On the mathematical formulation of renormalization in quantum field theory

*Class.Quant.Grav.***41**(2024)125009 and **arXiv:2502.16319**

András LÁSZLÓ

laszlo.andras@wigner.hun-ren.hu

HUN-REN Wigner RCP, Budapest

joint work with Zsigmond Tarcsay and Jobst Ziebell

Analysis Seminar ELTE, 7 April 2025

Outline

I. On Wilsonian RG flows of Feynman measures (Euclidean signature, flat spacetime, bosonic fields):

- $\exists$ of UV limit Feynman measure
- $\exists$ of UV limit interaction potential

[`arXiv:2502.16319`]

II. On Wilsonian RG flow of correlators (arbitrary signature):

- On manifolds: nice topological vector space behavior
- On flat spacetime for bosonic fields: $\exists$ of UV limit

[*Class.Quant.Grav.* **39**(2022)185004]

Part 0:

Notations, introduction

Distribution theory notations

Will consider only scalar valued fields for simplicity, see paper for vector valued case.

Will consider generic spacetime manifold, as well as flat spacetime case.

Usual distribution theory spaces:

only meaningful on flat spacetime

$$\underbrace{\mathcal{E}}_{\text{smooth fields}} \supset \underbrace{\mathcal{S}}_{\text{Schwartz fields}} \supset \underbrace{\mathcal{D}}_{\text{test fields}}$$

only meaningful on flat spacetime

$$\underbrace{\mathcal{E}'}_{\text{compactly supp distributions}} \subset \underbrace{\mathcal{S}'}_{\text{Schwartz distributions}} \subset \underbrace{\mathcal{D}'}_{\text{all distributions}}$$

Measure theory notations

(X, Σ, μ) is measure space iff:

- X is a set. [We only deal with $X = \mathcal{E}, \mathcal{S}, \mathcal{D}, \mathcal{E}', \mathcal{S}', \mathcal{D}'$.]
- Σ is a sigma-algebra of subsets of X.
  Usually, X carries topology $\rightarrow$ we take the Borel sigma-algebra. [Above X are Souslin.]
- $\mu : \Sigma \rightarrow \mathbb{R}^+_0 \cup \{\infty\}$ is a sigma-additive measure. Will only deal with finite measures.

Pushforward (or marginal) measure:

- Let (X, Σ, μ) be measure space and (Y, Δ) measurable space.
  Let $C : X \rightarrow Y$ be a measurable mapping.
- Pushforward (or marginal) measure $C_* \mu$ on Y is:
  for all $B \in \Delta$ one defines $(C_* \mu)(B) := \mu\big(\overset{-1}{C}(B)\big)$.
  One has $\int\limits_{\varphi \in \mathrm{Ran}(C)} f(\varphi)\,\mathrm{d}\big(C_*\mu\big)(\varphi) = \int\limits_{\phi \in X} f(C(\phi))\,\mathrm{d}\mu(\phi)$ for $f : \mathrm{Ran}(C) \rightarrow \mathbb{R} \cup \{\pm\infty\}$.

Fourier transform:

- Let μ be a finite measure e.g. on $X = \mathcal{E}, \mathcal{S}, \mathcal{D}, \mathcal{E}', \mathcal{S}', \mathcal{D}'$. Then $Z : X' \rightarrow \mathbb{C}$,
  $Z(j) := \int\limits_{\phi \in X} \mathrm{e}^{\mathrm{i}(j|\phi)}\,\mathrm{d}\mu(\phi)$ is its Fourier transform (partition function in QFT).

Ideology of Euclidean Wilsonian renormalization

- Take an Euclidean action $S = T + V$, with kinetic + potential term splitting.
  Say, $T(\varphi) = \int \varphi\,(-\Delta + m^2)\varphi$, and $V(\varphi) = g \int \varphi^4$.

- Then T, i.e. $(-\Delta + m^2)$ has fund.solution $K(\cdot,\cdot)$ which is positive semidefinite:
  - $(-\Delta + m^2)_x\, K(x, y) = \delta_y(x)$,
  - for all $j \in \mathcal{D}$ sources: $(K|j \otimes j) \geq 0$.

- Bochner-Minlos theorem: because of
  - continuity and positive definiteness of $\mathcal{D} \to \mathbb{C},\ j \mapsto \mathrm{e}^{-(K|j\otimes j)}$,
  - nuclear property of the space $\mathcal{D}$,

  $\exists|$ measure γ_T on $\mathcal{D}'$, whose Fourier transform is $j \mapsto \mathrm{e}^{-(K|j\otimes j)}$.
  It is the Feynman measure for free theory: $\int\limits_{\phi\in\mathcal{D}'} (\ldots)\,\mathrm{d}\gamma_T(\phi) = \int\limits_{\phi\in\mathcal{D}'} (\ldots)\,\mathrm{e}^{-T(\phi)}\,\text{“}\mathrm{d}\phi\text{”}$.

- Tempting definition for Feynman measure of interacting theory:

$$\int\limits_{\phi\in\mathcal{D}'} (\ldots)\,\mathrm{e}^{-V(\phi)}\,\mathrm{d}\gamma_T(\phi) \qquad \left[= \int\limits_{\phi\in\mathcal{D}'} (\ldots)\,\underbrace{\mathrm{e}^{-(T(\phi)+V(\phi))}}_{=\mathrm{e}^{-S(\phi)}}\,\text{“}\mathrm{d}\phi\text{”}\right]$$

- Problem: the interacting Feynman measure $\mu := \mathrm{e}^{-V} \cdot \gamma_T$ is undefined,

$$\int_{\phi\in\mathcal{D}'} (\dots) \underbrace{\mathrm{d}\mu(\phi)}_{\text{wannabe Feynman measure}} := \int_{\phi\in\mathcal{D}'} (\dots) \underbrace{\mathrm{e}^{-V(\phi)}}_{\text{lives on function sense fields}} \underbrace{\mathrm{d}\gamma_T(\phi)}_{\text{lives on distribution sense fields}}$$

  Because V is spacetime integral of pointwise product of fields, e.g. $V(\varphi) = \int \varphi^4$.
  How to bring e^{-V} and γ_T to common grounds?

- Physicist workaround: Wilsonian regularization.
  Take a continuous linear mapping C: (distributional fields) $\to$ (smooth fields).
  Take the pushforward Gaussian measure $C_*\gamma_T$, that lives on $\mathrm{Ran}(C)$.
  Those are functions, so safe to integrate e^{-V} there:

$$\int_{\varphi\in\mathrm{Ran}(C)} (\dots)\ \mathrm{e}^{-V(\varphi)}\ \mathrm{d}\big(C_*\gamma_T\big)(\varphi) \qquad \left[= \int_{\varphi\in\mathrm{Ran}(C)} (\dots)\ \mathrm{e}^{-(T_C(\varphi)+V(\varphi))}\ \text{“}\mathrm{d}\varphi\text{”} \right]$$

  a space of UV regularized fields

  Schwartz kernel theorem: C is a partial evaluation of some smooth kernel $\kappa(x, y)$.
  If we require C transationally invariant: it is a convolution by some test function $\eta \in \mathcal{D}$.

- What do we do with the C-dependence? What is the physics / mathematics behind?

