The Thin Sandwich Conjecture

Ph.D. Thesis

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Abstract

In this paper we give a proof of the Thin Sandwich Conjecture of general relativity for a restrictive but open set of initial data $(g_{ab}, \dot{g}_{ab})$. In this case we show the existence of unique solutions for the Thin Sandwich Equations and the ellipticity of these equations. As a prominent application we examine the Robertson-Walker spacetimes.

1 Introduction

The Thick and Thin Sandwich Conjectures were formulated by Baierlein, Sharp and Wheeler[2], and discussed in more detail by Wheeler[3][4][5]. From the viewpoint adopted in these papers, a spacetime is considered as a "geometrodynamical history" of three-geometries. One such three-geometry is characterized by a positive definite metric, (or by a diffeomorphism equivalence class of them,) that is by a point in superspace. This picture is also the basis of Feynman's path integral approach towards quantum gravity. However, given two points in superspace, there is no unique curve connecting them, which belongs to some solution of the Einstein's equations. The two points correspond to two hypersurfaces in the embedding spacetime, and if we choose different slicing between these two surfaces, we get different connecting paths in superspace. The Sandwich Conjecture claims that this is the only reason for this nonuniqueness.

Sandwich Conjecture: *Given two three dimensional geometries, $g^{(1)}_{ab}$ and $g^{(2)}_{ab}$, there exists a spacetime satisfying the vacuum Einstein's equations and containing two Cauchy hypersurfaces with the intrinsic geometry $g^{(1)}_{ab}$ and $g^{(2)}_{ab}$. This embedding spacetime is unique.*

Contrary to the classical mechanical and electrodynamical analogue of this conjecture, we do not have to specify the time difference between the two configuration. The time difference between two hypersurfaces is not even a globally and invariantly defined concept. On the other hand the proper time interval along any timelike curve is determined by the geometry of the

*This work has been done with the supervising of Professor Robert Bartnik, [1] spacetime, which is assumed to be fixed by the two spacelike geometries. This would mean that in general relativity two 3-geometries contain all the information about space and time. Or in other words, three-geometry is the carrier of information about time. Essentially nothing is known about the validity of the Sandwich Conjecture. One would expect that the above formulation is excessively optimistic, and that serious restrictions would be needed on $g_{ab}^{(1)}$ and $g_{ab}^{(2)}$ before the existence of a connecting spacetime could be asserted. But the nature of such restrictions and the mathematical methods by which the results could be demonstrated remain conjectural at best.

In view of the complexity of the problem, it is natural to investigate a simplified version first, where the two hypersurfaces are infinitesimally close to each other.

Thin Sandwich Conjecture: *Suppose that a positive definite metric g_{ab} and a symmetric tensor field $\dot{g}_{ab}$ are given on a three-dimensional manifold $\mathcal{M}$. Then there exists a spacetime satisfying the vacuum Einstein's equations, such that on a Cauchy hypersurface of this spacetime the induced metric is g_{ab}, and the time parameter derivative along a time flow vector field of the spacelike part of the metric is $\dot{g}_{ab}$. This spacetime is unique on the Cauchy development of the initial hypersurface.*

We note that here $\dot{g}_{ab}$ is just a notation for a tensor field independent of g_{ab}, the dot does not denote a time derivative. The Thin Sandwich Conjecture in this form is generally not true. It is possible to specify g_{ab} and $\dot{g}_{ab}$ such that no embedding spacetime exists, and there are cases when the imbedding spacetime is not unique. However we will prove that the conjecture is true for a large open subset of initial data.

Suppose that the induced spacelike metric g_{ab} and the extrinsic curvature K_{ab} is given on a three dimensional hypersurface, satisfying the gravitational constraint equations. To simplify the situation, we will assume that the hypersurface is compact and without boundary. Possibly this restriction may be dropped after future investigation. Any spacelike slice of a spacetime satisfying Einstein's equations generates such g_{ab} and K_{ab}. On the other hand, this data determine a unique spacetime on the Cauchy development[6][7]. We can choose a time flow vector field which is defined at least on the initial hypersurface. This vector field can be specified by its hypersurface tangent and orthogonal parts, which are called the shift vector field and the lapse function. Calculating the time parameter derivative of the spacelike metric along this vector field, we get a symmetric tensor field $\frac{d}{dt}g_{ab} \equiv \dot{g}_{ab}$ on the initial hypersurface. Assuming that $(g_{ab}, \dot{g}_{ab})$ is generated by a solution (g_{ab}, K_{ab}) of the constraints, we will prove that the Thin Sandwich Conjecture is true on an open neighbourhood of this $(g_{ab}, \dot{g}_{ab})$ if the following two conditions hold:

Condition 1a:

$$2\epsilon - R > 0, \tag{1}$$

where R is the scalar curvature belonging to the induced three-metric g_{ab} and ϵ is the energy density of the (apriori prescribed) matter fields on the initial hypersurface.

Condition 2a: *There is no choice of time flow vector field on the spacetime induced by (g_{ab}, K_{ab}), for which*

$$\frac{d}{dt}g_{ab} \equiv \dot{g}_{ab} = 0 \tag{2}$$

on the initial hypersurface.

We will see that both of these conditions can be stated in the following equivalent forms:

Condition 1b: *The trace of the extrinsic curvature K_{ab} dominates the trace-free part $\overset{\circ}{K}_{ab}=$*

$K_{ab} - \frac{1}{3}g_{ab}K_c^c$,

$$\frac{2}{3}(K_a^a)^2 > \overset{\circ}{K}_{ab}\overset{\circ}{K}^{ab} . \tag{3}$$

Condition 2b: *The equation*

$$M_{(a|b)} = \mu K_{ab} \tag{4}$$

for the vector field M^a and scalar field μ on the initial hypersurface has only the trivial solution $M^a = 0$ and $\mu = 0$.

The Thin Sandwich Conjecture can be viewed in another way, not just as the 'thin' version of the Sandwich Conjecture. It can be viewed as a parametrisation of the constraint manifold. The constraint manifold $\mathcal{C} \subset \{(g_{ab}, K_{ab})\}$ consists of those metric and extrinsic curvature function pairs (g_{ab}, K_{ab}) on a three dimensional manifold, which satisfies the gravitational constraint equations. Given $(g_{ab}, \dot{g}_{ab})$, supposing that the Thin Sandwich Conjecture is true, they determine a unique pair (g_{ab}, K_{ab}), satisfying the constraints. However, it is only a parametrisation modulo time flow vector field changes, since different time derivatives $\dot{g}_{ab}$ can belong to the same pair (g_{ab}, K_{ab}), as one changes the time flow vector field. It has been shown by Arms, Fisher, Marsden and Moncrief[8][9], that the constraint manifold $\mathcal{C}$ really is a submanifold of the space of all possible configurations (g_{ab}, K_{ab}), except in the cases when the imbedding spacetime admits a Killing vector field. At these nongeneric (g_{ab}, K_{ab}) pairs, $\mathcal{C}$ has a quadratic singularity. This shows that the Thin Sandwich Conjecture may also fail in these cases, and indeed, Condition 2 is false if the imbedding spacetime admits Killing vectors.

In this paragraph we outline the structure of the paper. In Section 2 we derive the equations of the Thin Sandwich problem and a variational principle leading to them. This was originally done by Baierlein, Sharp and Wheeler[2], and extended by Wheeler[3] to include matter fields. The variational principle was used by Belasco and Ohanian[10] to prove a uniqueness theorem under Conditions 1 and 2. We present the theorem and its proof in Section 3. We note that there is an error in the proof of the second part of the uniqueness theorem of [10], and the result claimed there is easily seen to be incorrect. In Section 4, using the Implicit Function theorem we prove the existence and uniqueness of solutions of the equations of the Thin Sandwich problem in a neighbourhood of a $(g_{ab}, \dot{g}_{ab})$ generated by a solution (g_{ab}, K_{ab}) of the constraint equations, under the same Conditions 1 and 2 as in the uniqueness theorem. Under somewhat less restrictive conditions we also prove the ellipticity of the linearised Thin Sandwich Equations. They constitute an elliptic system, if and only if the momentum tensor $\pi_{ab} = K_{ab} - g_{ab}K_c^c$ is positive or negative definite. We will show that this is the case if Condition 1 holds. In section 5 we examine the Conditions 1 and 2 of existence and uniqueness. We will show that Condition 2 fails to hold if the spacetime generated by (g_{ab}, K_{ab}) admits a vector field which is Killing at least on the initial hypersurface. The difficulty that arises is very similar to that occurring in the analysis of linearisation stability of the Einstein constraint equations[9], in a neighbourhood of an initial data for which the induced spacetime admits a Killing vector field. Probably one will be able to prove existence in the future even if Condition 2 is false, and the nonuniqueness in this case may due only to different choices of hypersurfaces. Condition 1 may seem too restrictive at first, but in section 6 we will see that it is satisfied by most of the homogeneous isotropic cosmology solutions. Following Belasco and Ohanian[10], we present a class of $(g_{ab}, \dot{g}_{ab})$ for which no embedding spacetime exists with time flow vector field pointing everywhere to the same 'future' side of the hypersurface. This does not contradict our existence result, since we prove existence only in a neighbourhood of a known solution. Finally, in section 7 we examine a Thin Sandwich formulation of general relativity with different variables, introduced by Komar[11], and discuss the possibility of proving results similar to ours for these equations.

We give a summary of some other important previous results. Komar[12] has derived the vacuum Thin Sandwich Equations using canonical methods. Bergmann[13] has calculated the linearised Thin Sandwich Equations, but he did not examine the conditions for their ellipticity. Christodoulou and Francaviglia[14] have given an analysis of the Thin Sandwich Equations in the case when the lapse function takes the constant 1 value on the whole manifold. Naturally enough they found nonexistence of solutions under this unnatural condition. However their analysis shows nonuniqueness if the three-dimensional initial hypersurface has flat regions. Pereira[15] has given a Cauchy-Kowalewski type proof for the local existence of unique analytic solutions of the Thin Sandwich Equations in a small open set of the three dimensional initial hypersurface. These solutions belong to analytic initial data, most of them are given on a two dimensional surface in the initial hypersurface, and one is given on a line in this surface. The initial data are the lapse and shift functions and their various partial derivatives. The analysis of Pereira requires implicit conditions on the initial data, and the approach is not natural either from geometrodynamical or from partial differential equations viewpoint. Teitelboim[16] has presented a discussion about the quantum mechanical origin of the Thin Sandwich Conjecture, including the case where the two sides of the sandwich intersects each other.

There is an obvious objection to the sandwich philosophy which is also a motivation for the Thin Sandwich formalism rather than the 'Thick' Sandwich formalism. There are problems even with the classical mechanical and electrodynamical analogue of the 'Thick' Sandwich Conjecture. One can look at the motion of a classical harmonic oscillator by specifying the coordinates at two moments. The problem with the oscillator arises when the time difference is a multiple of the half period. If it is exactly half period, and the two coordinates are not negative of each other, then no solution exists for the time development, but if they are negative of each other, then the solution is not unique. One can avoid this trouble by looking only at time differences smaller than the half period. However, in electrodynamics there are arbitrarily high frequency modes. In the electromagnetic version of the Thin Sandwich Conjecture, the vector potential (or the magnetic field) is given at two moments. In a coordinate system $x^\mu = (t, x^1, x^2, x^3)$, let the components of the vector potential be $A^0 = 0$, $A^1 = \sin(nt)\sin(nx^2)$, $A^2 = A^3 = 0$, for some $n > 0$ integer. This satisfies the $\partial^\mu A_\mu = 0$ Lorentz condition, and the $\partial^\nu \partial_\nu A^\mu = 0$ Maxwell's equation, so it can be considered as a standing wave solution on a 2π sized 3-torus. The vector potential vanishes on the whole space if $t = 0$ and $t = 2\pi/n$. This shows, that contrary to the other cases, if the time difference is $2\pi/n$, then it is not enough to give the vector potential on the two sides of the sandwich to fix the electromagnetic field inside. To overcome this problem it is natural to turn to the limit where the time difference goes to zero. This means the specification of the vector potential (or alternatively the magnetic field) and its time derivative at a moment. The analogue of this in general relativity is to give the spacelike metric and its time derivative on a hypersurface, to determine the structure of the spacetime. Thus we arrive at the Thin Sandwich Conjecture.

