^6 q(\frac{3}{16}p^4 - \frac{13}{14}p^2 + 1),$

$P_6 = M^7(\frac{7}{64}p^6 - \frac{2897}{3696}p^4 + \frac{137}{84}p^2 - 1),$

$P_7 = iM^8 q(\frac{1}{16}p^6 - \frac{1217}{2288}p^4 + \frac{4}{3}p^2 - 1),$

$P_8 = M^9(\frac{9}{256}p^8 - \frac{2647}{6864}p^6 + \frac{33661}{24024}p^4 - \frac{67}{33}p^2 + 1),$

$P_9 = iM^{10} q(\frac{5}{256}p^8 - \frac{9521}{38896}p^6 + \frac{8205}{8008}p^4 - \frac{19}{11}p^2 + 1),$

$P_{10} = M^{11}(\frac{11}{1024}p^{10} - \frac{113513}{695552}p^8 + \frac{84829}{96096}p^6 - \frac{17309}{8008}p^4 + \frac{1385}{572}p^2 - 1),$

$P_{11} = iM^{12} q(\frac{3}{512}p^{10} - \frac{57457}{578816}p^8 + \frac{44689}{74256}p^6 - \frac{20571}{12376}p^4 + \frac{55}{26}p^2 - 1),$

$P_{12} = M^{13}(\frac{13}{4096}p^{12} - \frac{21723435}{346131968}p^{10} + \frac{2944065}{6366976}p^8 - \frac{320289}{193648}p^6 + \frac{75833}{24752}p^4 - \frac{73}{26}p^2 + 1),$

$\delta = 3$:

$P_0 = M,$

$P_1 = iM^2 q,$

$P_2 = M^3(\frac{19}{27}p^2 - 1),$

$P_3 = iM^4 q(\frac{11}{27}p^2 - 1),$

$P_4 = M^5(\frac{17}{81}p^4 - \frac{202}{189}p^2 + 1),$

$P_5 = iM^6 q(\frac{73}{729}p^4 - \frac{46}{63}p^2 + 1),$

$P_6 = M^7(\frac{11}{243}p^6 - \frac{28015}{56133}p^4 + \frac{781}{567}p^2 - 1),$

$P_7 = iM^8 q(\frac{43}{2187}p^6 - \frac{7981}{28431}p^4 + \frac{83}{81}p^2 - 1),$

$P_8 = M^9(\frac{163}{19683}p^8 - \frac{1331452}{8444007}p^6 + \frac{1880210}{2189187}p^4 - \frac{1484}{891}p^2 + 1),$

$P_9 = iM^{10} q(\frac{67}{19683}p^8 - \frac{3797140}{47849373}p^6 + \frac{130210}{243243}p^4 - \frac{388}{297}p^2 + 1),$

$P_{10} = M^{11}(\frac{1}{729}p^{10} - \frac{108856393}{2727414261}p^8 + \frac{357738586}{1004836833}p^6 - \frac{85406}{66339}p^4 + \frac{7505}{3861}p^2 - 1),$

$P_{11} = iM^{12} q(\frac{289}{531441}p^{10} - \frac{292154117}{15620645313}p^8 + \frac{344130778}{1735627257}p^6 - \frac{971134}{1127763}p^4 + \frac{185}{117}p^2 - 1),$

$P_{12} = M^{13}(\frac{113}{531441}p^{12} - \frac{61728190}{7044604749}p^{10} + \frac{13790165875}{119758280733}p^8 - \frac{2665870724}{4049796933}p^6 + \frac{670687}{375921}p^4 - \frac{778}{351}p^2 + 1),$

$\delta = 4:$

$P_0 = M,$

$P_1 = iM^2 q,$

$P_2 = M^3(\frac{11}{16}p^2 - 1),$

$P_3 = iM^4 q(\frac{3}{8}p^2 - 1),$

$P_4 = M^5(\frac{45}{256}p^4 - \frac{57}{56}p^2 + 1),$

$P_5 = iM^6 q(\frac{19}{256}p^4 - \frac{37}{56}p^2 + 1),$

$P_6 = M^7(\frac{119}{4096}p^6 - \frac{24169}{59136}p^4 + \frac{433}{336}p^2 - 1),$