- Take a family V_C $(C \in \{$coarse-grainings$\})$ of interaction terms $\leftrightarrow \mu_C := \mathrm{e}^{-V_C} \cdot C_* \gamma_T$. We say that it is a Wilsonian renormalization group (RG) flow iff:

  $\forall$ coarse-grainings C, C', C'' with $C'' = C' C$:

  $$\mu_{C''} \;=\; C'_* \, \mu_C$$

- If $\mathcal{G}_C = (\mathcal{G}_C^{(0)}, \mathcal{G}_C^{(1)}, \mathcal{G}_C^{(2)}, \ldots)$ are the moments of μ_C, then

  $\forall$ coarse-grainings C, C', C'' with $C'' = C' C$:

  $$\mathcal{G}_{C''}^{(n)} \;=\; \otimes^n C' \, \mathcal{G}_C^{(n)} \qquad \text{(for all } n = 0, 1, 2, \ldots)$$

[Valid also in Lorentz signature and on manifolds, for formal moments (correlators).]

Part I:

On Wilsonian RG flows of Feynman measures (Euclidean signature, flat spacetime, bosonic fields)

[arXiv:2502.16319]

Wilsonian renormalization in Euclidean signature

We study Euclidean Feynman measures on flat spacetime, for bosonic fields.
[We work on $\mathcal{S}$ and $\mathcal{S}'$, because we can — and also a useful theorem holds there.]

Coarse-grainings: convolution $C_\eta = \eta \star (\cdot)$ by some $\eta \in \mathcal{S}$ Schwartz functions.

Take a family V_C ($C \in \{$coarse-grainings$\}$) of interaction terms $\leftrightarrow \mu_C := \mathrm{e}^{-V_C} \cdot C_* \gamma_T$.
Let it be a Wilsonian RG flow:

$\forall$ coarse-grainings C, C', C'' with $C'' = C' C$:

$$\mu_{C''} \quad = \quad C'_* \, \mu_C$$

Space of Wilsonian RG flow of Feynman measures is nonempty:

For any Feynman measure μ on $\mathcal{S}'$, the family

$$\mu_C \quad := \quad C_* \, \mu \qquad (*)$$

is a Wilsonian RG flow.

Theorem[A.László, Z.Tarcsay, J.Ziebell **`arXiv:2502.16319`**]:

1. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form $(*)$. $\leftarrow$ UV limit
2. There exists some measurable potential $V : \mathcal{S}' \to \mathbb{R} \cup \{\pm\infty\}$, such that $\mu = \mathrm{e}^{-V} \cdot \gamma_T$.
3. For $C = C_\eta$ with $F(\eta) > 0$, one has $V_C(C\,\phi) = V(\phi)$ for γ_T-a.e. $\phi \in \mathcal{S}'$.
4. If $V_C : \mathrm{Ran}(C) \to \mathbb{R} \cup \{\pm\infty\}$ bounded from below, then V is ess.bounded from below.

Sketch of proofs.

1. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form $\mu_C = C_* \, \mu$.

- We prove it for Fourier transforms (Z), and then use Bochner-Minlos.
  We use that $\mathcal{S} \star \mathcal{S} = \mathcal{S}$, moreover
  that for all $K \subset \mathcal{S}$ compact $\exists \, \chi \in \mathcal{S}$ and $L \subset \mathcal{S}$ compact such that $K = \chi \star L$.

2. There exists some measurable potential $V : \, \mathcal{S}' \to \mathbb{R} \cup \{\pm\infty\}$, such that $\mu = \mathrm{e}^{-V} \cdot \gamma_T$.

- We apply Radon-Nikodym theorem, the fact that $\mathcal{S}'$ is Souslin space,
  and that for $\eta \in \mathcal{S}$ with $F(\eta) > 0$ the coarse-graining $C_\eta := \eta \star (\cdot)$ is injective.

3. For $C = C_\eta$ with $F(\eta) > 0$, one has $V_C(C \, \phi) = V(\phi)$ for γ_T-a.e. $\phi \in \mathcal{S}'$.

- Fundamental formula of integration variable substitution vs pusforward, Souslin-ness of $\mathcal{S}'$,
  injectivity of coarse-graining $C_\eta := \eta \star (\cdot)$ with $\eta \in \mathcal{S}, \; F(\eta) > 0$.

4. If $V_C : \, \mathrm{Ran}(C) \to \mathbb{R} \cup \{\pm\infty\}$ bounded from below, then V is ess.bounded from below.
- Trivial from 3.

Part II:

On Wilsonian RG flow of correlators (arbitrary signature)

[*Class.Quant.Grav.***39**(2022)185004]

Wilsonian RG flow of correlators, rigorously

Definition:

A continuous linear operator C : (distributional fields) $\to$ (smooth fields)
is coarse-graining iff properly supported and injective on compactly supported distributions.
[Info: on $\mathbb{R}^N$, convolution by test functions are the translationally invariant coarse-grainings.]

A family of smooth correlators $\mathcal{G}_C$ ($C \in$ coarse-grainings) is Wilsonian RG flow iff
$\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$:
$\mathcal{G}^{(n)}_{C''} = \otimes^n C'\,\mathcal{G}^{(n)}_C$ holds $(n = 0, 1, 2, ...)$. $\longleftarrow$ rigorous RGE in any signature

Space of Wilsonian RG flows is nonempty:
For any distributional correlator G, the family

$$\mathcal{G}^{(n)}_C \quad := \quad \otimes^n C\, G^{(n)} \qquad (*)$$

is a Wilsonian RG flow.

Space of Wilsonian RG flows naturally topologized by coarse-graining-wise Tychonoff.

Theorem[A.Lászó, Z.Tarcsay *Class.Quant.Grav.***41**(2024)125009]:
1. On manifolds it is "quite nice" topological vector space, similar to distributions.
2. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form of $(*)$.
$\downarrow$
UV limit.

Sketch of proofs.

1. On manifolds it is “quite nice” topological vector space, similar to distributions.
[It is Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.]

- Coarse-grainings have a natural ordering of being less UV than an other: $C'' \preceq C$ iff $C''=C$ or $\exists\, C'$: $C''=C'\,C$.
- With this, the space of Wilsonian RG flows is seen to be projective limit of copies of $\mathcal{E}_n$.
- Check known properties of $\mathcal{E}_n$, some of them are preserved by projective limit.

2. On flat spacetime for bosonic fields, all Wilsonian RG flows are $\mathcal{G}^{(n)}_C = \otimes^n C\, G^{(n)}$.

- On flat spacetime, convolution ops by test functions $C_\eta := \eta \star (\cdot)$ exist and commute.
- Due to RGE, commutativity of convolution ops, and polarization formula for n-forms, for bosonic fields $\mathcal{G}^{(n)}_{C_\eta}$ is n-order homogeneous polynomial in η.
  That is, $\exists|\ \mathcal{G}^{(n)}_{\eta_1,\dots,\eta_n}$ symmetric n-linear map in $\eta_1,\dots,\eta_n$, such that $\mathcal{G}^{(n)}_{C_\eta} = \mathcal{G}^{(n)}_{\eta,\dots,\eta}$.
- Due to RGE, commutativity of convolution ops, and a Banach-Steinhaus thm variant, $\mathcal{G}^{(n)}_{\eta^t_1,\dots,\eta^t_n}\Big|_0$ extends to an n-variate distribution, it will do the job as $(G^{(n)}\,|\,\eta_1\otimes\dots\otimes\eta_n)$.

[A Banach-Steinhaus theorem variant (the key lemma – A.László, Z.Tarcsay):
If a sequence of n-variate distributions pointwise converge on $\otimes^n\mathcal{D}$, then also on full $\mathcal{D}_n$.]