We will use Greek indices for spacetime tensors. These indices will be raised and lowered with the spacetime metric $\tilde{g}_{\alpha\beta}$. We denote the future directed unit normal vector field of the spacelike hypersurfaces by n^α, $n^\alpha n_\alpha = -1$, and the metric orthogonal projection tensor into the hypersurfaces by $h^\alpha_\beta = \delta^\alpha_\beta + n^\alpha n_\beta$. Tensors orthogonal in all the indices to n^α and n_α are in one-one correspondence with the tensors in the three dimensional hypersurfaces. The easiest to see this in the natural coordinate system, where the x^0 coordinate is adapted to the hypersurfaces. There $n_1 = n_2 = n_3 = 0$, and a vector field v^α is orthogonal to n_α if and only if $v^0 = 0$. This is not true for an n^α orthogonal one-form u_α, but the value of u_0 is unimportant if we only contract with n_α orthogonal tensors. In equations that contain only objects defined in the three dimensional hypersurfaces we will use Latin indices. These equations can also be interpreted as equations for four dimensional spacetime tensors orthogonal in all indices to n^α and n_α. Latin indices will be raised and lowered by the spacelike hypersurface metric g_{ab}, which corresponds to the projection of the spacetime metric $g_{\alpha\beta} = h^\gamma_\alpha h^\delta_\beta \tilde{g}_{\gamma\delta}$.

2 The Thin Sandwich Equations

In this section we derive the Thin Sandwich Equations from the gravitational constraints. We also present a variational principle leading to them. The induced metric and the extrinsic curvature as initial data given on a hypersurface completely determine the spacetime metric in the Cauchy development. Four of the ten Einstein's equations, the hypersurface orthogonal components, do not contain any time derivative of these tensors, therefore they are regarded as constraints on the initial data. The energy and momentum constraints are

$$\frac{1}{2}\left[R + (K^a_a)^2 - K^{ab}K_{ab}\right] = \epsilon \tag{5}$$

and

$$K^{ab}{}_{|b} - K^{b|a}_b = S^a, \tag{6}$$

where $|$ and R denote respectively the covariant derivative and the scalar curvature associated to the metric g_{ab} of the spacelike hypersurface. K_{ab} is the extrinsic curvature, ϵ is the energy density and S^a is the density of the energy flow. The energy density and the energy flow are related to the hypersurface orthogonal components of the stress-energy tensor $T_{\alpha\beta}$ by

$$\epsilon = 8\pi n_\alpha n_\beta T^{\alpha\beta}, \qquad S^\alpha = 8\pi h^\alpha_\beta n_\gamma T^{\beta\gamma}. \tag{7}$$

In the following we suppose that ϵ and S^a is specified apriori on the initial hypersurface, as is the case for example with vacuum or with perfect fluids in a Robertson-Walker cosmological model.

We specify a time flow vector field by giving its hypersurface parallel and orthogonal parts. The hypersurface parallel part N^a is called shift vector, and the orthogonal part N is called lapse function. This means that in adapted coordinates (t, x^a) the spacetime metric has the form

$$ds^2 = -N^2 dt^2 + g_{ab}(dx^a + N^a dt)(dx^b + N^b dt). \tag{8}$$

The derivative $\frac{d}{dt} g_{ab} \equiv \dot{g}_{ab}$ of the spacelike part of the spacetime metric along this time flow vector field can be written in terms of the extrinsic curvature,

$$\dot{g}_{ab} = 2NK_{ab} + 2N_{(a|b)}\ . \tag{9}$$

Expressing the extrinsic curvature

$$K_{ab} = \frac{1}{N}\left(\frac{1}{2}\dot{g}_{ab} - N_{(a|b)}\right), \tag{10}$$

we can substitute into the energy and momentum constraints,

$$N^2(2\epsilon - R) = \left(\frac{1}{2}g^{ab}\dot{g}_{ab} - N^a{}_{|a}\right)^2 - \left(\frac{1}{2}\dot{g}_{ab} - N_{(a|b)}\right)\left(\frac{1}{2}g^{ac}g^{bd}\dot{g}_{cd} - N^{(a|b)}\right), \tag{11}$$

and

$$\left[\frac{1}{N}\left(\frac{1}{2}g^{ac}g^{bd}\dot{g}_{cd} - N^{(a|b)} - \frac{1}{2}g^{ab}g^{cd}\dot{g}_{cd} + g^{ab}N^c{}_{|c}\right)\right]_{|b} = S^a. \tag{12}$$

If the three-metric g_{ab} and instead of the extrinsic curvature the time derivative $\dot{g}_{ab}$ of the three-metric is given as initial data together with ϵ and S^a, then equations (11) and (12) are four differential equations for the lapse and shift, called the *Thin Sandwich Equations.* The basic form of the Thin Sandwich Conjecture is

Thin Sandwich Conjecture 1: *Suppose that a positive definite metric g_{ab}, a symmetric tensor field $\dot{g}_{ab}$, a non negative energy density ϵ and an energy flow density S^a is given on a three-dimensional manifold $\mathcal{M}$. Then the Thin Sandwich Equations (11) and (12) have a unique solution for N^a and N.*

N^a and N determine the extrinsic curvature K_{ab} of the spacelike hypersurface by equation (10). If the matter fields obey some physically reasonable conditions, (certainly vacuum is ok), $(g_{ab}, K_{ab}, \epsilon, S^a)$ and possibly some other initial data for the matter fields specify a spacetime containing the hypersurface, which is unique in the Cauchy development[6][7]. We can write the conjecture in a form which is equivalent at least in the vacuum case,

Thin Sandwich Conjecture 2: *Suppose that a positive definite metric g_{ab}, a symmetric tensor field $\dot{g}_{ab}$, a non negative energy density ϵ, an energy flow density S^a and appropriate initial data for the matter fields are given on a three-dimensional manifold $\mathcal{M}$. Then there exists a spacetime satisfying the Einstein's and the matter field equations, such that on a Cauchy hypersurface of this spacetime the induced metric is g_{ab}, and the time parameter derivative along a time flow vector field of the spacelike part of the metric is $\dot{g}_{ab}$. This spacetime is unique on the Cauchy development of the initial hypersurface.*

It is clear that there exist data $(g_{ab}, \dot{g}_{ab}, \epsilon, S^a, N^a, N)$ satisfying the Thin Sandwich Equations. Any choice of spacelike hypersurface in a known solution of Einstein's equations generates $(g_{ab}, K_{ab}, \epsilon, S^a)$ satisfying the constraint equations. Any further choice of shift N^a and positive lapse N defines the time derivative $\dot{g}_{ab}$ of the spacelike metric in the direction fixed by them. Obviously $(g_{ab}, \dot{g}_{ab}, \epsilon, S^a, N^a, N)$ satisfies the Thin Sandwich Equations. However, this does not answer the Thin Sandwich Conjecture, which starts at $(g_{ab}, \dot{g}_{ab}, \epsilon, S^a)$ and seeks N^a and N.

To reduce the size of the equations, we introduce as a notation the tensor

$$\gamma_{ab} = \gamma_{ab}(g_{ab}, \dot{g}_{ab}, N^a) = \frac{1}{2}\dot{g}_{ab} - N_{(a|b)} \ , \tag{13}$$

and the scalar

$$\gamma^* = \gamma^*(g_{ab}, \dot{g}_{ab}, N^a) = (\gamma^a_a)^2 - \gamma^{ab}\gamma_{ab} \ . \tag{14}$$

The Thin Sandwich Equations can be written in the compact form

$$N^2(2\epsilon - R) = \gamma^* \tag{15}$$

and

$$\left[\frac{1}{N}\left(\gamma^{ab} - g^{ab}\gamma^c_c\right)\right]_{|b} = S^a. \tag{16}$$

The second derivative of the lapse function is not present in the Thin Sandwich Equations. This makes it difficult to analyse the elliptic or hyperbolic character of the linearisation of these equations. It is possible to overcome this difficulty if

Assumption 1 *The three dimensional metric g_{ab} and the energy density ϵ is given such that*

$$2\epsilon - R \neq 0 \tag{17}$$

at all points of the initial three dimensional manifold $\mathcal{M}$.

If we know apriori that the lapse function N is strictly positive, then we can use the scalar Thin Sandwich Equation (15) to eliminate the lapse N from the vectorial Thin Sandwich Equations (16),

$$N = \sqrt{\frac{\gamma^*}{2\epsilon - R}} \ , \tag{18}$$

thereby obtain a nonlinear system of equations for the shift vector N^a alone,

$$\left[\sqrt{\frac{2\epsilon - R}{\gamma^*}}\left(\gamma^{ab} - g^{ab}\gamma^e_e\right)\right]_{|b} = S^a. \tag{19}$$

However, Assumption 1 does not apriori assure that $N > 0$. If N and N^a satisfies the Thin Sandwich Equations, then equation (15) implies that the square root in (18) is well defined, but in (19) we have to assume apriori that $\gamma^* > 0$.

If we substitute γ_{ab} and γ^* from equation (13) and (14), we get the detailed form

$$\left\{\frac{\frac{1}{2}g^{ac}g^{bd}\dot{g}_{cd} - N^{(a|b)} - \frac{1}{2}g^{ab}g^{cd}\dot{g}_{cd} + g^{ab}N^c{}_{|c}}{\sqrt{\frac{1}{2\epsilon - R}\left[\left(\frac{1}{2}g^{ef}\dot{g}_{ef} - N^e{}_{|e}\right)^2 - \left(\frac{1}{2}\dot{g}_{ef} - N_{(e|f)}\right)\left(\frac{1}{2}g^{eh}g^{fi}\dot{g}_{hi} - N^{(e|f)}\right)\right]}}\right\}_{|b} = S^a. \tag{20}$$

This is a nonlinear system of differential equations for the shift N^a, called the *Reduced Thin Sandwich Equations.* If Assumption 1 holds, and we know apriori that the lapse N is positive, then the Reduced Thin Sandwich Equations are equivalent to the Thin Sandwich Equations (11) and (12) . The cost of this simplification, besides the more complicated algebraic structure of the new equations, is that we cannot consider cases where $2\epsilon - R$ or N changes sign in our three dimensional manifold. The real advantage is that if these two conditions and a simple further condition hold, then the linearised Reduced Thin Sandwich Equations are elliptic, as we shall see later. The condition $N > 0$ on the solution is equivalent to the assumption that in the resulting spacetime the time flow vector field points out to the same side on the whole initial hypersurface, it cannot be tangent to this surface anywhere.

Calculating the derivative of the quotient in the vectorial Thin Sandwich Equation (16), and multiplying by N^2, we get

$$N\left(\gamma^{ab} - g^{ab}\gamma^c_c\right)_{|b} - N_{|b}\left(\gamma^{ab} - g^{ab}\gamma^c_c\right) = N^2 S^a. \tag{21}$$

Those solutions of this equation for which $N = 0$ at some points of the manifold $\mathcal{M}$ may be problematic, because at those points the extrinsic curvature is not defined by equation (10). If K^{ab} cannot be defined by continuous extension from a neighbourhood, then it is possible that no embedding spacetime exists, or if it exists then it may not be unique.

The apriori condition $N > 0$ is ultimately a restriction on the class of admissible $\dot{g}_{ab}$. Clearly, given a solution $(g_{ab}, K_{ab}, \epsilon, S^a)$ of the constraints, using equation (9), one can construct $\dot{g}_{ab}$ for which $N = 0$ somewhere. Hence, for the Thin Sandwich Equations to be solvable (with $N > 0$), $\dot{g}_{ab}$ must be restricted in some form.

Remark: Calculating the derivative of the square root in equation (19) we get

$$\frac{1}{2}\frac{1}{\sqrt{2\epsilon - R}}\left(\gamma^*\right)^{-\frac{3}{2}}\left[2\gamma^*(2\epsilon - R)(\gamma^{ab} - g^{ab}\gamma^c_c)_{|b} + \gamma^*(2\epsilon - R)_{|b} - (2\epsilon - R)\gamma^*{}_{|b}\right] = S^a. \tag{22}$$

This shows that in the vacuum case $S^a = 0$ the square root can be eliminated from the Reduced Thin Sandwich Equations.

The Thin Sandwich Conjecture generally can only be true if suitable boundary conditions are given. To avoid technical problems, in the rest of the paper we suppose

Assumption 2 *The three dimensional manifold $\mathcal{M}$ representing the initial hypersurface is compact, connected and without boundary.*

In this case we do not have to specify boundary conditions. The results in this paper are probably true in the noncompact case if suitable boundary conditions are given, but this needs further investigation. The most probable next step would be the assymptotically hyperboloidal spacetimes. The assymptotically flat vacuum spacetimes are unfortunately precluded by Assumption 1.

The Reduced Thin Sandwich Equations can be derived from a variational principle[3].

Lemma 1 *The Reduced Thin Sandwich Equations (19) are the Euler-Lagrange equations belonging to the action*

$$I = -2 \int_{\mathcal{M}} dv \ \left\{ \sqrt{\gamma^*(2\epsilon - R)} + S_a N^a \right\} , \tag{23}$$

where the integral is performed on the three dimensional initial manifold $\mathcal{M}$, with respect to the natural volume element dv belonging to the spacelike metric g_{ab}.

This can be verified directly by variation with respect to N^a, or can be obtained by the following argument.