$P_7 = iM^8 q(\frac{11}{1024}p^6 - \frac{22843}{109824}p^4 + \frac{11}{12}p^2 - 1),$

$P_8 = M^9(\frac{249}{65536}p^8 - \frac{44861}{439296}p^6 + \frac{265957}{384384}p^4 - \frac{203}{132}p^2 + 1),$

$P_9 = iM^{10} q(\frac{85}{65536}p^8 - \frac{112089}{2489344}p^6 + \frac{50733}{128128}p^4 - \frac{51}{44}p^2 + 1),$

$P_{10} = M^{11}(\frac{451}{1048576}p^{10} - \frac{3083243}{159318016}p^8 + \frac{619407}{2680832}p^6 - \frac{131221}{128128}p^4 + \frac{4065}{2288}p^2 - 1),$

$P_{11} = iM^{12} q(\frac{73}{524288}p^{10} - \frac{14976557}{1926299648}p^8 + \frac{3456585}{30098432}p^6 - \frac{125859}{198016}p^4 + \frac{145}{104}p^2 - 1),$

$P_{12} = M^{13}(\frac{741}{16777216}p^{12} - \frac{63890511}{20849360896}p^{10} + \frac{28016609035}{487353810944}p^8 - \frac{89936695}{210689024}p^6 + \frac{556417}{396032}p^4 - \frac{209}{104}p^2 + 1).$

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[4] The commutator of the covariant derivatives of a 3-vector v_i defines our sign convention for the curvature quantities by $(D_i D_j - D_j D_i)v_k = R_{ijkr}v^r$ and $R_{ik} = R^r{}_{irk}$

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[10] REDUCE [A. Hearn; Reduce user's manual, the Rank Co., Santa Monica, 1983]

[11] To reduce the size of computations, it is often advantageous to drop terms containing $\rho^i z^j$ whenever $i + j > n$ for some positive n. When one uses a REDUCE code, this can be achieved by declaring WEIGHT $\rho = 1, z = 1$, WTLEVEL n. To enable differentiation again, one can cancel truncation by CLEAR ρ, z

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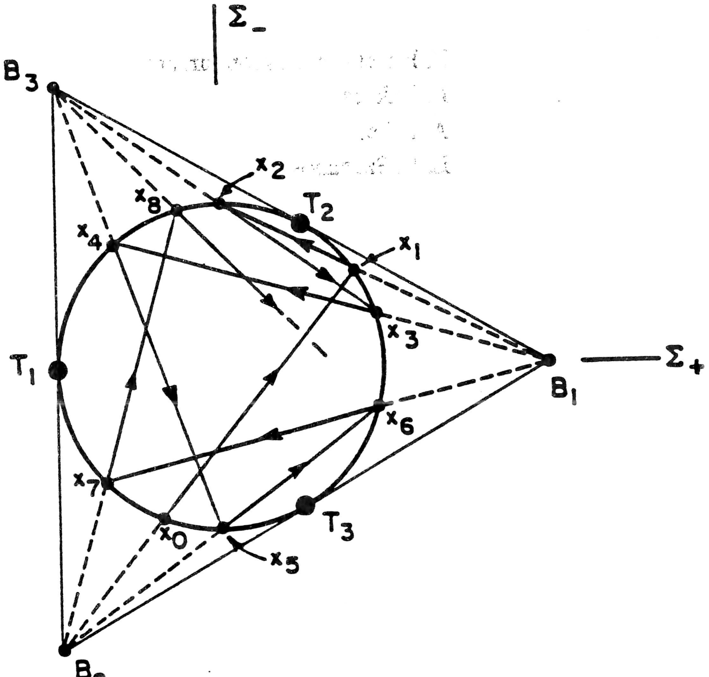

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