Summary

- Defined Wilsonian RG flow of Feynman measures,
  for Euclidean signature field theories on $\mathbb{R}^N$.
  Theorem: they originate from a measure on distributions.
  Absolute continuity preserved in UV.
  Lower bound of interaction term preserved in UV.

- Wilsonian RG flow of correlators can be defined
  in any signature and on manifolds.
  Theorem: they have "nice" properties.
  On flat spacetime, for bosonic fields,
  they originate from a distributional correlator (UV limit).

  Maybe generically true? (on manifolds, in any signature)

Backup slides

Appendix:

On Wilsonian regularized Feynman functional integral formulation

The classical field theory scene

$\mathcal{M}$ a smooth orientable oriented manifold (wannabe spacetime, but no metric, yet).

$V(\mathcal{M})$ a vector bundle over it (its smooth sections are matter fields + metric if dynamical).

Field configurations:

$$\underbrace{(v,\nabla)}_{=:\ \psi} \in \underbrace{\Gamma\Big(V(\mathcal{M})\times_{\mathcal{M}} \mathrm{CovDeriv}\big(V(\mathcal{M})\big)\Big)}_{=:\ \boldsymbol{\mathcal{E}}}$$

Real topological affine space with the $\mathcal{E}$ smooth function topology.

Field variations:

$$\underbrace{(\delta v,\delta C)}_{=:\ \delta\psi} \in \underbrace{\Gamma\Big(V(\mathcal{M})\times_{\mathcal{M}} T^{*}(\mathcal{M})\otimes V(\mathcal{M})\otimes V^{*}(\mathcal{M})\Big)}_{=:\ \mathcal{E}}$$

Real topological vector space with the $\mathcal{E}$ smooth function topology.

Test field variations: $\delta\psi_{T} \in \mathcal{D}$, compactly supported ones from $\mathcal{E}$ with $\mathcal{D}$ test funct. top.

Informal Feynman functional integral in Lorentz signature

Fix a reference field $\psi_0 \in \boldsymbol{\mathcal{E}}$ for bringing the problem from $\boldsymbol{\mathcal{E}}$ to $\mathcal{E}$, and take $J_1, \ldots, J_n \in \mathcal{E}'$. Then, $\psi \mapsto (J_1|\psi{-}\psi_0) \cdot \ldots \cdot (J_n|\psi{-}\psi_0)$ defines a $\boldsymbol{\mathcal{E}} \to \mathbb{R}$ polynomial observable.

Feynman type quantum vacuum expectation value of this is postulated as:

$$\int_{\psi\in\boldsymbol{\mathcal{E}}} (J_1|\psi{-}\psi_0) \cdot \ldots \cdot (J_n|\psi{-}\psi_0) \;\; \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)} \, \mathrm{d}\lambda(\psi) \Bigg/ \int_{\psi\in\boldsymbol{\mathcal{E}}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)} \, \mathrm{d}\lambda(\psi)$$

Partition function often invoked to book-keep these (formal Fourier transform of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\lambda$):

$$Z_{\psi_0} : \quad \mathcal{E}' \longrightarrow \mathbb{C}, \quad J \longmapsto Z_{\psi_0}(J) := \int_{\psi\in\boldsymbol{\mathcal{E}}} \mathrm{e}^{\mathrm{i}\,(J|\psi-\psi_0)} \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)} \, \mathrm{d}\lambda(\psi),$$

and from this one can define

$$G^{(n)}_{\psi_0} \quad := \quad \left((-\mathrm{i})^n \, \frac{1}{Z_{\psi_0}(J)} \, \partial^{(n)}_J Z_{\psi_0}(J) \right)\Bigg|_{J=0}$$

n-field correlator, and their collection $G_{\psi_0} := (G^{(0)}_{\psi_0}, G^{(1)}_{\psi_0}, \ldots, G^{(n)}_{\psi_0}, \ldots) \in \bigoplus_{n\in\mathbb{N}_0} \overset{n}{\otimes} \mathcal{E}$.

Above quantum expectation value expressable via distribution pairing: $\big(J_1 \otimes \ldots \otimes J_n \,\big|\, G^{(n)}_{\psi_0}\big)$.

Well known problems:

- No "Lebesgue" measure λ in infinite dimensions.
- Neither $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$ is meaningful. (Can be repaired to some extent in Euclidean signature.)
- Neither the Fourier transform of this undefined measure is meaningful.

Rules in informal QFT:

- as if λ existed as *translation invariant* (Lebesgue) measure,
- as if $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$ existed as *finite measure*, with *finite moments* and *analytic Fourier transform*.

Textbook "theorem": because of above rules, one has

$Z:\,\mathcal{E}'\to\mathbb{C}$ is Fourier transform of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$ "$\Longleftrightarrow$" it satisfies master-Dyson-Schwinger eq

$$\Big(\,\mathbf{E}\big((-\mathrm{i})\partial_J+\psi_0\big)\;+\;\hbar\,J\,\Big)Z(J)\quad=\quad 0\quad(\forall J\in\mathcal{E}')$$

where $E(\psi):=DS(\psi)$ is the Euler-Lagrange functional at $\psi\in\boldsymbol{\mathcal{E}}$.

Does this informal PDE have a meaning? [Yes, on the correlators $G=\big(G^{(0)},G^{(1)},\ldots\big)$.]

Rigorous definition of Euler-Lagrange functional

- Let a Lagrange form be given, which is

$$\mathrm{L}: \; V(\mathcal{M}) \;\oplus\; T^*(\mathcal{M}) \otimes V(\mathcal{M}) \;\oplus\; T^*(\mathcal{M}) \wedge T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M}) \;\longrightarrow\; \overset{\dim(\mathcal{M})}{\wedge} T^*(\mathcal{M})$$

pointwise bundle homomorphism.

- Lagrangian expression:

$$\Gamma\big(V(\mathcal{M}) \times_{\mathcal{M}} \mathrm{CovDeriv}\big(V(\mathcal{M})\big)\big) \;\longrightarrow\; \Gamma\big(\overset{\dim(\mathcal{M})}{\wedge} T^*(\mathcal{M})\big), \quad (v, \nabla) \;\longmapsto\; \mathrm{L}(v, \nabla v, F(\nabla))$$

where $F(\nabla)$ is the curvature tensor.

- Action functional:

$$S: \quad \underbrace{\Gamma\big(V(\mathcal{M}) \times_{\mathcal{M}} \mathrm{CovDeriv}(V(\mathcal{M}))\big)}_{=:\; \boldsymbol{\mathcal{E}}} \;\longrightarrow\; \mathrm{Meas}(\mathcal{M}, \mathbb{R}), \quad \underbrace{(v, \nabla)}_{=:\; \psi} \;\longmapsto\; \Big(\mathcal{K} \mapsto S_{\mathcal{K}}(v, \nabla)\Big)$$

where $S_{\mathcal{K}}(v, \nabla) := \int_{\mathcal{K}} \mathrm{L}(v, \nabla v, F(\nabla))$ for all $\mathcal{K} \subset \mathcal{M}$ compact.

Action functional $S : \boldsymbol{\mathcal{E}} \to \mathrm{Meas}(\mathcal{M}, \mathbb{R})$ Fréchet differentiable, its Fréchet derivative

$$DS : \quad \boldsymbol{\mathcal{E}} \times \mathcal{E} \longrightarrow \mathrm{Meas}(\mathcal{M}, \mathbb{R}), \quad (\psi, \delta\psi) \longmapsto \Big(\mathcal{K} \mapsto \big(DS_{\mathcal{K}}(\psi) \,\big|\, \delta\psi\big)\Big)$$

is the usual Euler-Lagrange integral on $\mathcal{K}$ $+$ usual boundary integral on $\partial\mathcal{K}$.
Jointly continuous in its variables, linear in second variable.