As all the field equations, the constraints also can be obtained by extremizing the action $I = I_G + I_M$, where I_M is the action associated to the matter fields, $I_G = \int d^4x \sqrt{-\tilde{g}} \tilde{R}$, $\tilde{g}$ is the determinant of the spacetime metric $\tilde{g}_{\alpha\beta}$, and $\tilde{R}$ is the scalar curvature belonging to $\tilde{g}_{\alpha\beta}$. After dropping boundary terms I_G can be written in terms of three dimensional quantities,

$$I_G = \int d^4x \sqrt{g} N \left[R + K_{ab} K^{ab} - (K_a^a)^2 \right] = \int d^4x \sqrt{g} \left[NR - \frac{1}{N} \gamma^* \right] . \tag{24}$$

The action does not contain time derivatives of N and N^a, and there are no second time derivatives of the other variables either, so by varying with respect to N and N^a we will get constraints. The variation of I_M with respect to N and N^a is

$$\frac{\delta I_M}{\delta N} = \frac{\delta I_M}{\delta \tilde{g}^{\alpha\beta}} \frac{\partial \tilde{g}^{\alpha\beta}}{\partial N} , \qquad \frac{\delta I_M}{\delta N^\alpha} = \frac{\delta I_M}{\delta \tilde{g}^{\alpha\beta}} \frac{\partial \tilde{g}^{\alpha\beta}}{\partial N^\alpha} . \tag{25}$$

Using the definition of the stress-energy tensor

$$T_{\alpha\beta} = -\frac{1}{8\pi} \frac{\delta I_M}{\delta \tilde{g}^{\alpha\beta}} , \tag{26}$$

and the decomposition of the spacetime metric

$$\tilde{g}^{\alpha\beta} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^b}{N^2} \\ \frac{N^a}{N^2} & g^{ab} - \frac{1}{N^2} N^a N^b \end{pmatrix} , \tag{27}$$

we get

$$\frac{\delta I_M}{\delta N} = -16\pi \sqrt{g} n^\alpha n^\beta T_{\alpha\beta} = -2\sqrt{g}\epsilon , \tag{28}$$

$$\frac{\delta I_M}{\delta N^\alpha} = -16\pi \sqrt{g} h^\beta_\alpha n^\gamma T_{\beta\gamma} = -2\sqrt{g} S_\alpha . \tag{29}$$

We assume that ϵ and S^a are prescribed functions on the initial manifold $\mathcal{M}$, hence are independent of N and N^a, so if we are only interested in the variations with respect to N and N^a, we can write

$$I_M = \int d^4x \left(\frac{\delta I_M}{\delta N} N + \frac{\delta I_M}{\delta N^a} N^a \right) = -2 \int d^4x \sqrt{g} \left(\epsilon N + S_a N^a \right) . \tag{30}$$

Since there are no time derivatives of N and N^a in the action, we obtain the same equations at the variations by dropping the time integral and integrating only in the hypersurface,

$$I = \int\limits_{\mathcal{M}} dx^3 \sqrt{g} \left[-(2\epsilon - R)N - \frac{1}{N}\gamma^* - 2S_a N^a \right]. \tag{31}$$

Easy to check that the variation with respect to N and N^a indeed gives the Thin Sandwich Equations (11) and (12). Using the form (18) of the energy constraint to eliminate N, gives the form (23) of the action, involving only the shift N^a.

3 Uniqueness

In this section we present the proof of the uniqueness part of the Thin Sandwich Conjecture, under the Assumptions 1a and 2b of the Introduction. It is easy to see from the form (19) of the Reduced Thin Sandwich Equations that if we have a solution N^a with

$$\gamma_{ab} = \frac{1}{2}\dot{g}_{ab} - N_{(a|b)} \ , \tag{32}$$

and if $\tilde{N}^a$ is a vector field such that

$$\tilde{\gamma}_{ab} = \frac{1}{2}\dot{g}_{ab} - \tilde{N}_{(a|b)} = \alpha\gamma_{ab} \tag{33}$$

for some positive function α on the three-dimensional manifold $\mathcal{M}$, then $\tilde{N}^a$ also satisfies the Reduced Thin Sandwich Equations. Belasco and Ohanian[10] have proved the converse if Condition 1a holds.

Theorem 1 *Suppose $2\epsilon - R > 0$. If N^a and $\tilde{N}^a$ are two solutions of the Reduced Thin Sandwich Equations (20), there exists a function $\alpha > 0$ such that $\tilde{\gamma}_{ab} = \alpha\gamma_{ab}$. Conversely, if N^a is a solution and $\tilde{N}^a$ satisfies (33), then $\tilde{N}^a$ also satisfies (20).*

Remark: If $2\epsilon - R > 0$, then solution of the Reduced Thin Sandwich Equations means that $\gamma^* > 0$ apriori, and the lapse N defined by (18) is positive.
Proof: Let

$$\tilde{\gamma}^* = (\tilde{\gamma}^a_a)^2 - \tilde{\gamma}_{ab}\tilde{\gamma}^{ab}. \tag{34}$$

Since N^a and $\tilde{N}^a$ are supposed to be solutions of the Reduced Thin Sandwich Equations (19),

$$\frac{2\epsilon - R}{\gamma^*} > 0, \qquad \frac{2\epsilon - R}{\tilde{\gamma}^*} > 0. \tag{35}$$

The condition $2\epsilon - R > 0$ implies that

$$\gamma^* > 0, \qquad \tilde{\gamma}^* > 0. \tag{36}$$

Since the metric is positive definite, $\gamma_{ab}\gamma^{ab} \geq 0$, so γ^a_a, and similarly $\tilde{\gamma}^a_a$, cannot be zero at any point of the manifold $\mathcal{M}$. Using their definition (32) and (33),

$$\int\limits_{\mathcal{M}} dv \ (\gamma^a_a - \tilde{\gamma}^a_a) = \int\limits_{\mathcal{M}} dv \ \left(\tilde{N}^a - N^a\right)_{|a} = 0, \tag{37}$$

since our manifold $\mathcal{M}$is compact without boundary. Because of the conectedness of the manifold this means that at some point $\gamma^a_a = \tilde{\gamma}^a_a$, and since they cannot change sign, it follows that γ^a_a and $\tilde{\gamma}^a_a$ have the same sign at every point of the manifold $\mathcal{M}$.

We define

$$\Gamma_{ab}(u) = u\gamma_{ab} + (1-u)\tilde{\gamma}_{ab}\ , \tag{38}$$

and

$$\Gamma^*(u) = (\Gamma^a_a(u))^2 - \Gamma_{ab}(u)\Gamma^{ab}(u)\ , \tag{39}$$

for $0 \leq u \leq 1$. Since we will use the square root of $\Gamma^*(u)$ in the proof, we first show that it is positive. Substituting (38) into (39),

$$\Gamma^*(u) = u^2\left[\gamma^* + \tilde{\gamma}^* - 2(\gamma\tilde{\gamma})^*\right] + 2u\left[(\gamma\tilde{\gamma})^* - \tilde{\gamma}^*\right] + \tilde{\gamma}^*, \tag{40}$$

where we have introduced the notation

$$(\gamma\tilde{\gamma})^* = \gamma^a_a\tilde{\gamma}^b_b - \gamma_{ab}\tilde{\gamma}^{ab}. \tag{41}$$

Straightforward to check, that since $\gamma^* > 0$ and $\tilde{\gamma}^* > 0$, $\Gamma^*(u)$ has root in the $0 \leq u \leq 1$ interval if and only if both of the following conditions hold,

$$(\gamma\tilde{\gamma})^* < 0, \tag{42}$$

$$\left[(\gamma\tilde{\gamma})^*\right]^2 - \gamma^*\tilde{\gamma}^* \geq 0. \tag{43}$$

We fix a point on the manifold and consider $\Gamma^a_b(u)$ there, denoting its eigenvalues by $G_1(u)$, $G_2(u)$, $G_3(u)$. These can be calculated for example in a local (coordinate) frame in which the metric is Euclidean, $g_{ab} = \delta_{ab}$. We can introduce a three-dimensional space of eigenvalues, in which the above eigenvalues are represented by the point with coordinates $(G_1(u), G_2(u), G_3(u))$. Easy to check that the condition $\Gamma^*(u) > 0$ means that the eigenpoints lie inside a double cone whose apex is the direction $(1,1,1)$, and which has the three coordinate axes lying on its surface. The two halves of the cone are distinguished by the value of $\Gamma^a_a(u) = G_1(u) + G_2(u) + G_3(u)$: it is positive in one cone and negative in the other. We have seen that γ^a_a and $\tilde{\gamma}^a_a$ are both negative or both positive, so the eigenpoints of $\Gamma^a_b(0)$ and $\Gamma^a_b(1)$ lie in the same cone. We can define another function $\bar{\Gamma}_{ab}(u)$ by substituting $-\tilde{\gamma}_{ab}$ instead of $\tilde{\gamma}_{ab}$ into the definition (38),

$$\bar{\Gamma}_{ab}(u) = u\gamma_{ab} + (1-u)(-\tilde{\gamma}_{ab}), \tag{44}$$

and defining

$$\bar{\Gamma}^*(u) = \left(\bar{\Gamma}^a_a(u)\right)^2 - \bar{\Gamma}_{ab}(u)\bar{\Gamma}^{ab}(u) \tag{45}$$

in the same way as in (39). Now the eigenvalues of $\bar{\Gamma}^a_b(0)$ and $\bar{\Gamma}^a_b(1)$ lie in the opposite cones, therefore the curve of the eigenvalues of $\bar{\Gamma}^a_b(u)$ for $0 \leq u \leq 1$ must intersect the surface of the cones somewhere. This means that $\bar{\Gamma}^*(u)$ has root in the $0 \leq u \leq 1$ interval. Since $(-\tilde{\gamma})^* = \tilde{\gamma}^* > 0$, the conditions of existence of roots are quite similar to (42) and (43); we have only changed the sign of $\tilde{\gamma}_{ab}$. From the existence of the root it follows that the following two inequalities hold:

$$(\gamma\tilde{\gamma})^* > 0, \tag{46}$$

$$\left[(\gamma\tilde{\gamma})^*\right]^2 - \gamma^*\tilde{\gamma}^* \geq 0. \tag{47}$$

Comparing the first with condition (42) we see that $\Gamma^*(u)$ cannot have root in the interval $0 \leq u \leq 1$. Since $\Gamma^*(0) = \gamma^* > 0$, it follows that $\Gamma^*(u) > 0$.

We can now define the following integral,

$$I(u) = -2\int\limits_{\mathcal{M}} dv\ \left\{\sqrt{\Gamma^*(u)(2\epsilon - R)} + S_a N^a(u)\right\}, \tag{48}$$

where

$$N^a(u) = uN^a + (1-u)\tilde{N}^a, \tag{49}$$

for $0 \le u \le 1$. Since by Lemma 1 N^a and $\tilde{N}^a$ satisfies the Euler-Lagrange equation (19) for the action (23), and $\Gamma_{ab}(0) = \gamma_{ab}$, $\Gamma_{ab}(1) = \tilde{\gamma}_{ab}$, obviously

$$I'(0) = I'(1) = 0, \tag{50}$$

where the prime denotes derivative with respect to u. The second derivative of the integral $I(u)$ is

$$I''(u) = -2\int\limits_{\mathcal{M}} dv\ \sqrt{(2\epsilon - R)}\frac{d^2}{du^2}\sqrt{\Gamma^*}\ . \tag{51}$$

Using the expression (40) for $\Gamma^*(u)$ we get

$$I''(u) = 2\int\limits_{\mathcal{M}} dv\ \sqrt{(2\epsilon - R)}(\Gamma^*(u))^{-\frac{3}{2}}\left\{[(\gamma\tilde{\gamma})^*]^2 - \gamma^*\tilde{\gamma}^*\right\}. \tag{52}$$

From (47) we see that $I''(u) \ge 0$. This together with $I'(0) = I'(1) = 0$ means that $I''(u)$ must vanish on the interval $0 \le u \le 1$. Since the integrand is non negative, it must be zero at every point of the manifold $\mathcal{M}$, so we have

$$[(\gamma\tilde{\gamma})^*]^2 - \gamma^*\tilde{\gamma}^* = 0. \tag{53}$$

We introduce the traceless tensor

$$q_{ab} = \gamma_{ab} - \frac{\gamma^c_c}{\tilde{\gamma}^d_d}\tilde{\gamma}_{ab}\ . \tag{54}$$

Expressing γ_{ab} and substituting into the equation (53), we get

$$[(\gamma\tilde{\gamma})^*]^2 - \gamma^*\tilde{\gamma}^* = \left(\tilde{\gamma}_{ab}q^{ab}\right)^2 + \tilde{\gamma}^* q_{ab}q^{ab} = 0. \tag{55}$$

Because of the positive definiteness of the metric this can be zero only if $q_{ab} = 0$, that is

$$\gamma_{ab} = \frac{\gamma^c_c}{\tilde{\gamma}^d_d}\tilde{\gamma}_{ab} = \alpha\tilde{\gamma}_{ab}\ , \qquad \alpha = \frac{\gamma^c_c}{\tilde{\gamma}^d_d} > 0. \tag{56}$$

Easy to check that if this condition holds then (53) is true and $I''(u) = 0$. The proof is complete.□

Remark: Belasco and Ohanian also claim that α should be 1, but there is a mistake in that part of the proof, because they forgot the cases where the region where $\alpha \ne 1$ consists of disjoint sets. We will present a counterexample when examining Robertson-Walker spacetimes, which shows that the result claimed there is incorrect.

If we introduce the notation $M^a = \tilde{N}^a - N^a$ and $\mu = N(1-\alpha)$, we can write the condition $\tilde{\gamma}_{ab} = \alpha\gamma_{ab}$ into the form

$$M_{(a|b)} = \mu K_{ab}\ . \tag{57}$$

The condition $\alpha > 0$ means $\mu < N$. This is not particularly restrictive because of the linearity of equation (57). We can reformulate the previous theorem as

Theorem 2 *Suppose that $2\epsilon - R > 0$, and there exists a solution N^a of the Reduced Thin Sandwich Equations (20). The solution is unique if and only if the equation (57) has only the trivial solution $M^a = 0$, $\mu = 0$. Furthermore, M^a satisfies (57) with $\mu < N$, if and only if $N^a + M^a$ is also a solution of the Reduced Thin Sandwich Equations (20).*

4 Existence

In this section we will apply the Implicit Function theorem to prove the existence of solutions of the Reduced Thin Sandwich Equations nearby a known solution, under the Conditions 1a and 2b of the Introduction. Since we assume that the original solution is a solution of the Reduced Thin Sandwich Equations, it satisfies $\gamma^* > 0$ and $N > 0$, and the solutions in a neighbourhood will also satisfy these conditions. Let $H^n(T^i_j)$ denote the set of tensors of valence (i,j) belonging to the Sobolev space $H^n = W^{n,2}$ for $n \geq 2$. We define the mapping

$$F^a\left((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c\right) = \left[\sqrt{\frac{2\epsilon - R}{\gamma^*}}\left(\gamma^{ab} - g^{ab}\gamma^e_e\right)\right]_{|b} - S^a, \tag{58}$$

which assigns a vector field $F^a \in H^n(T^1_0)$ to any positive definite symmetric tensor field $g_{ab} \in H^{n+2}(T^0_2)$, symmetric tensor field $\dot{g}_{ab} \in H^{n+1}(T^0_2)$, scalar function $\epsilon \in H^{n+1}$, vector field $S^a \in H^n(T^1_0)$, and vector field $N^a \in H^{n+2}(T^1_0)$, where $\gamma_{ab} = \frac{1}{2}\dot{g}_{ab} - N_{(a|b)}$, and $\gamma^* = (\gamma^a_a)^2 - \gamma_{ab}\gamma^{ab}$. If we assume $2\epsilon - R > 0$ and $\gamma^* > 0$, then $F^a\left((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c\right)$ is continuous on these function spaces. We will need the partial derivative with respect to N^a, which is a linear map that assigns the vector field $D_2F^a\left((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c\right)(M^b) \in H^n(T^1_0)$ to any vector field $M^a \in H^{n+2}(T^1_0)$,

$$\begin{aligned} D_2F^a\left((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c\right)(M^b) = &\left\{\sqrt{\frac{2\epsilon - R}{\gamma^*}}\left(-M^{(a|b)} + g^{ab}M^e{}_{|e}\right)\right. \\ &\left.-\frac{1}{2}\sqrt{\frac{2\epsilon - R}{(\gamma^*)^3}}\left[-2\gamma^c_c M^c{}_{|c} + 2\gamma_{cd}M^{c|d}\right]\left(\gamma^{ab} - g^{ab}\gamma^e_e\right)\right\}_{|b}. \end{aligned} \tag{59}$$

Defining the non-density momentum tensor, as function of g_{ab}, $\dot{g}_{ab}$ and N^a,

$$\pi_{ab} = K_{ab} - g_{ab}K^c_c = \frac{1}{N}\left(\gamma_{ab} - g_{ab}\gamma^c_c\right), \tag{60}$$

where now N denotes the function defined by (18), we can write the partial derivative as

$$D_2F^a\left((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c\right)(M^b) = \left(-\frac{1}{N}M^{(a|b)} + \frac{1}{N}g^{ab}M^c{}_{|c} - \frac{1}{N(2\epsilon - R)}\pi^{ab}\pi_{cd}M^{c|d}\right)_{|b}. \tag{61}$$

Easy to verify that

Theorem 3 *The Reduced Thin Sandwich Operator $D_2F^a\left((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c\right)$ is self adjoint with respect to the L^2 inner product of the metric g_{ab} on $\mathcal{M}$.*

Proof: Let M^a and $\bar{M}^a$ be elements of $H^2(T^1_0)$. Then the L^2 inner product

$$\begin{aligned} \left(\bar{M}^a, D_2F^a(M^b)\right) = &\int_{\mathcal{M}} dv\, \left\{\bar{M}_a\left(-\frac{1}{N}M^{(a|b)} + \frac{1}{N}g^{ab}M^c{}_{|c} - \frac{1}{N(2\epsilon - R)}\pi^{ab}\pi_{cd}M^{c|d}\right)_{|b}\right\} = \\ &\int_{\mathcal{M}} dv\, \left\{\left(-\frac{1}{N}\bar{M}^{(a|b)} + \frac{1}{N}g^{ab}\bar{M}^c{}_{|c} - \frac{1}{N(2\epsilon - R)}\pi^{ab}\pi_{cd}\bar{M}^{c|d}\right)_{|b} M_a\right\} = \left(D_2F^a(\bar{M}^b), M^a\right), \end{aligned} \tag{62}$$

where we dropped divergence terms, since the three dimensional manifold $\mathcal{M}$ is compact and without boundary.□

Theorem 4 *The differential operator $D_2F^a\left((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c\right)$ is elliptic, if and only if $N \neq 0$, $2\epsilon - R \neq 0$ and the momentum tensor π_{ab} belonging to g_{ab}, $\dot{g}_{ab}$ and N^a is positive or negative definite.*

Proof: We have to look at the leading second order terms, which are

$$\frac{1}{N}M^{[b,a]}{}_{,b} - \frac{1}{N(2\epsilon - R)}\pi^{ab}\pi_{cd}M^{c,d}{}_{,b} =$$
$$\left(\frac{1}{2N}g^{ac}\partial_b\partial_c - \frac{1}{2N}\delta^a_b g^{cd}\partial_c\partial_d - \frac{1}{N(2\epsilon - R)}\pi^{ac}\pi^d_b\partial_c\partial_d\right) M^b. \tag{63}$$

Hence the leading symbol is

$$f^a_b(\xi) = -\frac{1}{2N}\xi^a\xi_b + \frac{1}{2N}\delta^a_b\xi^c\xi_c + \frac{1}{N(2\epsilon - R)}\pi^a_c\pi^d_b\xi^c\xi_d\ . \tag{64}$$

The operator $D_2F^a\left((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c\right)$ is elliptic, if the determinant of the leading symbol is not zero when $\xi^a \neq 0$. It is easier to calculate the determinant using an orthonormal frame, in which ξ^a has only one nonvanishing component, $\xi^a = (\xi, 0, 0)$. This frame is not necessarily continuous, but we perform only algebraic operations at a point. Then $g_{ab} = \delta_{ab}$, and

$$\det\ f^a_b(\xi) = \det\begin{pmatrix} \frac{1}{N(2\epsilon-R)}\xi^2(\pi^1_1)^2 & \frac{1}{N(2\epsilon-R)}\xi^2\pi^1_1\pi^1_b \\ \frac{1}{N(2\epsilon-R)}\xi^2\pi^a_1\pi^1_1 & \frac{1}{2N}\delta^a_b\xi^2 + \frac{1}{N(2\epsilon-R)}\xi^2\pi^a_1\pi^1_b \end{pmatrix} = \frac{1}{4N^3(2\epsilon - R)}\xi^6(\pi^1_1)^2. \tag{65}$$

Thus, in general,

$$\det\ f^a_b(\xi) = \frac{1}{4N^3(2\epsilon - R)}\xi^e\xi_e(\pi_{cd}\xi^c\xi^d)^2, \tag{66}$$

which is not zero for all $\xi^a \neq 0$ if and only if π_{ab} is definite.□

We can also look at the ellipticity of the linearised Reduced Thin Sandwich Equations in the vicinity of a $(g_{ab}, \dot{g}_{ab}, \epsilon, S^a)$ for which there exists a solution N^a. Since N^a is a solution of the Reduced Thin Sandwich Equations, $\gamma^* > 0$ and $N > 0$. By equation (9), $(g_{ab}, \dot{g}_{ab}, \epsilon, S^a)$ can be generated by a solution $(g_{ab}, K_{ab}, \epsilon, S^a)$ of the constraint equations and arbitrary choice of shift and positive lapse.

Theorem 5 *Suppose that given g_{ab}, $\dot{g}_{ab}$, ϵ and S^a such that N^a is a solution of the Reduced Thin Sandwich Equations (20), $F^a\left((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c\right) = 0$. Then for small h_{ab}, $\dot{h}_{ab}$, $\tilde{\epsilon}$, $\tilde{S}^a$ and M^a, in the linearised approximation, the condition of $N^a + M^a$ being a solution with the metric $g_{ab} + h_{ab}$, its time derivative $\dot{g}_{ab} + \dot{h}_{ab}$, energy density $\epsilon + \tilde{\epsilon}$, and density of the energy flow $S^a + \tilde{S}^a$, gives a system of linear second order partial differential equations for the small M^a,*

$$F^a\left((g_{cd} + h_{ab}, \dot{g}_{cd} + \dot{h}_{ab}, \epsilon + \tilde{\epsilon}, S^a + \tilde{S}^a), N^c + M^a\right) = 0. \tag{67}$$

This system is elliptic if and only if the momentum tensor π_{ab} defined by (60) and (10) is positive or negative definite.

Proof: If we are interested only in ellipticity, then we only have to look at the leading second order terms, which are independent of the small h_{ab}, $\dot{h}_{ab}$, $\tilde{\epsilon}$ and $\tilde{S}^a$, and are obviously the same as in (63), so the condition is the same as in the previous theorem.□

Lemma 2 *Suppose that g_{ab}, K_{ab} and ϵ satisfies the energy constraint (5). If $2\epsilon - R > 0$, then π_{ab} defined by (60) and (10) is positive or negative definite.*

Proof: From the definition (60) of the momentum tensor, $K^a_a = -\frac{1}{2}\pi^a_a$ and $K_{ab} = \pi_{ab} - \frac{1}{2}g_{ab}\pi^c_c$. Substituting into the energy constraint

$$2\epsilon - R = (K^a_a)^2 - K^{ab}K_{ab} = \frac{1}{2}(\pi^a_a)^2 - \pi^{ab}\pi_{ab}\ . \tag{68}$$

Hence it is enough to prove that if $\frac{1}{2}(\pi^a_a)^2 - \pi^{ab}\pi_{ab} > 0$, then π_{ab} is positive or negative definite. It is the easiest to do this using an orthonormal frame in which π_{ab} is diagonal,

$$\pi_{ab} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \tag{69}$$

Since now $g_{ab} = \delta_{ab}$,

$$\begin{aligned} \frac{1}{2}(\pi^a_a)^2 - \pi^{ab}\pi_{ab} &= \frac{1}{2}(\alpha+\beta+\gamma)^2 - \left(\alpha^2+\beta^2+\gamma^2\right) \\ &= -\frac{1}{2}\left(\alpha^2+\beta^2+\gamma^2\right) + \alpha\beta + \alpha\gamma + \beta\gamma = 2\alpha\beta - \frac{1}{2}(\alpha+\beta-\gamma)^2, \end{aligned} \tag{70}$$

which can only be positive if $\alpha\beta$ is positive, that is if α and β have the same sign. Similarly, α and γ must have the same sign, so π_{ab} must be positive or negative definite.□

Remark: This lemma is not true in the other direction, there exist π_{ab} which is positive (or negative) definite, but for which $\frac{1}{2}(\pi^a_a)^2 - \pi^{ab}\pi_{ab} < 0$. For example if $\alpha = \beta = 1$ and $\gamma = 6$ then $\frac{1}{2}(\pi^a_a)^2 - \pi^{ab}\pi_{ab} = -6$.

We have the corollary

Corollary 1 *If N^c is a solution of the Reduced Thin Sandwich Equations belonging to the initial data $(g_{cd}, \dot{g}_{cd}, \epsilon, S^c)$, such that $2\epsilon - R > 0$ and $N > 0$, then the linearised Reduced Thin Sandwich Operator $D_2F^a\left((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c\right)$ is elliptic.*

In order to apply the Implicit Function theorem, we need to show that $D_2F^a((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c)$ is an isomorphism. Since it is self adjoint, in the elliptic case this is equivalent to the vanishing of its kernel.