Euler-Lagrange functional:
We restrict DS from $\boldsymbol{\mathcal{E}} \times \mathcal{E}$ to $\boldsymbol{\mathcal{E}} \times \mathcal{D}$, to make the EL integral over full $\mathcal{M}$ finite.

$$E : \quad \boldsymbol{\mathcal{E}} \times \mathcal{D} \longrightarrow \mathbb{R}, \quad \big(\psi,\, \delta\psi_T\big) \longmapsto \big(E(\psi) \,\big|\, \delta\psi_T\big) := \big(DS_{\mathcal{M}}(\psi) \,\big|\, \delta\psi_T\big)$$

Bulk Euler-Lagrange integral remains, no boundary term. Meaningful on full $\mathcal{M}$, real valued.
Jointly sequentially continuous, linear in second variable. (Also, $E : \boldsymbol{\mathcal{E}} \to \mathcal{D}'$ continuous.)

Classical field equation is

$$\psi \in \boldsymbol{\mathcal{E}}\ ? \qquad \forall\, \delta\psi_T \in \mathcal{D} :\ \big(E(\psi) \,\big|\, \delta\psi_T\big) = 0.$$

Observables are the $O : \boldsymbol{\mathcal{E}} \to \mathbb{R}$ continuous maps.

Rigorous definition of master Dyson-Schwinger equation

- Want to rephrase informal MDS operator on Z to n-field correlators $G=\big(G^{(0)},G^{(1)},\ldots\big)$.
  These sit in the tensor algebra $\mathcal{T}(\mathcal{E}):=\bigoplus_{n\in\mathbb{N}_0}\hat{\otimes}_\pi^n\,\mathcal{E}$ of field variations.
  More precisely, they sit in a graded-symmetrized subspace, e.g. $\bigvee(\mathcal{E})$ or $\bigwedge(\mathcal{E})$ of $\mathcal{T}(\mathcal{E})$.
  Naturally topologized: with Tychonoff topology, similar to $\mathcal{E}$, i.e. nuclear Fréchet.

- Algebraic tensor algebra $\mathcal{T}_a(\mathcal{E}'):=\bigoplus_{n\in\mathbb{N}_0}\hat{\otimes}_\pi^n\,\mathcal{E}'$ of sources.
  Naturally topologized: loc.conv. direct sum topology, similar to $\mathcal{E}'$, i.e. dual nuclear Fréchet.

- Schwartz kernel thm gives some simplification: $\hat{\otimes}_\pi^n\,\mathcal{E}\equiv\mathcal{E}_n$ and $\hat{\otimes}_\pi^n\,\mathcal{E}'\equiv\mathcal{E}_n'$ (n-variate).

- One has $\big(\mathcal{T}(\mathcal{E})\big)'\equiv\mathcal{T}_a(\mathcal{E}')$ and $\big(\mathcal{T}(\mathcal{E})\big)''\equiv\mathcal{T}(\mathcal{E})$ etc, "nice" properties.
  Moreover, tensor algebra of field variations is topological unital bialgebra.
  Unity $\mathbb{1}:=(1,0,0,0,\ldots)$.
  Left-multiplication L_x by a fix element x meaningful and continuous linear.
  Left-insertion ι_p (tracing out) by $p\in\big(\mathcal{T}(\mathcal{E})\big)'\equiv\mathcal{T}_a(\mathcal{E}')$ also meaningful, continuous linear.
  Usual graded-commutation: $\big(\,\iota_p\,L_{\delta\psi}\,\pm\,L_{\delta\psi}\,\iota_p\,\big)\,G\;=\;(p|\delta\psi)\,G\quad(\forall p\in\mathcal{E}',\;\delta\psi\in\mathcal{E},\;G\,)$.

Take a classical observable $O:\ \boldsymbol{\mathcal{E}} \to \mathbb{R},\ \psi \mapsto O(\psi)$, let $O_{\psi_0} := O \circ (\mathrm{I}_{\mathcal{E}} + \psi_0)$.

That is, $O_{\psi_0}(\psi - \psi_0) \stackrel{!}{=} O(\psi) \quad (\forall \psi \in \boldsymbol{\mathcal{E}})$, with some fixed reference field $\psi_0 \in \boldsymbol{\mathcal{E}}$.

We say that O is multipolynomial iff for some $\psi_0 \in \boldsymbol{\mathcal{E}}$ there exists $\mathbf{O}_{\psi_0} \in \mathcal{T}_a(\mathcal{E}')$, such that

$$\forall\, \psi \in \boldsymbol{\mathcal{E}}: \quad \underbrace{O_{\psi_0}(\psi{-}\psi_0)}_{=\,O(\psi)} \;=\; \Big(\mathbf{O}_{\psi_0}\,\Big|\,\big(1,\ \overset{1}{\otimes}(\psi{-}\psi_0),\ \overset{2}{\otimes}(\psi{-}\psi_0),\ \ldots\big)\Big).$$

Similarly $E:\,\boldsymbol{\mathcal{E}}\to\mathcal{D}'$, $\psi\mapsto E(\psi)$, let $E_{\psi_0}:=E\circ(\mathrm{I}_{\mathcal{E}}+\psi_0)$ the same re-expressed on $\mathcal{E}$.

That is, $E_{\psi_0}(\psi-\psi_0)\overset{!}{=}E(\psi)\quad(\forall\psi\in\boldsymbol{\mathcal{E}})$, with some fixed reference field $\psi_0\in\boldsymbol{\mathcal{E}}$.

We say that E is multipolynomial iff $\exists\,\mathbf{E}_{\psi_0}\in\mathcal{T}_a(\mathcal{E}')\hat{\otimes}_\pi\mathcal{D}'$, such that

$$\forall\,\psi\in\boldsymbol{\mathcal{E}},\,\delta\psi_T\in\mathcal{D}:\;\underbrace{\Big(E_{\psi_0}(\psi-\psi_0)\,\Big|\,\delta\psi_T\Big)}_{=\big(E(\psi)\,\big|\,\delta\psi_T\big)}\;=\;\Big(\mathbf{E}_{\psi_0}\,\Big|\,\big(1,\,\overset{1}{\otimes}(\psi-\psi_0),\,\overset{2}{\otimes}(\psi-\psi_0),\,\ldots\big)\otimes\delta\psi_T\Big).$$

For fixed $\delta\psi_T\in\mathcal{D}$ one has $(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)\in\mathcal{T}_a(\mathcal{E}')$, i.e. one can left-insert with it:

$\iota_{(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)}$ meaningfully acts on $\mathcal{T}(\mathcal{E})$.

The master Dyson-Schwinger (MDS) equation is:

we search for (ψ_0, G_{ψ_0}) such that:

$$\underbrace{G^{(0)}_{\psi_0}}_{=:\, b\, G_{\psi_0}} \quad = \quad 1,$$

$$\forall\, \delta\psi_T \in \mathcal{D} : \qquad \underbrace{\Big(\boldsymbol{\iota}_{(\mathbf{E}_{\psi_0} \,|\, \delta\psi_T)} \;-\; \mathrm{i}\,\hbar\, L_{\delta\psi_T} \Big)}_{=:\, \mathbf{M}_{\psi_0,\,\delta\psi_T}} G_{\psi_0} \quad = \quad 0.$$

This substitutes Feynman functional integral formulation, signature independently.
Also, no fixed background causal structure etc needed.

[Feynman type quantum vacuum expectation value of O is then $\big(\mathbf{O}_{\psi_0} \,\big|\, G_{\psi_0}\big)$.]

Example: ϕ^4 model.

Euler-Lagrange functional is

$$E: \quad \boldsymbol{\mathcal{E}} \times \mathcal{D} \longrightarrow \mathbb{R}, \quad (\psi,\, \delta\psi_T) \longmapsto \int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\,\Box_y\psi(y)\,\mathrm{v}(y) \;+\; \int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\,\psi^3(y)\,\mathrm{v}(y).$$

$$\Downarrow$$

MDS operator at $\psi_0 = 0$ reads

$$\big(\mathbf{M}_{\psi_0,\delta\psi_T}\, G\big)^{(n)}(x_1,\ldots,x_n) =$$

$$\int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\,\Box_y G^{(n+1)}(y,x_1,\ldots,x_n)\,\mathrm{v}(y) \;+\; \int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\,G^{(n+3)}(y,y,y,x_1,\ldots,x_n)\,\mathrm{v}(y)$$

$$-\mathrm{i}\,\hbar\; \underbrace{n\,\tfrac{1}{n!}\textstyle\sum_{\pi\in\Pi_n}\delta\psi_T(x_{\pi(1)})\,G^{(n-1)}(x_{\pi(2)},\ldots,x_{\pi(n)})}_{=\,(L_{\delta\psi_T}\,G)^{(n)}(x_1,\ldots,x_n)}$$

Pretty much well-defined, and clear recipe, if field correlators were *functions*.

Theorem: no solutions with high differentiability (e.g. as smooth functions).

Theorem: for free Minkowski KG case, distributional solution only,
namely $G_{\psi_0} = \exp(K_{\psi_0})$, where

$$
\begin{aligned}
K_{\psi_0}^{(0)} &= 0, \\
K_{\psi_0}^{(1)} &= 0, \\
K_{\psi_0}^{(2)} &= \mathrm{i}\,\hbar\,\mathtt{K}_{\psi_0}^{(2)} && \longleftarrow \text{(symmetric propagator)} \\
K_{\psi_0}^{(n)} &= 0 && (n \geq 2)
\end{aligned}
$$

So we expect distributional solutions only, at best.

How can one extend to distributions interaction term like $G^{(n+3)}(y, y, y, x_1, ..., x_n)$?

With sufficiency condition of Hörmander? (Theorem: not workable.)

Via approximation with functions, i.e. sequential closure? (Theorem: not workable.)

Workaround in QFT: Wilsonian regularization using coarse-graining (UV damping).

Wilsonian regularized master Dyson-Schwinger equation

- When $\mathcal{E}$ (resp $\mathcal{D}$) are smooth sections of some vector bundle, denote by $\mathcal{E}^{\times}$ (resp $\mathcal{D}^{\times}$) the smooth sections of its densitized dual vector bundle. Then, distributional sections are $\mathcal{D}^{\times\prime}$ (resp $\mathcal{E}^{\times\prime}$).

- A continuous linear map $C:\,\mathcal{E}^{\times\prime}\to\mathcal{E}$ is called smoothing operator. Schwartz kernel theorem: $C \;\longleftrightarrow$ its Schwartz kernel κ which is section over $\mathcal{M}\times\mathcal{M}$.

- C_κ is properly supported iff $\forall\,\mathcal{K}\subset\mathcal{M}$ compact: $\kappa|_{\mathcal{M}\times\mathcal{K}}$ and $\kappa|_{\mathcal{K}\times\mathcal{M}}$ has compact supp. It extends to $\mathcal{E}^{\times\prime}$, $\mathcal{E}$, $\mathcal{D}$, $\mathcal{D}^{\times\prime}$ and preserves compact support (the transpose similarly).

- A properly supported smoothing operator is coarse-graining iff injective as $\mathcal{E}^{\times\prime}\to\mathcal{E}$ and its transpose similarly. E.g. ordinary convolution by a nonzero test function over affine (Minkowski) spacetime.

Coarse-graining ops are natural generalization of convolution by test functions to manifolds.

Originally: Feynman integral "$\Longleftrightarrow$" MDS equation.

Wilsonian regularized Feynman integral:
integrate only on the image space $C_\kappa[\mathcal{D}^{\times\prime}] \subset \mathcal{E}$ of some coarse-graining operator C_κ.

Wilsonian regularized Feynman integral "$\Longleftrightarrow$" Wilsonian regularized MDS equation:

we search for $(\psi_0, \gamma(\kappa), \mathcal{G}_{\psi_0,\kappa})$ such that:

$$\underbrace{\mathcal{G}^{(0)}_{\psi_0,\kappa}}_{=:\, b\,\mathcal{G}_{\psi_0,\kappa}} = 1,$$

$$\forall\, \delta\psi_T \in \mathcal{D} : \quad \underbrace{\Big(\boldsymbol{\iota}_{\gamma(\kappa)\,(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)} - \mathrm{i}\,\hbar\, L_{C_\kappa \delta\psi_T} \Big)}_{=:\, \mathbf{M}_{\psi_0,\kappa,\delta\psi_T}} \mathcal{G}_{\psi_0,\kappa} = 0.$$

Brings back problem from distributions to smooth functions, but depends on regulator κ.

Smooth function solution to free KG regularized MDS eq: $\mathcal{G}_{\psi_0,\kappa} = \exp(\mathcal{K}_{\psi_0,\kappa})$ where

$$\begin{aligned}
\mathcal{K}^{(0)}_{\psi_0,\kappa} &= 0, \\
\mathcal{K}^{(1)}_{\psi_0,\kappa} &= 0, \\
\mathcal{K}^{(2)}_{\psi_0,\kappa} &= \mathrm{i}\,\hbar\,\mathtt{K}^{(2)}_{\psi_0,\kappa} && \longleftarrow \text{(smoothed symmetric propagator)} \\
\mathcal{K}^{(n)}_{\psi_0,\kappa} &= 0 && (n \geq 2)
\end{aligned}$$

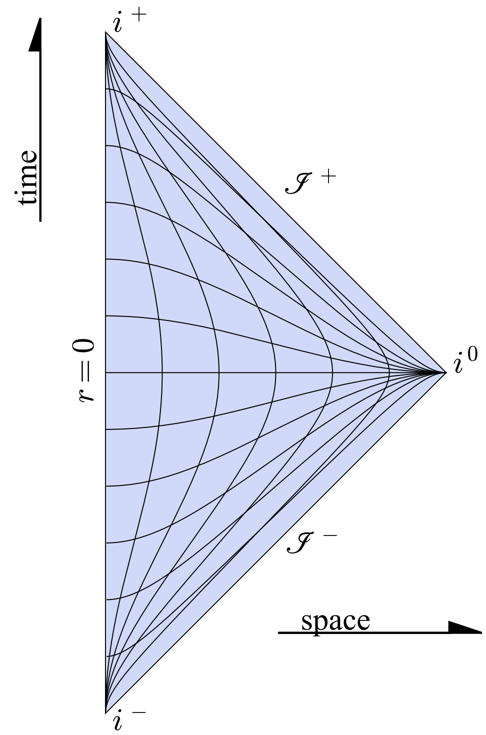
No problem to evaluate interaction term like $\mathcal{G}^{(n+3)}(y, y, y, x_1, ..., x_n)$ on functions.

[We proved a convergent iterative solution method at fix κ, see the paper or ask.]

But what we do with κ dependence? (Rigorous Wilsonian renormalization?)