Lemma 3 *Suppose that N^c is a solution of the Reduced Thin Sandwich Equations belonging to the initial data $(g_{cd}, \dot{g}_{cd}, \epsilon, S^c)$, such that $2\epsilon - R > 0$ and $N > 0$. Then M^a is an element of the kernel of the operator $D_2F^a\left((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c\right)$, if and only if there exists a scalar function μ such that M^a and μ are solutions of the equation*

$$M_{(a|b)} = \mu K_{ab} \ . \tag{71}$$

Proof: Suppose that M^a is in the kernel of $D_2F^a\left((g_{cd}, \dot{g}_{cd}, \epsilon, S^c), N^c\right)$. We have to look for the nontrivial solutions of the following equation,

$$\left(-\frac{1}{N}M^{(a|b)} + \frac{1}{N}g^{ab}M^c{}_{|c} - \frac{1}{N(2\epsilon - R)}\pi^{ab}\pi_{cd}M^{c|d}\right)_{|b} = 0. \tag{72}$$

Multiplying with M_a, integrating on the three dimensional manifold $\mathcal{M}$, and dropping divergence terms,

$$\int_{\mathcal{M}} dv \, \left[\frac{1}{N}M_{(a|b)}M^{a|b} - \frac{1}{N}\left(M^a{}_{|a}\right)^2 + \frac{1}{N(2\epsilon - R)}\left(\pi^{ab}M_{a|b}\right)^2\right] = 0. \tag{73}$$

Using equation (68), we get a form for the integral, which is more homogeneous in the variable π_{ab},

$$\int_{\mathcal{M}} dv \, \frac{1}{N(2\epsilon - R)}\left\{\left[\frac{1}{2}(\pi^c_c)^2 - \pi^{cd}\pi_{cd}\right]\left[M_{(a|b)}M^{a|b} - \left(M^a{}_{|a}\right)^2\right] + \left(\pi^{ab}M_{a|b}\right)^2\right\} = 0. \tag{74}$$

We can choose an orthonormal frame in which the extrinsic curvature is diagonal. Let

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad K_{ab} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \tag{75}$$

Then the momentum is also diagonal,

$$\pi_{ab} = \begin{pmatrix} -\beta-\gamma & 0 & 0 \\ 0 & -\alpha-\gamma & 0 \\ 0 & 0 & -\alpha-\beta \end{pmatrix}. \tag{76}$$

From the energy constraint (68),

$$2\epsilon - R = 2(\alpha\beta + \alpha\gamma + \beta\gamma) \neq 0. \tag{77}$$

We denote the components of $M_{(a|b)}$ by

$$M_{(a|b)} = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}. \tag{78}$$

It can be verified by a simple but lengthy calculation, that the expression in the integral can be written in the form

$$\begin{aligned}
&\left[\frac{1}{2}(\pi^c_c)^2 - \pi^{cd}\pi_{cd}\right]\left[M_{(a|b)}M^{a|b} - \left(M^a{}_{|a}\right)^2\right] + \left(\pi^{ab}M_{a|b}\right)^2 = \\
&= \left[\frac{1}{2}(2\alpha+2\beta+2\gamma)^2 - (\beta+\gamma)^2 - (\alpha+\gamma)^2 - (\alpha+\beta)^2\right] \\
&\quad \left[a^2+b^2+c^2+2d^2+2e^2+2f^2-(a+b+c)^2\right] \\
&\quad - \left[a(\beta+\gamma)+b(\alpha+\gamma)+c(\alpha+\beta)\right]^2 = \\
&= \alpha^2b^2 - 2a\alpha b\beta + a^2\beta^2 + 2\alpha^2bc - 2a\alpha\beta c - 2\alpha b\beta c + 2a\beta^2c + \alpha^2c^2 \\
&\quad +2\alpha\beta c^2 + \beta^2c^2 - 2a\alpha b\gamma + 2\alpha b^2\gamma + 2a^2\beta\gamma - 2ab\beta\gamma - 2a\alpha c\gamma - 2\alpha bc\gamma \\
&\quad -2a\beta c\gamma - 2b\beta c\gamma + a^2\gamma^2 + 2ab\gamma^2 + b^2\gamma^2 + 4\alpha\beta d^2 + 4\alpha\gamma d^2 + 4\beta\gamma d^2 \\
&\quad +4\alpha\beta e^2 + 4\alpha\gamma e^2 + 4\beta\gamma e^2 + 4\alpha\beta f^2 + 4\alpha\gamma f^2 + 4\beta\gamma f^2 = \\
&= \frac{1}{(\beta+\gamma)^2}\left\{\left[(\beta+\gamma)^2a + \gamma^2b + \beta^2c - (\alpha\beta+\alpha\gamma+\beta\gamma)(b+c)\right]^2\right. \\
&\quad \left. +4(\alpha\beta+\alpha\gamma+\beta\gamma)\left[(\beta+\gamma)^2(d^2+e^2+f^2) + (\gamma b - \beta c)^2\right]\right\}.
\end{aligned} \tag{79}$$

If $2\epsilon - R = 2(\alpha\beta + \alpha\gamma + \beta\gamma) > 0$, this expression cannot be negative, so it must vanish on the whole manifold $\mathcal{M}$. Hence $d = e = f = 0$, $\beta c = \gamma b$ and $(\beta+\gamma)^2a + \gamma^2b + \beta^2c - (\alpha\beta+\alpha\gamma+\beta\gamma)(b+c) = 0$. Multiplying with β and eliminating βc,

$$0 = \beta\left[(\beta+\gamma)^2a + \gamma^2b + \beta^2c - (\alpha\beta+\alpha\gamma+\beta\gamma)(b+c)\right] = (\beta a - \alpha b)(\beta+\gamma)^2. \tag{80}$$

Similarly, multiplying with γ and eliminating γb,

$$0 = \gamma\left[(\beta+\gamma)^2a + \gamma^2b + \beta^2c - (\alpha\beta+\alpha\gamma+\beta\gamma)(b+c)\right] = (\gamma a - \alpha c)(\beta+\gamma)^2. \tag{81}$$

If $\beta+\gamma=0$, then $2\epsilon-R=-\beta^2\leq 0$. Hence, if $2\epsilon-R>0$, then $\beta+\gamma\neq 0$, and the expression (79) is zero if and only if

$$\alpha b=\beta a\ ,\quad \alpha c=\gamma a\ ,\quad \beta c=\gamma b\ ,\quad d=e=f=0. \tag{82}$$

This means that in the case $2\epsilon-R>0$ the integral can be zero, only if there exists a function μ on the manifold, such that

$$M_{(a|b)}=\mu K_{ab}\ . \tag{83}$$

Substituting this into the form (61) of the partial derivative, we see that if M^a satisfies this equation, then it is really in the kernel. The relationship between the solutions of this equation and spacetime Killing vectors is described in the next section.□

The main theorem of the paper is

Theorem 6 *Suppose that given $g_{ab}\in H^{n+2}(T^0_2)$, $\dot g_{ab}\in H^{n+1}(T^0_2)$, $\epsilon\in H^{n+1}$ and $S^a\in H^n(T^1_0)$, such that $2\epsilon-R>0$, where R is the scalar curvature belonging to g_{ab}. Suppose that $N^a\in H^{n+2}(T^1_0)$ is a solution of the Reduced Thin Sandwich Equations with $N>0$, that is $F^a\left((g_{cd},\dot g_{cd},\epsilon,S^c),N^c\right)=0$. Let K_{ab} be defined by (10). If the equation $M_{(a|b)}=\mu K_{ab}$ has only the trivial solution, then there exists a unique continuous map on an open neighbourhood of $(g_{cd},\dot g_{cd},\epsilon,S^c)$ in $H^{n+2}(T^0_2)\times H^{n+1}(T^0_2)\times H^{n+1}\times H^n(T^1_0)$, which assigns $\bar N^a\in H^{n+2}(T^1_0)$ to $(\bar g_{cd},\dot{\bar g}_{cd},\bar\epsilon,\bar S^c)$, such that $\bar N^a$ is a solution of the Reduced Thin Sandwich Equations with $(\bar g_{cd},\dot{\bar g}_{cd},\bar\epsilon,\bar S^c)$, that is $F^a\left((\bar g_{cd},\dot{\bar g}_{cd},\bar\epsilon,\bar S^c),\bar N^c\right)=0$.*

In other words, the Thin Sandwich Equations are solvable in a neighbourhood of this $(g_{cd},\dot g_{cd},\epsilon,S^c)$, in which still $2\epsilon-R>0$. The solution is unique, and $N>0$.

Proof: We have proved that if $2\epsilon-R>0$ and $M_{(a|b)}=\mu K_{ab}$ has only the trivial solution, then N^a is unique by Theorem 2, $D_2F^a\left((g_{cd},\dot g_{cd},\epsilon,S^c),N^c\right)$ is elliptic by Corollary 1 , it is self adjoint by Theorem 3, and its kernel vanishes by Lemma 3, so it is a linear isomorphism. The result follows by direct application of the Implicit Function theorem[17].□

There is a natural way to generate some $(g_{ab},\dot g_{ab},\epsilon,S^a)$ and a solution N^a of the Reduced Thin Sandwich Equations belonging to them. Any choice of spacelike hypersurface in an arbitrary spacetime solution of Einstein's equations induces a solution $(g_{ab},K_{ab},\epsilon,S^a)$ of the constraint equations. Then by equation (9), an arbitrary choice of shift vector N^a and positive lapse function N fixes the time derivative $\dot g_{ab}$ of the spacelike metric along the time flow vector field given by this shift and lapse. Obviously, this N^a is a solution of the reduced Thin Sandwich Equations with $(g_{ab},\dot g_{ab},\epsilon,S^a)$. In view of this we can rewrite our main theorem into the more natural form

Theorem 7 *Suppose that $g_{ab}\in H^{n+2}(T^0_2)$, $K_{ab}\in H^{n+1}(T^0_2)$, $\epsilon\in H^{n+1}$ and $S^a\in H^n(T^1_0)$ satisfies the constraint equations (5) and (6), and $2\epsilon-R>0$, where R is the scalar curvature belonging to g_{ab}. For any choice of shift vector $N^a\in H^{n+2}(T^1_0)$ and positive lapse function $N\in H^{n+1}$ we define the time derivative $\dot g_{ab}$ of the spacelike metric by (9). If the equation $M_{(a|b)}=\mu K_{ab}$ has only the trivial solution, then there exists a unique continuous map on an open neighbourhood of $(g_{cd},\dot g_{cd},\epsilon,S^c)$ in $H^{n+2}(T^0_2)\times H^{n+1}(T^0_2)\times H^{n+1}\times H^n(T^1_0)$, which assigns $\bar N^a\in H^{n+2}(T^1_0)$ to $(\bar g_{cd},\dot{\bar g}_{cd},\bar\epsilon,\bar S^c)$, such that $\bar N^a$ is a solution of the Reduced Thin Sandwich Equations with $(\bar g_{cd},\dot{\bar g}_{cd},\bar\epsilon,\bar S^c)$, that is $F^a\left((\bar g_{cd},\dot{\bar g}_{cd},\bar\epsilon,\bar S^c),\bar N^c\right)=0$.*

5 Conditions

If N^a and $N\neq 0$ are solutions of the Thin Sandwich Equations for the initial values $(g_{cd},\dot g_{cd},\epsilon,S^c)$, then by equation (10) these determine the extrinsic curvature K_{ab}, such that $(g_{cd},K_{cd},\epsilon,S^c)$ satisfies the constraint equations. On the other hand, if we choose a solution $(g_{cd},K_{cd},\epsilon,S^c)$

of the constraints, then by equation (9) any choice of lapse N and shift N^a generates $(g_{cd}, \dot{g}_{cd}, \epsilon, S^c)$, for which N and N^a are solutions of the Thin Sandwich Equations. If $2\epsilon - R \neq 0$, and we are interested only in solutions with $N > 0$, then the Thin Sandwich Equations are equivalent to the Reduced Thin Sandwich Equations.

Suppose that $(g_{cd}, \dot{g}_{cd}, \epsilon, S^c)$ is given together with a solution N^a and $N > 0$ of the Thin Sandwich Equations, or equivalently, $(g_{cd}, \dot{g}_{cd}, \epsilon, S^c)$ can be obtained from a solution $(g_{cd}, K_{cd}, \epsilon, S^c)$ of the constraint equations by choosing $N > 0$ and N^a. Theorem 7 means that there exists an open neighbourhood of $(g_{cd}, \dot{g}_{cd}, \epsilon, S^c)$ in which for any $(\bar{g}_{cd}, \dot{\bar{g}}_{cd}, \bar{\epsilon}, \bar{S}^c)$ the form 1 of the Thin Sandwich Conjecture is true if the following two conditions hold:

Condition 1a:

$$2\epsilon - R > 0, \tag{84}$$

where R is the scalar curvature belonging to the three-metric g_{ab}.

Condition 2b: *The equation*

$$M_{(a|b)} = \mu K_{ab} \tag{85}$$

for the vector field M^a and scalar field μ on the initial hypersurface has only the trivial solution $M^a = 0$ and $\mu = 0$.