Existence condition for regularized MDS solutions

If Euler-Lagrange functional $E : \boldsymbol{\mathcal{E}} \to \mathcal{D}'$ conformally invariant: re-expressable on Penrose conformal compactification.



That is always a compact manifold, with cone condition boundary.

$E : \boldsymbol{\mathcal{E}} \to \mathcal{D}'$ reformulable over this base manifold.

In such situation, $\mathcal{E} = \mathcal{D}$ and have nice properties:
countably Hilbertian nuclear Fréchet (CHNF) space.

$$F_0 \ \supset \ F_1 \ \supset \ \ldots \ \supset \ F_m \ \supset \ \ldots \ \supset \ \mathcal{E}$$

(Intersection of shrinking Hilbert spaces F_m with Hilbert-Schmidt embedding.)

Theorem [Dubin,Hennings:*P.RIMS* **25**(1989)971]:
without penalty, one can equip $\mathcal{T}(\mathcal{E})$ with a better topology, inheriting CHNF topology.

$$H_0 \ \supset \ H_1 \ \supset \ \ldots \ \supset \ H_m \ \supset \ \ldots \ \supset \ \mathcal{T}_h(\mathcal{E})$$

Regularized MDS operator is then a Hilbert-Schmidt linear map

$$\mathbf{M}_{\psi_0,\kappa} : \quad H_m \otimes F_m \longrightarrow H_0, \quad \mathcal{G} \otimes \delta\psi_T \longmapsto \mathbf{M}_{\psi_0,\kappa,\delta\psi_T} \mathcal{G}$$

Theorem: one can legitimately trace out $\delta\psi_T$ variable to form

$$\hat{\mathbf{M}}^2_{\psi_0,\kappa} : \quad H_m \longrightarrow H_m, \quad \mathcal{G} \longmapsto \sum_{i \in \mathbb{N}_0} \mathbf{M}^\dagger_{\psi_0,\kappa,\delta\psi_{T\,i}} \mathbf{M}_{\psi_0,\kappa,\delta\psi_{T\,i}} \mathcal{G}$$

By construction: $\mathcal{G}$ is κ-regularized MDS solution $\Longleftrightarrow$ $b\,\mathcal{G} = 1$ and $\hat{\mathbf{M}}^2_{\psi_0,\kappa}\mathcal{G} = 0$.

Theorem [A.L.]:

(i) the iteration

$$\mathcal{G}_0 \;:=\; \mathbb{1} \;\text{ and }\; \mathcal{G}_{l+1} \;:=\; \mathcal{G}_l - \tfrac{1}{T}\,\hat{\mathbf{M}}^2_{\psi_0,\kappa}\mathcal{G}_l \qquad (l = 0, 1, 2, ...)$$

is always convergent if $T >$ trace norm of $\hat{\mathbf{M}}^2_{\psi_0,\kappa}$.

(ii) the κ-regularized MDS solution space is nonempty iff

$$\lim_{l\to\infty} b\,\mathcal{G}_l \;\neq\; 0.$$

(iii) and in this case

$$\lim_{l\to\infty} \mathcal{G}_l$$

is an MDS solution, up to normalization factor.

Use for lattice-like numerical method in Lorentz signature?
(Treatment can be adapted to flat spacetime also, because Schwartz functions are CHNF.)

Structure of model building in fundamental physics

Relativistic or non-relativistic point mechanics:

- Take Newton equation over a fixed spacetime and fixed potentials.
- Solution space to the equation turns out to be a symplectic manifold.
- One can play classical probability theory on the solution space:
  - Elements of solution space X are elementary events.
  - Collection of Borel sets Σ of X are composite events.
  - A state is a probability measure W on Σ, i.e. (X, Σ, W) is classical probability space.

Relativistic or non-relativistic quantum mechanics:

- Take Dirac etc. equation over a fixed spacetime and fixed potentials.
- Finite charge weak solution space to the equation turns out to be a Hilbert space.
- One can play quantum probability theory on the solution space:
  - One dimensional subspaces of the solution space $\mathcal{H}$ are elementary events, X.
  - Collection of all closed subspaces Σ of $\mathcal{H}$ are composite events.
  - A state is a probability measure W on Σ, i.e. (X, Σ, W) is quantum probability space.

Fréchet derivative in top.vector spaces

Let F and G real top.affine space, Hausdorff.
Subordinate vector spaces: $\mathbb{F}$ and $\mathbb{G}$.

A map $S: F \to G$ is Fréchet-Hadamard differentiable at $\psi \in F$ iff:
there exists $DS(\psi) : \mathbb{F} \to \mathbb{G}$ continuous linear, such that for all sequence $n \mapsto h_n$ in $\mathbb{F}$, and nonzero sequence $n \mapsto t_n$ in $\mathbb{R}$ which converges to zero,

$$(\mathbb{G}) \lim_{n \to \infty} \left(\frac{S(\psi + t_n\, h_n) - S(\psi)}{t_n} - DS(\psi)\, h_n \right) \quad = \quad 0$$

holds.

Fréchet derivative of action functional

Fréchet derivative of $S : \boldsymbol{\mathcal{E}} \longrightarrow \mathrm{Meas}(\mathcal{M}, \mathbb{R})$ is

$$DS : \boldsymbol{\mathcal{E}} \times \mathcal{E} \longrightarrow \mathrm{Meas}(\mathcal{M}, \mathbb{R}), \; (\psi, \delta\psi) \longmapsto \Big(\mathcal{K} \mapsto \big(DS_{\mathcal{K}}(\psi)\big|\delta\psi\big)\Big)$$

For $\underbrace{(v, \nabla)}_{=:\psi} \in \boldsymbol{\mathcal{E}}$ given,

$$\underbrace{(\delta v, \delta C)}_{=:\delta\psi} \mapsto \big(DS_{\mathcal{K}}(v,\nabla) \,\big|\, (\delta v, \delta C)\big) =$$

$$\int_{\mathcal{K}} \Big(D_1 \mathrm{L}(v, \nabla v, P(\nabla))\, \delta v + D_2^a \mathrm{L}(v, \nabla v, P(\nabla))\, (\nabla_a \delta v + \delta C_a v) + 2\, D_3^{[ab]} \mathrm{L}(v, \nabla v, P(\nabla))\, \tilde{\nabla}_{[a} \delta C_{b]}\Big)$$

$$= \int_{\mathcal{K}} \Big(D_1 \mathrm{L}(v, \nabla v, P(\nabla))_{[c_1 \ldots c_m]}\, \delta v - \big(\tilde{\nabla}_a D_2^a \mathrm{L}(v, \nabla v, P(\nabla))_{[c_1 \ldots c_m]}\big)\, \delta v\Big) +$$

$$\Big(D_2^a \mathrm{L}(v, \nabla v, P(\nabla))_{[c_1 \ldots c_m]}\, \delta C_a v - 2\, \big(\tilde{\nabla}_a D_3^{[ab]} \mathrm{L}(v, \nabla v, P(\nabla))_{[c_1 \ldots c_m]}\big)\, \delta C_b\Big)$$

$$+ \; m \int_{\partial\mathcal{K}} \Big(D_2^a \mathrm{L}(v, \nabla v, P(\nabla))_{[a c_1 \ldots c_{m-1}]}\, \delta v + 2\, D_3^{[ab]} \mathrm{L}(v, \nabla v, P(\nabla))_{[a c_1 \ldots c_{m-1}]}\, \delta C_b\Big)$$

$$(m := \dim(\mathcal{M}))$$

[usual Euler-Lagrange bulk integral + boundary integral]

Distributions on manifolds

$W(\mathcal{M})$ vector bundle, $W^\times(\mathcal{M}) := W^*(\mathcal{M}) \otimes \overset{\dim(\mathcal{M})}{\wedge} T^*(\mathcal{M})$ its densitized dual.
$W^{\times\times}(\mathcal{M}) \equiv W(\mathcal{M})$.

Correspondingly: $\mathcal{E}^\times$ and $\mathcal{D}^\times$ are densitized duals of $\mathcal{E}$ and $\mathcal{D}$.

$\mathcal{E} \times \mathcal{D}^\times \to \mathbb{R}$, $(\delta\psi, p_T) \mapsto \int\limits_{\mathcal{M}} \delta\psi\, p_T$ and $\mathcal{D} \times \mathcal{E}^\times \to \mathbb{R}$, $(\delta\psi_T, p) \mapsto \int\limits_{\mathcal{M}} \delta\psi_T\, p$ jointly sequentially continuous.

Therefore, continuous dense linear injections $\mathcal{E} \to \mathcal{E}^{\times\prime}$ and $\mathcal{D} \to \mathcal{D}^{\times\prime}$.
(hance the name, distributional sections)

Let $A:\ \mathcal{E} \to \mathcal{E}$ continuous linear.
It has formal transpose iff there exists $A^t:\ \mathcal{D}^\times \to \mathcal{D}^\times$ continuous linear, such that
$\forall \delta\psi \in \mathcal{E}$ and $p_T \in \mathcal{D}^\times$: $\int\limits_{\mathcal{M}} (A\,\delta\psi)\, p_T = \int\limits_{\mathcal{M}} \delta\psi\, (A^t\, p_T)$.
Topological transpose of formal transpose $(A^t)':\ (\mathcal{D}^\times)' \to (\mathcal{D}^\times)'$ is the distributional extension of A. Not always exists.

Fundamental solution on manifolds

Let $E:\ \boldsymbol{\mathcal{E}} \times \mathcal{D} \to \mathbb{R}$ be Euler-Lagrange functional, and $J \in \mathcal{D}'$.

$\mathtt{K}_{(J)} \in \boldsymbol{\mathcal{E}}$ is solution with source J, iff $\forall \delta\psi_T \in \mathcal{D}:\ \left(E(\mathtt{K}_{(J)})\,|\,\delta\psi_T\right) = (J|\delta\psi_T)$.

Specially: one can restrict to $J \in \mathcal{D}^{\times} \subset \mathcal{E}^{\times} \subset \mathcal{D}'$.

A continuous map $\mathtt{K}:\ \mathcal{D}^{\times} \to \boldsymbol{\mathcal{E}}$ is fundamental solution, iff for all $J \in \mathcal{D}^{\times}$ the field $\mathtt{K}(J) \in \boldsymbol{\mathcal{E}}$ is solution with source J.

May not exists, and if does, may not be unique.

If $\mathtt{K}_{\psi_0}:\ \mathcal{D}^{\times} \to \mathcal{E}$ vectorized fundamental solution is linear (e.g. for linear $E_{\psi_0}:\ \mathcal{E} \to \mathcal{D}'$):
$\mathtt{K}_{\psi_0} \in \mathcal{L}in(\mathcal{D}^{\times}, \mathcal{E}) \subset (\mathcal{D}^{\times})' \otimes (\mathcal{D}^{\times})'$ is distribution.

Particular solutions to the free MDS equation

Distributional solutions to free MDS equation: $G_{\psi_0} = \exp(K_{\psi_0})$ where

$$\begin{aligned} K^{(0)}_{\psi_0} &= 0, \\ K^{(1)}_{\psi_0} &= 0, \\ K^{(2)}_{\psi_0} &= \mathrm{i}\,\hbar\,\mathtt{K}^{(2)}_{\psi_0} \\ K^{(n)}_{\psi_0} &= 0 \qquad\qquad (n \geq 2) \end{aligned}$$

Smooth function solutions to free regularized MDS equation: $G_{\psi_0} = \exp(K_{\psi_0,\kappa})$ where

$$\begin{aligned} K^{(0)}_{\psi_0,\kappa} &= 0, \\ K^{(1)}_{\psi_0,\kappa} &= 0, \\ K^{(2)}_{\psi_0,\kappa} &= \mathrm{i}\,\hbar\,(C_\kappa \otimes C_\kappa)\mathtt{K}^{(2)}_{\psi_0} \\ K^{(n)}_{\psi_0,\kappa} &= 0 \qquad\qquad (n \geq 2) \end{aligned}$$

[Here $C_\kappa(\cdot) := \eta \star (\cdot)$ is convolution by a test function η.]

Renormalization from functional analysis p.o.v.

Let $\mathbb{F}$ and $\mathbb{G}$ real or complex top.vector space, Hausdorff loc.conv complete.

Let $M : \mathbb{F} \rightarrowtail \mathbb{G}$ densely defined linear map (e.g. MDS operator).

Closed: the graph of the map is closed.

Closable: there exists linear extension, such that its graph closed (unique if exists).

Closable $\Leftrightarrow$ where extendable with limits, it is unique.

Multivalued set:
$\mathrm{Mul}(M) := \{ y \in \mathbb{G} \,|\, \exists\,(x_n)_{n\in\mathbb{N}}$ in $\mathrm{Dom}(M)$ such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} M x_n = y \}$.

$\mathrm{Mul}(M)$ always closed subspace.

Closable $\Leftrightarrow \mathrm{Mul}(M) = \{0\}$.

Maximally non-closable $\Leftrightarrow \mathrm{Mul}(M) = \overline{\mathrm{Ran}(M)}$. Pathological, not even closable part.

Polynomial interaction term of MDS operator maximally non-closable!

MDS operator:

$$\mathbf{M}: \quad \mathcal{D} \otimes \mathcal{T}(\mathcal{E}) \to \mathcal{T}(\mathcal{E}), \quad G \mapsto \mathbf{M}\, G$$

linear, everywhere defined continuous. So,

$$\mathbf{M}: \quad \mathcal{T}(\mathcal{D}^{\times\prime}) \rightarrowtail \mathcal{D}' \otimes \mathcal{T}(\mathcal{D}^{\times\prime}), \quad G \mapsto \mathbf{M}\, G$$

linear, densely defined.

Similarly: $\mathbf{M}_\kappa$ regularized MDS operator (κ: a fix regularizator).

Not good equation:

$$G \in \mathcal{T}(\mathcal{D}^{\times\prime})\,? \qquad G^{(0)} = 1 \;\text{ and }\; \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that } : \quad \lim_{\kappa\to\delta} \mathbf{M}\,\mathcal{G}_\kappa = 0.$$

All G would be selected, because $\mathrm{Mul}()$ set of interaction term is full space.

Not good equation:

$$G \in \mathcal{T}(\mathcal{D}^{\times\prime})\,? \qquad G^{(0)} = 1 \;\text{ and }\; \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that } : \quad \lim_{\kappa\to\delta} \mathbf{M}_\kappa\,\mathcal{G}_\kappa = 0.$$

All G would be selected, because $\mathrm{Mul}()$ set of interaction term is full space.

Can be good:

$$G \in \mathcal{T}(\mathcal{D}^{\times\prime})\ ? \qquad G^{(0)} = 1 \text{ and } \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that :}$$
$$\forall\, \kappa : \ \mathbf{M}_\kappa\, \mathcal{G}_\kappa = 0.$$

That is, as implicit function of κ, not as operator closure kernel.
But is this what we want? Why would there be some distinguished approximator?