Introducing the trace-free part of the extrinsic curvature, $\overset{\circ}{K}_{ab} = K_{ab} - \frac{1}{3} g_{ab} K^c_c$, the energy constraint (5) becomes

$$2\epsilon - R = (K^a_a)^2 - K_{ab}K^{ab} = \frac{2}{3}(K^a_a)^2 - \overset{\circ}{K}_{ab} \overset{\circ}{K}{}^{ab} . \tag{86}$$

Hence condition 1a is equivalent to

Condition 1b: *The trace of the extrinsic curvature K_{ab} dominates the trace-free part*

$$\overset{\circ}{K}_{ab} = K_{ab} - \frac{1}{3} g_{ab} K^c_c \tag{87}$$

in the following way:

$$\frac{2}{3}(K^a_a)^2 > \overset{\circ}{K}_{ab} \overset{\circ}{K}{}^{ab} . \tag{88}$$

Obviously the same is true for the shift tensor γ_{ab}. An advantage of the $2\epsilon - R > 0$ form of the condition is that it is expressed only in terms of the initial data in the hypersurfaces, while the exterior curvature is only known after solving the Thin Sandwich Equations for the lapse and shift. The extrinsic curvature form is useful to examine the condition of when various slicings of a spacetime induce appropriate initial data for the Thin Sandwich Equations.

We can interpret (9) as an equation determining the time derivative of the spacelike part of the metric in the direction of the time flow vector field specified by the shift N^a and lapse N. Comparing with equation (85) we get the following equivalent form of Condition 2b:

Condition 2a: *There is no choice of time flow vector field on the spacetime induced by $(g_{ab}, K_{ab}, \epsilon, S^a)$, for which*

$$\dot{g}_{ab} = 0 \tag{89}$$

on the whole initial hypersurface. (Now $\dot{g}_{ab} = \frac{d}{dt} g_{ab}$ is the derivative in the direction of the time flow vector field.)

In view of this the following theorem is not surprising.

Theorem 8 *Suppose $\tilde{M}^\alpha$ is a spacetime vector field. Denote its projection into a spacelike hypersurface by $M^\alpha = h^\alpha_\beta \tilde{M}^\beta$ and its hypersurface orthogonal part by $\mu = \tilde{M}^\alpha n_\alpha$, where n^α is the unit normal of the hypersurface and $h^\alpha_\beta = \delta^\alpha_\beta + n^\alpha n_\beta$ is the metric orthogonal projection into the hypersurface. Then the following are equivalent:*

1. *$M_{(a|b)} = \mu K_{ab}$ in the hypersurface*
2. *$\tilde{M}^\alpha$ satisfies the hypersurface projected part of the Killing equations on the hypersurface.*

Proof: We can express $\tilde{M}^\alpha$ by its components,

$$\tilde{M}^\alpha = \delta^\alpha_\beta \tilde{M}^\beta = (h^\alpha_\beta - n^\alpha n_\beta)\tilde{M}^\beta = M^\alpha - \mu n^\alpha. \tag{90}$$

Denoting the spacetime covariant derivative related to $\tilde{g}_{ab}$ by ∇_α, the projected Killing equation is

$$0 = h^\gamma_\alpha h^\delta_\beta \nabla_{(\gamma}\tilde{M}_{\delta)} = h^\gamma_\alpha h^\delta_\beta \nabla_{(\gamma} M_{\delta)} - h^\gamma_\alpha h^\delta_\beta \nabla_{(\gamma}\left(n_{\delta)}\mu\right). \tag{91}$$

Using the relation $M_{\alpha|\beta} = h^\gamma_\alpha h^\delta_\beta \nabla_\delta M_\gamma$ between the two covariant derivatives and the definition of the extrinsic curvature $K_{\alpha\beta} = h^\gamma_\alpha h^\delta_\beta \nabla_\gamma n_\delta$, we get

$$0 = M_{(\alpha|\beta)} - \mu K_{ab}\ , \tag{92}$$

which is exactly the condition (85).□

As a special case we have

Corollary 2 *If $\tilde{M}^\alpha$ is a spacetime Killing vector field, its projection into a hypersurface with extrinsic curvature K_{ab} satisfies the equation $M_{(a|b)} = \mu K_{ab}$ with some function μ.*

This means that Condition 2b is false if the spacetime generated by $(g_{ab}, K_{ab}, \epsilon, S^a)$ admits a Killing vector, or admits a vector field which satisfies at least the spacelike part of the Killing equations on the initial hypersurface. Let $(g_{ab}, \dot{g}_{ab}, \epsilon, S^a)$ be generated by such $(g_{ab}, K_{ab}, \epsilon, S^a)$ and some choice of N^a and N. Then by Theorem 2, N^a and N is not the unique solution of the Thin Sandwich Equations belonging to this $(g_{ab}, \dot{g}_{ab}, \epsilon, S^a)$. Furthermore, our existence proof does not work either in the neighbourhood of such $(g_{ab}, \dot{g}_{ab}, \epsilon, S^a)$. It is possible that existence can be proved even in these cases, and the different lapse and shift solutions generate the same spacetime with different foliations, but this needs further investigation.

An important question is whether the converse of Corollary 2 is true in the following sense. Suppose that M^a satisfies equation $M_{(a|b)} = \mu K_{ab}$ on the initial hypersurface. Is there a Killing vector $\tilde{M}^\alpha$ on the induced spacetime such that M^a is the projection of $\tilde{M}^\alpha$ on the initial hypersurface? In view of Theorem 8 this appears to be false, generally one can only find a vector field which satisfies only the spacelike part of the Killing equations.

However there is a large class of initial data which satisfies Condition 2b.

Theorem 9 *If the Ricci tensor belonging to the hypersurface metric g_{ab} is negative definite, the equation $M_{(a|b)} = \mu K_{ab}$ has only the trivial solution $M^a = 0$, $\mu = 0$.*

Proof: The proof is very similar to the well known argument that a compact manifold with negative definite Ricci tensor has no nontrivial Killing vectors. Our sign conventions for the Riemann and Ricci tensors are $M_{c|ba} - M_{c|ab} = R_{abc}{}^d M_d$, and $R_{ab} = R_{acb}{}^c$. We have

$$M_{b|a}{}^{|b} - M_b{}^{|b}{}_{|a} = R_{ab} M^b. \tag{93}$$

Differentiating condition (85), we get

$$M_{a|b}{}^b + M_{b|a}{}^b = 2(\mu K_{ab})^{|b}. \tag{94}$$

Comparing these and using that $M^a{}_{|a} = \mu K^a_a$,

$$M_{a|b}{}^b + R_{ab}M^b - 2(\mu K_{ab})^{|b} + (\mu K^b_b)_{|a} = 0. \tag{95}$$

We multiply with M^a and integrate on the three dimensional manifold $\mathcal{M}$. We can drop divergence terms, since our manifold does not have boundary and it is compact, so all integrals are finite. Substituting for $M_{(a|b)}$ from (85), and then using the form (5) of the energy constraint, we get

$$\begin{aligned}
0 &= \int_{\mathcal{M}} dv\ M^a \left[M_{a|b}{}^b + R_{ab}M^b - 2(\mu K_{ab})^{|b} + (\mu K^b_b)_{|a}\right] \\
&= \int_{\mathcal{M}} dv\ \left[-M^{a|b}M_{a|b} + M^a M^b R_{ab} + 2\mu M^{a|b}K_{ab} - \mu M^a{}_{|a}K^b_b\right] \\
&= \int_{\mathcal{M}} dv\ \left[-M^{[a|b]}M_{[a|b]} - \mu^2 K^{ab}K_{ab} + M^a M^b R_{ab} + 2\mu^2 K^{ab}K_{ab} - \mu^2\left(K^a_a\right)^2\right] \\
&= \int_{\mathcal{M}} dv\ \left[-M^{[a|b]}M_{[a|b]} + M^a M^b R_{ab} - \mu^2(2\epsilon - R)\right].
\end{aligned} \tag{96}$$

Since the Ricci curvature is negative definite, the second term is negative if $M^a \neq 0$. As the energy density ϵ must be non negative, the last term cannot be positive either. Since the metric is Riemannian, the first term is non positive, so the only solution is $M^a = 0$, $\mu = 0$.□

The proof also shows that

Theorem 10 *If the initial hypersurface has a flat intrinsic geometry and $\epsilon \neq 0$, then the equation $M_{(a|b)} = \mu K_{ab}$ has only the Killing vector solutions with $\mu = 0$.*

A linear combination of solutions of equation (85) is also a solution. We prove

Theorem 11 *If the determinant of K_{ab} is not zero, the space of the solutions of the equation $M_{(a|b)} = \mu K_{ab}$ is finite dimensional.*

Proof: We follow the proof of Taub[18] for conformal Killing vectors. Suppose now that we are looking for solutions on an n dimensional manifold. Substituting (85) into the definition of the Riemann tensor, we get

$$M_{c|ba} + M_{a|cb} = R_{abc}{}^d M_d + 2(\mu K_{ac})_{|b}\ . \tag{97}$$

Adding and subtracting the same equation with permuted indices and using the symmetry property $R_{[abc]}{}^d = 0$ of the Riemann tensor, we have

$$M_{c|ba} = -R_{bca}{}^d M_d + (\mu K_{ac})_{|b} + (\mu K_{bc})_{|a} - (\mu K_{ab})_{|c}\ . \tag{98}$$

By the definition of the curvature,

$$M_{c|bad} - M_{c|bda} = R_{dac}{}^e M_{e|b} + R_{dab}{}^e M_{c|e}\ . \tag{99}$$

Substituting $M_{c|ba}$ and $M_{c|bd}$ from (98), we get

$$\begin{aligned}
&(\mu K_{ac})_{|bd} + (\mu K_{bc})_{|ad} + (\mu K_{db})_{|ca} - (\mu K_{ab})_{|cd} - (\mu K_{dc})_{|ba} - (\mu K_{bc})_{|da} \\
&\quad + R_{bcd}{}^e M_{e|a} - R_{dac}{}^e M_{e|b} - R_{dab}{}^e M_{c|e} - R_{bca}{}^e M_{e|d} + R_{bcd}{}^e{}_{|a} M_e - R_{bca}{}^e{}_{|d} M_e = 0.
\end{aligned} \tag{100}$$

We wish to use this equation to express $\mu_{|ab}$. If the inverse of K_{ab} exists, that is the determinant is nonvanishing, then we can multiply by K^{-1cd}, and the μ second derivative terms will be

$$(2-n)\mu_{|ab} - K_{ab}K^{-1cd}\mu_{|cd}. \tag{101}$$

After multiplying also by K^{-1ab}, we get an expression for $2(1-n)K^{-1cd}\mu_{|cd}$. If $n>2$ we can substitute into the previous equation to get an expression for $\mu_{|ab}$ in terms of μ, $\mu_{|a}$, M_a, $M_{a|b}$ and g_{ab}. Introducing the condensed notation

$$\Theta_A = (\mu, \mu_{|a}, M_a, M_{a|b}), \qquad A = 1, \ldots, (n+1)^2, \tag{102}$$

the equation (98) and the expression for $\mu_{|ab}$, together with

$$\frac{\partial \mu}{\partial x^a} = \mu_{|a} \tag{103}$$

and

$$\frac{\partial M_a}{\partial x^b} = M_{a|b} + \Gamma_{ab}{}^{c} M_c, \tag{104}$$

where $\Gamma_{ab}{}^{c}$ is the Christoffel symbol belonging to g_{ab}, form a system of first order partial differential equations

$$\frac{\partial \Theta_A}{\partial x^a} = f_{aA}(x, \Theta), \qquad A = 1, \ldots, (n+1)^2, \quad a = 1, \ldots, n, \tag{105}$$

where $f_{aA}(x,\Theta)$ are functions of the coordinates x^b and the variables Θ_A. The $(n+1)^2$ components of Θ_A at a point determines all the partial derivatives of Θ_A at that point. However there are $n(n+1)/2$ algebraic relations of the form (85) between the values Θ_A, hence the maximum number of independent parameters is $(n+1)(n+2)/2$. We proved that the space of solutions of equation (85) is finite dimensional, and maximum $(n+1)(n+2)/2 = 10$ parameters given at a point determine the solution uniquely. It is interesting that the maximum number of independent Killing vector fields in the embedding spacetime is also $(n+1)(n+2)/2$.□

6 Examples

If the Ricci tensor belonging to the spacelike metric g_{ab} of a given spacetime solution is negative definite, then $2\epsilon - R > 0$, and we proved in Theorem 9, that $M_{(a|b)} = \mu K_{ab}$ has only the trivial solution. The existence of unique solutions of the Thin Sandwich Equations in a neighbourhood of the initial data follows by Theorem 7. Any known spacetime which has a Cauchy surface with negative scalar curvature generates such initial data. An obvious example with a compact Cauchy surface without boundary is the spatially compactified hyperbolic Robertson-Walker spacetime. Hence

Corollary 3 *The Thin Sandwich Equations are solvable with initial data $(g_{ab}, \dot{g}_{ab}, \epsilon, S^a)$ in a neighbourhood of the initial data $(g^{RW}_{ab}, \dot{g}^{RW}_{ab}, \epsilon_{RW}, S^a_{RW})$ generated by the standard slicing of the spatially compactified hyperbolic Robertson-Walker spacetime. The solution for N^a and $N>0$ is unique.*

We look at the spherical Robertson-Walker solutions in detail. The spacetime metric is

$$ds^2 = -d\tau^2 + a^2\left[d\psi^2 + \sin^2\psi(d\vartheta^2 + \sin^2\vartheta d\varphi)\right], \tag{106}$$

where the coordinates are $x^\alpha = (\tau, \psi, \vartheta, \varphi)$, and a is a function of τ. Since the shift is zero, the extrinsic curvature is $K_{ab} = \frac{1}{2}\dot{g}_{ab} = \frac{\dot{a}}{a} g_{ab}$, where the dot denotes derivative with respect to τ. Equation (85) becomes the conformal Killing equation $M_{(a|b)}$. From the energy constraint,

$$2\epsilon - R = (K^a_a)^2 - K_{ab}K^{ab} = 6\frac{\dot{a}^2}{a^2} \geq 0. \tag{107}$$

$2\epsilon - R$ is zero, if and only if $\dot{a} = 0$. Then the universe is maximal in size, turning from expansion to contraction. It follows by Theorem 5 and Lemma 2, that except from the maximal sized case, the linearisation of the Reduced Thin Sandwich Equations is elliptic at this solution. However, the zero shift is not the only solution of the Thin Sandwich Equations induced on the initial hypersurface by this spacetime. In view of Corollary 2, equation (85) has non trivial solutions, the hypersurface projections of the spacetime Killing vectors. This includes the Killing vectors in the spacelike hypersurfaces, with $\mu = 0$. Although $2\epsilon - R > 0$, contrary to the assertion of Belasco and Ohanian[10], there exist non Killing vector solutions of this equation. For example the vector field $M^a = (\sin\psi, 0, 0)$ is a solution with $\mu = -\frac{a}{\dot{a}}\cos\psi$. $M^a{}_{|a} = -3\cos\psi$ is positive on one half of the manifold, and negative on the other, but if $M^a{}_{|a} = 0$, then also $\mu = 0$. In view of the second part of Theorem 2, this example also shows that generally the solutions of the Thin Sandwich Equations are not unique, and this nonuniqueness is not only because of the possible isometries of the spacelike hypersurfaces generated by the Killing vectors belonging to the three-metric in the initial data. This is also contrary to one might expect from the analysis of Wheeler[4] of the spherical cosmology solution in the small shift approximation.

We have seen in Theorem 11, that the space of solutions of the equation $M_{(a|b)} = \mu K_{ab}$ is finite dimensional and completely determined by the values of M^a and μ and their first derivatives at any point. It has been shown by Arms, Fisher, Marsden and Moncrief[8][9], that those (g_{ab}, K_{ab}) which satisfies the constraint equations constitute a smooth submanifold of the Hilbert space of all possible configurations (g_{ab}, K_{ab}), except in the nongeneric case when the spacetime induced by (g_{ab}, K_{ab}) admits Killing vectors. In these cases there are also solutions of the constraints in any neighbourhood of such (g_{ab}, K_{ab}), but instead of forming a submanifold, they have a quadratic singularity at the point (g_{ab}, K_{ab}) of the configuration space. These considerations show, that probably there exists $(g_{ab}, K_{ab}, \epsilon, S^a)$ close to the Robertson-Walker initial data, for which the equation $M_{(a|b)} = \mu K_{ab}$ has only the trivial solution on the initial hypersurface, and $2\epsilon - R$ is still positive. By Theorem 7, the Thin Sandwich Equations would have a unique solution in a neighbourhood of $(g_{ab}, \dot{g}_{ab}, \epsilon, S^a)$ generated by this $(g_{ab}, K_{ab}, \epsilon, S^a)$ and any choice of shift and positive lapse.

Conjecture 1 *There exists a neighbourhood of the initial data* $(g^{RW}_{ab}, \dot{g}^{RW}_{ab}, \epsilon_{RW}, S^a_{RW})$ *generated by the standard slicing of the spherical Robertson-Walker spacetime, in which those* $(g_{ab}, \dot{g}_{ab}, \epsilon, S^a)$ *for which the Thin Sandwich Equations are uniquely solvable form a dense open subset.*

It is possible to specify g_{ab} and $\dot{g}_{ab}$ on the initial hypersurface, such that no solution of the Thin Sandwich Equations (15) and (16) exists with positive lapse. Suppose that g_{ab} and ϵ is given such that $2\epsilon - R > 0$, on a compact connected three dimensional manifold $\mathcal{M}$ without boundary. Then from the energy constraint (15), if $N > 0$ then

$$(\gamma^a_a)^2 = N^2(2\epsilon - R) + \gamma_{ab}\gamma^{ab} > 0. \tag{108}$$

Using the expression (13) for γ_{ab} and dropping boundary terms,

$$\int_{\mathcal{M}} dv\ \gamma^a_a = \frac{1}{2}\int_{\mathcal{M}} dv\ g^{ab}\dot{g}_{ab}\ . \tag{109}$$

If $\dot{g}_{ab}$ is specified such that $\int_{\mathcal{M}} dv g^{ab}\dot{g}_{ab} = 0$, then γ^a_a must vanish at some point of the manifold, which is impossible if $N > 0$, because of the condition $2\epsilon - R > 0$. This means that if $2\epsilon - R > 0$, then

$$\int\limits_{\mathcal{M}} dv \; g^{ab}\dot{g}_{ab} \neq 0 \tag{110}$$

is a necessary condition to the existence of solutions with positive lapse of the Thin Sandwich Equations. Belasco and Ohanian[10] has given two examples to this, the first is $\dot{g}_{ab} = L_{(a|b)}$ for some vector field L^a, and the second is $g^{ab}\dot{g}_{ab} = 0$.

There is another counterexample given by Belasco and Ohanian[10], which is rather a restriction on the admissible energy flow densities. Let our three dimensional manifold $\mathcal{M}$ be compact without boundary. Suppose that it admits a Killing vector field X^a. The momentum constraint (6) tells us that the energy flow density is the divergence of the momentum tensor,

$$S^a = \pi^{ab}{}_{|b} \; . \tag{111}$$

Because of the symmetry of the momentum tensor and the $X_{(a|b)} = 0$ Killing-equation,

$$\int\limits_{\mathcal{M}} dv \; S^a X_a = \int\limits_{\mathcal{M}} dv \; \pi^{ab}{}_{|b} X_a = -\int\limits_{\mathcal{M}} dv \; \pi^{ab} X_{(a|b)} = 0. \tag{112}$$

This shows that if the initial manifold $\mathcal{M}$ is compact without boundary and admits a Killing vector X^a, then

$$\int\limits_{\mathcal{M}} dv \; S^a X_a = 0 \tag{113}$$

is a necessary condition on the energy flow density S^a in order to have a solution of the Thin Sandwich Equations.

7 Other variables

In this section we investigate a different approach towards the Thin Sandwich formulation of general relativity, introduced by Komar[11]. The new metric variable is conformally related to the spacelike metric. The conformal factor is the trace of the extrinsic curvature. This formalism leads to elliptic differential equations in more general cases than the usual formalism, but because of the complicated structure of the equations we cannot apply the Implicit Function theorem to get existence and uniqueness results. Komar has introduced the new variables using a canonical transformation[19]. It has been shown by Bergmann[20], that all canonical transformations involving the extrinsic curvature lead to elliptic differential equations under this less restrictive condition.

We define the momentum density tensor

$$p^{ab} = \sqrt{g}\pi^{ab} = \sqrt{g}\left(K^{ab} - K^c_c g^{ab}\right) , \tag{114}$$

where

$$g = \det(g_{ab}) \tag{115}$$

is the determinant of the spacelike metric. We denote the trace of the momentum by

$$p = p^{ab} g_{ab} \; , \tag{116}$$

and the covariant derivative associated to g_{ab} by D_a. The momentum and energy constraints in the vacuum case are

$$\mathcal{K}^a = -2\sqrt{g}D_b\left(\frac{1}{\sqrt{g}}p^{ab}\right) = 0, \tag{117}$$

and

$$\mathcal{K} = \frac{1}{\sqrt{g}}p^{ab}p_{ab} - \frac{1}{2\sqrt{g}}p^2 - \sqrt{g}R = 0. \tag{118}$$

The Hamiltonian is

$$H = \int\limits_{\mathcal{M}} dx^3 \left(N_a\mathcal{K}^a + N\mathcal{K}\right). \tag{119}$$

We introduce the new variables

$$\bar{g}_{ab} = \frac{p^2}{4g}g_{ab} \tag{120}$$

and

$$\bar{p}^{ab} = \frac{4g}{p^2}\left(p^{ab} - \frac{1}{2}pg^{ab}\right). \tag{121}$$

These can be obtained by a canonical transformation with the generator[19]

$$G = -\int\limits_{\mathcal{M}} dx^3\ g\left(\bar{p}^{ab}g_{ab}\right), \tag{122}$$

but we do not necessarily need to use this fact in the following calculations. The determinant of the new metric is

$$\bar{g} = \det(\bar{g}_{ab}) = \frac{p^6}{2^6g^2}\quad , \tag{123}$$

and the trace of the new canonically conjugate momentum tensor is

$$\bar{p} = \bar{p}^{ab}\bar{g}_{ab} = -\frac{1}{2}p\ . \tag{124}$$

Since we assume both g_{ab} and $\bar{g}_{ab}$ to be positive definite,

$$g = \frac{|\bar{p}|^3}{\sqrt{\bar{g}}}\quad . \tag{125}$$

The inverse relations to the old variables are

$$g_{ab} = \frac{|\bar{p}|}{\sqrt{\bar{g}}}\bar{g}_{ab}\ , \tag{126}$$

and

$$p^{ab} = \frac{\sqrt{\bar{g}}}{|\bar{p}|}\left(\bar{p}^{ab} - \bar{p}\bar{g}^{ab}\right). \tag{127}$$

Denoting the covariant derivative belonging to the new metric $\bar{g}_{ab}$ by $\bar{D}_a$, we have the following relation between the two kind of covariant derivatives of a one-form u_a,

$$D_au_b = \bar{D}_au_b - C^c_{ab}u_c\ , \tag{128}$$

where

$$C^c_{ab} = \frac{\sqrt{\bar{g}}}{2\bar{p}}\left(\delta^c_a\bar{D}_b\frac{\bar{p}}{\sqrt{\bar{g}}} + \delta^c_b\bar{D}_a\frac{\bar{p}}{\sqrt{\bar{g}}} - \bar{g}_{ab}\bar{g}^{cd}\bar{D}_d\frac{\bar{p}}{\sqrt{\bar{g}}}\right). \tag{129}$$

The relation between the two curvature scalars, R belonging to g_{ab}, and $\bar{R}$ belonging to $\bar{g}_{ab}$, is

$$R = \frac{\sqrt{g}}{|\bar{p}|}\bar{R} - 2\frac{\bar{g}}{\bar{p}^2}\bar{g}^{ab}\bar{D}_a\bar{D}_b\frac{|\bar{p}|}{\sqrt{\bar{g}}} + \frac{3}{2}\frac{\bar{g}^{\frac{3}{2}}}{|\bar{p}|^3}\bar{g}^{ab}\left(\bar{D}_a\frac{\bar{p}}{\sqrt{\bar{g}}}\right)\bar{D}_b\frac{\bar{p}}{\sqrt{\bar{g}}} \ . \tag{130}$$

Using these transformation formulas we get the following forms for the constraints in terms of the new variables

$$\mathcal{K}^a = -2\frac{\bar{g}}{|\bar{p}|}\bar{D}_b\left(\frac{1}{\sqrt{\bar{g}}}\bar{p}^{ab}\right) = 0, \tag{131}$$

and

$$\mathcal{K} = \frac{\bar{g}^{\frac{1}{4}}}{|\bar{p}|^{\frac{3}{2}}}\left[\bar{p}^{ab}\bar{p}_{ab} - \bar{p}^2 - \bar{p}^2\bar{R} + 2\sqrt{\bar{g}}\bar{p}\bar{g}^{ab}\bar{D}_a\bar{D}_b\frac{\bar{p}}{\sqrt{\bar{g}}} - \frac{3}{2}\bar{g}\bar{g}^{ab}\left(\bar{D}_a\frac{\bar{p}}{\sqrt{\bar{g}}}\right)\bar{D}_b\frac{\bar{p}}{\sqrt{\bar{g}}}\right] = 0. \tag{132}$$