Can be good:

$$G \in \mathcal{T}(\mathcal{D}^{\times\prime})\ ? \qquad G^{(0)} = 1 \text{ and } \forall\, \kappa \to \delta \text{ Dirac delta approximator sequence :}$$
$$\lim_{\kappa\to\delta} \mathbf{M}\, C_\kappa G = 0.$$

Extension of $\mathbf{M}$ by δ-closure.

Some complications on topological vector spaces

Careful with tensor algebra! Schwartz kernel theorems:

$$
\begin{array}{ccccccc}
\hat{\otimes}_\pi^n \mathcal{E} & \equiv & \mathcal{E}_n & \equiv & (\hat{\otimes}_\pi^n \mathcal{E}')' & \equiv & \mathcal{L}in(\mathcal{E}', \hat{\otimes}_\pi^{n-1} \mathcal{E}) \\
(\hat{\otimes}_\pi^n \mathcal{E})' & \equiv & \mathcal{E}'_n & \equiv & \hat{\otimes}_\pi^n \mathcal{E}' & \equiv & \mathcal{L}in(\mathcal{E}, \hat{\otimes}_\pi^{n-1} \mathcal{E}') \\
\hat{\otimes}_\pi^n \mathcal{D} & \leftarrow & \mathcal{D}_n & \equiv & (\hat{\otimes}_\pi^n \mathcal{D}')' & & \\
 & \text{cont.bij.} & & & & & \\
(\hat{\otimes}_\pi^n \mathcal{D})' & \rightarrow & \mathcal{D}'_n & \equiv & \hat{\otimes}_\pi^n \mathcal{D}' & \equiv & \mathcal{L}in(\mathcal{D}, \hat{\otimes}_\pi^{n-1} \mathcal{D}')
\end{array}
$$

$\mathcal{E} \times \mathcal{E} \to F$ separately continuous maps are jointly continuous.
$\mathcal{E}' \times \mathcal{E}' \to F$ separately continuous bilinear maps are jointly continuous.
For mixed, no guarantee.
For $\mathcal{D}$ or $\mathcal{D}'$ spaces, joint continuity from separate continuity of bilinear forms not automatic.
For mixed, even less guarantee.

But as convergence vector spaces, everything is nice with mixed $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{D}$, $\mathcal{D}'$ multilinears (separate sequential continuity $\Leftrightarrow$ joint sequential continuity).

Relation to usual RG theory:

Fix some $\eta \in \mathcal{S}$ such that $F(\eta) > 0$ and $F(\eta)$ is 1 around 0 momentum.
Introduce scaled η, that is $\eta_\Lambda(x) := \Lambda^N \eta(\Lambda\, x)$ (for all $x \in \mathbb{R}^N$ and scaling $1 \leq \Lambda < \infty$).
One has $\eta_\Lambda \xrightarrow{\mathcal{S}'} \delta$ as $\Lambda \longrightarrow \infty$.

By our theorem, for all Λ, one has $V_{C_{\eta_\Lambda}}(C_{\eta_\Lambda}\, \phi) = V(\phi)$ for γ_T-a.e. $\phi \in \mathcal{S}'$.

$\Downarrow$

Informally: ODE for $V_{C_{\eta_\Lambda}}$, namely $\frac{\mathrm{d}}{\mathrm{d}\Lambda} V_{C_{\eta_\Lambda}}(C_{\eta_\Lambda}\, \phi) = 0$ for $1 \leq \Lambda < \infty$.

QFT people try to solve such flow equation, given initial data $V_{C_\Lambda}\big|_{\Lambda=1}$.

But why bother? By our theorem, all RG flows of such kind has some V at the UV end.
Look directly for V?

What really the game is about?

Original problem:
- We had $\mathcal{V} : \{\text{function sense fields}\} \to \mathbb{R} \cup \{\pm\infty\}$, say $\mathcal{V}(\varphi) = g \int \varphi^4$.
- We would need to integrate it against γ_T, but that lives on $\mathcal{S}'$ fields.
- γ_T known to be supported "sparsely", i.e. not on function fields, but really on $\mathcal{S}'$.
- So, we really need to extend $\mathcal{V}$ at least γ_T-a.e. to make sense of $\mu := \mathrm{e}^{-V} \cdot \gamma_T$.

Caution by physicists: this may be impossible.
- We are afraid that V on $\mathcal{S}'$ might not exist.
- For safety, we push γ_T to smooth fields by C, do there $\mu_C := \mathrm{e}^{-V_C} \cdot C_* \gamma_T$.
- Then, get rid of C-dependence of μ_C by concept of Wilsonian RG flow.
  Maybe even $\mu_C \to \mu$ could exist as $C \to \delta$ if we are lucky...

Our result: we are back to the start.
- The UV limit Feynman measure μ then indeed exists.
- But we just proved that then there must exist some extension V of $\mathcal{V}$ to $\mathcal{S}'$, γ_T-a.e.
- So, we'd better look for that ominous extension V.
- For bounded from below $\mathcal{V}$, ess.bounded from below measurable V needed.
  If we find one, $\mu := \mathrm{e}^{-V} \cdot \gamma_T$ is then finite measure automatically.
  Only pathology: overlap integral of e^{-V} and γ_T expected small, maybe zero.
  We only need to make sure that $\int_{\phi \in \mathcal{S}'} \mathrm{e}^{-V(\phi)}\, \mathrm{d}\gamma_T(\phi) > 0$!

A natural extension[A.László, Z.Tarcsay, J.Ziebell **`arXiv:2502.16319`**]:

If $\mathcal{V}$ is bounded from below, there is an optimal extension, the "greedy" extension.

$$V(\cdot) \;:=\; \underset{\{\eta_n\to\delta\}}{(\gamma_T)\mathrm{inf}}\;\liminf_{n\to\infty}\;\mathcal{V}(\eta_n\star\;\cdot\;)$$

This is the lower bound of extensions, i.e. overlap of e^{-V} and γ_T largest.

But is V measurable at all? Not evident.

Theorem[A.László, Z.Tarcsay, J.Ziebell **`arXiv:2502.16319`**]:

1. The "greedy extension" is measurable.
2. The interacting Feynman measure $\mu := \mathrm{e}^{-V}\cdot\gamma_T$ by greedy extension is nonzero iff

$$\exists\;\eta_n\to\delta\;:\qquad \int\limits_{\phi\in\mathcal{S}'}\limsup_{n\to\infty}\;\mathrm{e}^{-\mathcal{V}(\eta_n\star\phi)}\;\mathrm{d}\gamma_T(\phi)\quad>\quad 0.$$

E.g. a sufficient condition:

$$\exists\;\eta_n\to\delta\;:\qquad \lim_{n\to\infty}\int\limits_{\phi\in\mathcal{S}'}\mathrm{e}^{-\mathcal{V}(\eta_n\star\phi)}\;\mathrm{d}\gamma_T(\phi)\quad>\quad 0.$$

This should be a calculable condition for concrete models!

So, it turns out that Wilsonian RG flow of correlators $\leftrightarrow$ distributional correlators.
(under mild conditions)

Executive summary:
- In QFT, the fundamental objects of interest are distributional field correlators.
- Physical ones selected by a "field equation", the master Dyson-Schwinger equation.
  Through their smoothed (Wilsonian regularized) instances [*CQG* **39**(2022)185004].

Academic question:
- What about Wilsonian RG flow of measures? (In Euclidean signature QFT.)
  [**`arXiv:2502.16319`**].