The time derivative of the new metric variable is

$$\dot{\bar{g}}_{ab} = \left(\frac{p^2}{4g}g_{ab}\right)^{\cdot} = \frac{p}{2g}g_{ab}\left(\dot{p}^{cd}g_{cd} + p^{cd}\dot{g}_{cd}\right) - \frac{p^2}{4g}g_{ab}g^{cd}\dot{g}_{cd} + \frac{p^2}{4g}\dot{g}_{ab} \ . \tag{133}$$

Using the dynamical equations

$$\dot{g}_{ab} = \frac{2}{\sqrt{g}}N\left(p_{ab} - \frac{1}{2}pg_{ab}\right) + 2D_{(a}N_{b)}, \tag{134}$$

$$\begin{aligned}\dot{p}^{ab} = &-N\sqrt{g}\left(R^{ab} - \frac{1}{2}Rg^{ab}\right) + \frac{1}{2\sqrt{g}}Ng^{ab}\left(p^{cd}p_{cd} - \frac{1}{2}p^2\right) \\ &-\frac{2}{\sqrt{g}}N\left(p^{ac}p_c^b - \frac{1}{2}pp^{ab}\right) + \sqrt{g}\left(D^aD^bN - g^{ab}D^cD_cN\right) \\ &+\sqrt{g}D_c\left(\frac{1}{\sqrt{g}}N^cp^{ab}\right) - 2p^{c(a}D_cN^{b)} \ ,\end{aligned} \tag{135}$$

and eliminating the curvature term using the energy constraint (118), we get for the time derivative

$$\dot{\bar{g}}_{ab} = \frac{p^2}{2g}D_{(a}N_{b)} + \frac{p^2}{2g^{\frac{3}{2}}}Np_{ab} + \frac{p^2}{g^{\frac{3}{2}}}Ng_{ab}\left(p^{cd}p_{cd} - \frac{1}{2}p^2\right) + \frac{p}{2\sqrt{g}}g_{ab}N^cD_c\frac{p}{\sqrt{g}} - \frac{p}{\sqrt{g}}g_{ab}D^cD_cN. \tag{136}$$

There are now two one-forms associated to the shift vector N^a, the first is $N_a = g_{ab}N^b$, and the second is

$$\bar{N}_a = \bar{g}_{ab}N^b = \frac{\sqrt{\bar{g}}}{|\bar{p}|}N_a \ . \tag{137}$$

For ease of notation we also use $\bar{N}^a = N^a$. In place of the lapse function N we introduce a new variable,

$$\lambda = \frac{\bar{g}^{\frac{1}{4}}}{|\bar{p}|^{\frac{3}{2}}}\bar{p}N \ . \tag{138}$$

Using the transformation rules (124-128), we can express the time derivative of the new metric in terms of the new variables,

$$\begin{aligned}\dot{\bar{g}}_{ab} = &2\bar{D}_{(a}\bar{N}_{b)} + 2\bar{g}_{ab}\bar{D}^c\bar{D}_c\lambda + 3\frac{\sqrt{\bar{g}}}{\bar{p}}\bar{g}_{ab}\bar{g}^{cd}\left(\bar{D}_c\lambda\right)\bar{D}_d\frac{\bar{p}}{\sqrt{\bar{g}}} \\ &+\lambda\left[\frac{2}{\bar{p}}\bar{p}_{ab} - \frac{2}{\bar{p}^2}\bar{g}_{ab}\bar{p}^{cd}\bar{p}_{cd} + \frac{\sqrt{\bar{g}}}{\bar{p}}\bar{g}_{ab}\bar{D}^c\bar{D}_c\frac{\bar{p}}{\sqrt{\bar{g}}}\right] \ .\end{aligned} \tag{139}$$

This is the formula where the differences between the new and the usual formalisms are most apparent. To the derivation of equation (139) we had to use the dynamical field equations, while the corresponding formula with the usual variables

$$\dot{g}_{ab} = 2D_{(a}N_{b)} + \frac{2}{\sqrt{g}} N p_{ab} - \frac{1}{\sqrt{g}} N g_{ab} p_c^c \tag{140}$$

follows simply from the definition of the extrinsic curvature. The other major difference is that $\bar{p}_{ab}$ cannot be expressed from equation (139) as we shall see later.

Using the fact that the new variables emerge from a canonical transformation, there is another way of getting the formula (139) for the time derivative of the new metric variable. The Hamiltonian (119) takes the form

$$\begin{aligned} H = \int_{\mathcal{M}} dx^3 \Bigg\{ \lambda \left[\frac{1}{\bar{p}} \bar{p}^{ab} \bar{p}_{ab} - \bar{p} - \bar{p}\bar{R} + 2\sqrt{\bar{g}} \bar{D}^a \bar{D}_a \frac{\bar{p}}{\sqrt{\bar{g}}} - \frac{3}{2\bar{p}} \bar{g} \bar{g}^{ab} \left(\bar{D}_a \frac{\bar{p}}{\sqrt{\bar{g}}} \right) \bar{D}_b \frac{\bar{p}}{\sqrt{\bar{g}}} \right] \\ -2\sqrt{\bar{g}} \bar{N}_a \bar{D}_b \left(\frac{1}{\sqrt{\bar{g}}} \bar{p}^{ab} \right) \Bigg\}. \end{aligned} \tag{141}$$

Since $\bar{p}^{ab}$ is the canonically conjugate variable to $\bar{g}_{ab}$, variation of this Hamiltonian with respect to $\bar{p}^{ab}$ gives the same result for $\dot{\bar{g}}_{ab}$ as in (139), apart from terms vanishing by the energy constraint. We do not have to consider the $\bar{p}^{ab}$ dependence of λ, since at the variation it would only yield an expression proportional to the energy constraint.

We introduce the notation

$$l_{ab} = \dot{\bar{g}}_{ab} - 2\bar{D}_{(a}\bar{N}_{b)} \ . \tag{142}$$

The trace of equation (139) is

$$l = \bar{g}^{ab} l_{ab} = 6\bar{D}^a \bar{D}_a \lambda + 9 \frac{\sqrt{\bar{g}}}{\bar{p}} \bar{g}^{ab} \left(\bar{D}_a \lambda \right) \bar{D}_b \frac{\bar{p}}{\sqrt{\bar{g}}} + \lambda \left[2 - \frac{6}{\bar{p}^2} \bar{p}^{ab} \bar{p}_{ab} + 3 \frac{\sqrt{\bar{g}}}{\bar{p}} \bar{D}^a \bar{D}_a \frac{\bar{p}}{\sqrt{\bar{g}}} \right] . \tag{143}$$

The trace-free part of (139) gives an equation, from which the trace-free part of p^{ab} can be expressed in terms of the trace,

$$\bar{p}^{ab} = \bar{p} \left(\frac{1}{2\lambda} l^{ab} - \frac{1}{6\lambda} l \bar{g}^{ab} + \frac{1}{3} \bar{g}^{ab} \right) . \tag{144}$$

However, in contrast with the formalism with the usual variables, the trace of $\bar{p}$ cannot be expressed algebraically from the equation. Instead, we get one more differential equation, and we have to treat $\bar{p}$ as a fifth unknown function together with the shift and lapse functions. Multiplying equation (144) by itself,

$$\frac{1}{\bar{p}^2} \bar{p}^{ab} \bar{p}_{ab} = \frac{1}{4\lambda^2} l^{ab} l_{ab} - \frac{1}{12\lambda^2} l^2 + \frac{1}{3} \ . \tag{145}$$

Substituting this into the form (132) of the energy constraint, we can rearrange to express the only secondary derivative term, the Laplacian of $\bar{p}/\sqrt{\bar{g}}$,

$$\bar{D}^a \bar{D}_a \frac{\bar{p}}{\sqrt{\bar{g}}} = \frac{\bar{p}}{\sqrt{\bar{g}}} \left[\frac{1}{2} \bar{R} + \frac{1}{3} - \frac{1}{8\lambda^2} l^{ab} l_{ab} + \frac{l^2}{24\lambda^2} + \frac{3}{4} \frac{\bar{g}}{\bar{p}^2} \bar{g}^{ab} \left(\bar{D}_a \frac{\bar{p}}{\sqrt{\bar{g}}} \right) \bar{D}_b \frac{\bar{p}}{\sqrt{\bar{g}}} \right] . \tag{146}$$

Substituting this and the expression (145) into the equation (143), we can express the Laplacian of λ,

$$\bar{D}^a \bar{D}_a \lambda = \frac{l}{6} - \frac{\lambda}{6} - \frac{\lambda}{4} \bar{R} + \frac{5}{16\lambda} l^{ab} l_{ab} - \frac{5}{48\lambda} l^2 - \frac{3}{2} \frac{\sqrt{\bar{g}}}{\bar{p}} \bar{g}^{ab} \left(\bar{D}_a \lambda \right) \bar{D}_b \frac{\bar{p}}{\sqrt{\bar{g}}} - \frac{3}{8} \frac{\bar{g}}{\bar{p}^2} \bar{g}^{ab} \left(\bar{D}_a \frac{\bar{p}}{\sqrt{\bar{g}}} \right) \bar{D}_b \frac{\bar{p}}{\sqrt{\bar{g}}} . \tag{147}$$

Substituting the form (144) of $\bar{p}^{ab}$ into the momentum constraint (117), and using the formula

$$\bar{D}^b\bar{D}_a\bar{N}_b - \bar{D}_a\bar{D}^b\bar{N}_b = \bar{R}_{ab}\bar{N}^b \tag{148}$$

for commuting the second derivatives, we can rearrange to express the second derivative terms of $\bar{N}^a$,

$$\begin{aligned}\bar{D}^b\bar{D}_b\bar{N}_a + \frac{1}{3}\bar{D}_a\bar{D}_b\bar{N}^b = &\left(l_{ab} - \frac{1}{3}l\bar{g}_{ab}\right)\left(\frac{\sqrt{\bar{g}}}{\bar{p}}\bar{D}^b\frac{\bar{p}}{\sqrt{\bar{g}}} - \frac{1}{\lambda}\bar{D}^b\lambda\right)\\ &+\frac{2}{3}\lambda\frac{\sqrt{\bar{g}}}{\bar{p}}\bar{D}_a\frac{\bar{p}}{\sqrt{\bar{g}}} + \bar{D}^b\dot{\bar{g}}_{ab} - \frac{1}{3}\bar{g}^{bc}\bar{D}_a\dot{\bar{g}}_{bc} - \bar{R}_{ab}\bar{N}^b.\end{aligned} \tag{149}$$

Suppose that $\bar{g}_{ab}$ and $\dot{\bar{g}}_{ab}$ is given on the initial three dimensional manifold $\mathcal{M}$. Then after the substitution of l_{ab} and $l = \bar{g}^{ab}l_{ab}$ from (142), the differential equations (146), (147) and (149), form a quasilinear system of five partial differential equations for the determination of the five variables

$$\left(\frac{\bar{p}}{\sqrt{\bar{g}}}, \lambda, \bar{N}^1, \bar{N}^2, \bar{N}^3\right). \tag{150}$$

All the second derivative terms are on the right hand side of these equations, so the symbol of the leading second order terms is

$$f = \begin{pmatrix} \xi^a\xi_a & 0 & 0 & 0 & 0 \\ 0 & \xi^a\xi_a & 0 & 0 & 0 \\ 0 & 0 & \xi^a\xi_a + \frac{1}{3}\xi^1\xi_1 & \frac{1}{3}\xi^1\xi_2 & \frac{1}{3}\xi^1\xi_3 \\ 0 & 0 & \frac{1}{3}\xi^2\xi_1 & \xi^a\xi_a + \frac{1}{3}\xi^2\xi_2 & \frac{1}{3}\xi^2\xi_3 \\ 0 & 0 & \frac{1}{3}\xi^3\xi_1 & \frac{1}{3}\xi^3\xi_2 & \xi^a\xi_a + \frac{1}{3}\xi^3\xi_3 \end{pmatrix}. \tag{151}$$

The determinant is

$$\det(f) = \frac{4}{3}\left(\xi^a\xi_a\right)^5. \tag{152}$$

Because of the positive definiteness of the spacelike metric this is not zero if $\xi^a \neq 0$. We have proved that this system of differential equations is elliptic, and the only restriction we had to make is that $\bar{p}$ cannot be zero anywhere on the manifold $\mathcal{M}$.

As with the usual variables, we can define a mapping

$$F^A\left(\left(\bar{g}_{ab}, \dot{\bar{g}}_{ab}\right), \left(\frac{\bar{p}}{\sqrt{\bar{g}}}, \lambda, \bar{N}^a\right)\right), \quad A = 1\ldots5, \tag{153}$$

which maps to zero when the equations (146), (147) and (149) are satisfied. However the partial derivative operator with respect to the variables $\left(\frac{\bar{p}}{\sqrt{\bar{g}}}, \lambda, \bar{N}^a\right)$ is not self adjoint. It does not seem to exist any obvious way of determining when the kernel of the partial derivative and the kernel of its adjoint vanishes. Probably, similarly to the formalism with the usual variables, this condition is much more restrictive than the condition of ellipticity.

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